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Estimating a class of diffusion models: An evaluation of the effects of sampled discrete observations

Eugene M. Cleur

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Eugene M. Cleur
Facoltà di Economia, Via Cosimo Ridolfi 10,
56124 Pisa, Italia.
email: cleur@ec.unipi.it

Abstract

This paper considers, via a Monte-Carlo experiment, the effects of using sampled discrete data. Maximum likelihood, conditional least squares and indirect estimation procedures are considered. The strong relationship between the estimates of the drift and the diffusion coefficients is evidenced.The Monte-Carlo experiments are conducted on the well known Square Root and Ornstein-Uhlenbeck processes, both of which admit an "exact" solution in the distribution domain. The "exact" solutions may be exploited for correctly generating the underlying processes.

Some key words: diffusion processes, discrete maximum likelihood estimator, exact solution, Monte-Carlo Methods, stochastic differential equations.

1 Introduction

Of late, diffusion models expressed in the form of a Stochastic Differential Equation (SDE) have increasingly been considered and their estimation using data observed at discrete time points is currently attracting much attention. The application of simulation based inference in the estimation of such models has been particularly explored in the financial literature (see, for ex., Broze et al. (1995), Bianchi and Cleur (1996), Calzolari et al. (1998)), but much has yet to be done in terms of the computational problems involved in such a procedure.

Thus consider the estimation of the following class of one-dimensional SDE models :

$$dX_t = k(\vartheta - X_t)dt + \sigma X_t^{\beta} dW_t \qquad (1)$$

(1) implies a mean reversion towards the long-term mean, ϑ, with speed of adjustment given by k. σ is a scale parameter and β is the variance elasticity parameter which measures the sensitivity of relative changes to the level of the stochastic process X. k, ϑ and σ are strictly positive and $t \in [0, T]$. The parameter k also establishes the degree of convexity of the mean solution of the process.

Model (1) is of course a continuous time model. However, the best one can achieve on a digital computer is a representation of that process at discrete points in time. Such a representation may be achieved either through the exact analytical solution, when it is available, or through a numerical approximation. In Cleur and Manfredi (1999), this problem was analysed to some detail when the whole process is observed, rather than being sampled, and the importance of a higher order approximation scheme, such as the order 1.5 strong Taylor scheme, rather than the commonly used Euler approximation, was evidenced.

This paper addresses the problem of estimating a SDE process from a discrete approximation which is subsequently sampled. Such a problem arises from the widely used practice (see for ex. Bianchi and Cleur (1996), Ball and Torous (1996), Shoji and Ozaki (1997) and Calzolari et al (1998)) of approximating a SDE model by the Euler scheme approximation given by

$$X_{t_i} = X_{t_{i-1}} + k(\vartheta - X_{t_{i-1}})\delta + \sigma\sqrt{\delta} X_{t_{i-1}}^{\beta} W_{t_i} \ , \qquad (2)$$

where $\delta_i = t_i - t_{i-1}$ is the discretization step which, for simplicity, is assumed constant. However, since the Euler scheme gives a very poor approximation to the underlying continuous process when δ is not sufficiently small, it is argued that many observations must be generated on a very fine grid of time points in order to obtain an acceptable approximation to the underlying process at a much larger grid of values; for ex., in Ball and Torous (1996) interest rate data are generated from the Square Root process by setting δ

to 1/360, corresponding to daily data, which are then sampled every 30 observations in order to obtain so-called monthly data, i.e. data corresponding to time points with step 1/12, or in Calzolari et al. (1998), who consider the estimation of the Square Root and Ornstein-Uhelnbeck processes, the approximation is generated with $\delta = 1/20$ which is then sampled every 20 observations to obtain data at time points t = 1, 2, ... , T. The correctness of such an approach could be evaluated in those cases for which the exact solution is available; infact, in this paper data are generated using the exact solutions of the two processes considered and the results thereby obtained are compared with those obtained from data generated using the Euler scheme approximation. The discrete maximum likelihood estimator defined in Cleur and Manfredi (1999) is used for the comparison.

Results from a Monte Carlo experiment conducted on the estimation of the Square Root (SR henceforth) process, for which $\beta = 0.5$, and the Ornstein-Ulhenbeck (O-U henceforth) process, for which $\beta = 0$, are reported below.

2 Estimation Methods

2.1 A Discrete Maximum Likelihood Estimator

Consider the following general representation of equation (1)

$$dX_t = a(X_t, \Theta)dt + b(X_t, \sigma)dW_t \quad . \tag{3}$$

In continuous time, the following log likelihood ratio function defined as the Radon-Nikodym derivative dP_X/dP_W of the ratio of the probability measures P_X and P_W, corresponding to the processes X and W, is used for deriving maximum likelihood estimates (see, for ex., Kloeden et al (1994)):

$$\log L(\Theta) = \int_0^T \frac{a(X_t, \Theta)}{\{b(X_t, \sigma)\}^2} dX_t - \frac{1}{2} \int_0^T \frac{\{a(X_t, \Theta)\}^2}{\{b(X_t, \sigma)\}^2} dt \tag{4}$$

Maximizing (4) leads to the so-called Continuous Time Maximum Likelihood Estimator (see Liptser and Shiryayev (1981) and Kloeden et al. (1994)). In particular, if the SDE is defined as in (1), estimates of k and ϑ may be obtained conditional on given values of σ and β.

On the other hand, when the continuous signal is observed at equidistant time points t_0 , $t_1 = t_0 + \delta$, $t_0 + 2\delta$, ... , T, various approaches are available for obtaining Discrete Maximum Likelihood Estimators (DMLE henceforth) of k and ϑ for given values of σ and β; it is not possible to obtain maximum likelihood estimates of all the parameters simultaneously. One such approach, applied in Cleur and Manfredi (1999), consists in the derivation, for a given value of β and known σ, of explicit expressions for the estimators of k and ϑ in (4), which inevitably involve stochastic integrals.

In the two processes considered, β is fixed and hence does not have to be estimated. As for σ, the effect of substituting the following quadratic variation estimate as the initial value in the maximization of the likelihood is evaluated

$$\hat{\sigma}^2 = \sum_{t_i \leq T} \frac{(X_{t_i} - X_{t_{i-1}})^2}{X_{t_{i-1}}^{2\beta}} \tag{5}$$

This estimate has often been proposed in the literature on SDEs (see, for ex., Polson and Roberts (1994), Shoji and Ozaki (1996)) when the discretization step is sufficiently small.

The expressions for the estimates of k and ϑ are given in Cleur and Manfredi (1999).

The final estimate of σ is obtained, as for ex. in Shoji and Ozaki (1996), from the residuals calculated from the following reparametrization of (2)

$$\nu_{t_i} = \frac{X_{t_i} - X_{t_{i-1}}}{X_{t_{i-1}}^{\beta}} - \frac{\hat{k}\hat{\vartheta}\delta}{X_{t_{i-1}}^{\beta}} + \hat{k}X_{t_{i-1}}^{1-\beta} , \tag{6}$$

2.2 A Conditional Least Squares Estimator

The conditional least squares estimator (CLSE henceforth) considered in this paper is that presented in Overbeck and Ryden (1997). In their paper, Overbeck and Ryden (1997) consider the estimation of the Square Root process and results from a Monte-Carlo experiment are reported for data reproduced by the exact solution, i.e. the non-central chi-square conditional distribution, and a step size equal to $\delta = \Delta = 1.0$. They did not consider the problem of a sampled process and despite the (large) discretiszation step their simulation results indicate in the CLSE a potentially good estimator. The expressions for the estimates of k and ϑ, which do not depend from σ, are given by

$$\hat{k} = -b, \qquad \hat{\vartheta} = -a/b \tag{7}$$

where the expressions for a and b are given in Overbeck and Ryden (1997) and $\hat{\sigma}$ is obtained through the pseudo-likelihood method defined by equation (10) for the Square Root process in the same paper.

For the Ornstein-Uhlenbeck process the same definitions hold except for the conditional variance which is given by equation (9) of this paper.

2.3 An Indirect Estimator

The indirect estimator used in this paper was presented in Gourieroux et al. (1994) and has been widely experimented (see, for ex., Bianchi and Cleur (1996), Broze et al. (1995), Cleur and Manfredi (1999), Calzolari et al. (1998)). Roughly speaking, given an observed series from which a certain model has to be estimated, the indirect estimator consists in simulating a series of data from that model such that the difference between the real data and the simulated data is as small as possible according to some statistical criterion.

3 Simulation Experiments

The SR and O-U processes are part of a class of SDE processes which admit an exact solution. In particular, the conditional distribution, p(X(t)/X(s)), of the SR process has a closed form solution which is given by the stationary non-central chi-square distribution (for its generation on the computer see, for ex., Johnson and Kotz (1992)) with $2q$ degrees of freedom and noncentrality parameter u, where

$$u(s) = 2c(s)X(s)e^{-k(t-s)}; \qquad c(s) = \frac{2k}{\sigma^2(1-e^{-k(t-s)})}; \qquad q = \frac{2k\vartheta}{\sigma^2}$$

The O-U process too has a conditional distribution which is a stationary Gaussian distribution with

$$E\left\{X(t)\,/\,X(s)\right\} = \left\{\vartheta + (x_s - \vartheta)e^{-k(t-s)}\right\} \tag{8}$$

and

$$Var\{X(t)\,/\,X(s)\} = \sigma^2(1 - e^{-2k(t-s)})\,/\,(2k) \tag{9}$$

It should be mentioned that the O-U process also has an exact solution in the trajectory domain which was used in Cleur and Manfredi (1999).

For both the SR and O-U processes, a Monte-Carlo experiment is carried out by first generating the data using their exact solutions as well as the corresponding Euler scheme approximations with the following constellation of parameter values: $k = 0.8$, $\vartheta = 0.10$, $\sigma = 0.06$, and discretization step $\delta = 0.01$. These are subsequently sampled by taking every fifth, tenth, twenyfifth, fiftieth and hundreth generated value; in other words, five different sample steps are examined, i.e. $\Delta = 0.05$, 0.10, 0.25, 0.50 and 1.00. Processes of length T = 100 and T = 2000 are considered. Each combination between the above constellation of parameter values, sample steps and generation schemes is replicated 10000 times.

The starting value for each replication, X_0, is always set to the long-term mean of the process, i.e. $X_0 = \vartheta = 0.1$.

The parameters are estimated using the procedures outlined in the previous Section.

3.1 Estimation of the SR process with known σ

In this Section, the importance of having a good initial value for σ in the DMLE is evidenced by considering the performance of this estimator in the special case when σ is known.

Tables 1 and 2 summarize the results when the generated process is sampled at regular intervals, and k and ϑ are calculated from the expressions in Cleur and Manfredi (1999) with σ taken as known. The standard errors refer to the 10000 estimates, i.e. what is commonly known as the Monte Carlo standard error.

The estimation of ϑ does not pose any problems, as will be observed in the rest of this paper, and will therefore not be commented any further.

Table 1. DMLE of the Square Root process. T = 100
True Values: k = 0.8, $\vartheta = 0.10$. $\sigma = 0.06$ taken as known,
$\delta = 0.01$ (standard errors in brackets)

	Δ = 0.01	Δ = 0.05	Δ = 0.10	Δ = 0.25	Δ = 0.50	Δ = 1.00
$\hat{k}$	.8403	.8403	.8407	.8406	.8414	.8499
	(.1319)	(.1319)	(.1339)	(.1339)	(.1377)	(.1521)
$\hat{\vartheta}$	.1000	.1000	.1000	.1000	.1000	.9999E-1
	(.2359E-2)	(.2360E-2)	(.2392E-2)	(.2380E-2)	(.2385E-2)	(.2444E-2)

Table 2. DMLE of the Square Root process. T = 2000
True Values: k = 0.8, $\vartheta = 0.10$. $\sigma = 0.06$ taken as known,
$\delta = 0.01$ (standard errors in brackets)

	Δ = 0.01	Δ = 0.05	Δ = 0.10	Δ = 0.25	Δ = 0.50	Δ = 1.00
$\hat{k}$	.8018	.8019	.8019	.8019	.8016	.8022
	(.2852E-1)	(.2856E-1)	(.2862E-1)	(.2829E-1)	(.2910E-1)	(.3168E-1)
$\hat{\vartheta}$	.1000	.1000	.1000	.1000	.1000	.9999E-1
	(.5454E-3)	(.5440E-3)	(.5428E-3)	(.5403E-3)	(.5482E-3)	(.5520E-3)

The differences in the estimates of k for varying sample steps are very limited thereby suggesting that, when σ is known, it does not matter whether the whole process is observed or whether it is sampled, because the resulting DMLE behaves in much the same way in both cases. Further, the estimate of k is not significantly biased only if the observed process is sufficiently long (T=2000). In other words, for short series of data, whether they are sampled or not, it will not be possible to obtain unbiased estimates of k even though σ is known.

3.2 Estimation of the SR process with unknown σ

As mentioned above, in the published literature the quadratic variation estimator (5) has often been used for estimating σ. Before undertaking the estimation of the parameters of the two models considered in this paper, a preliminary simulation experiment, based on 10000 replications, is carried out in order to evaluate the behaviour of this estimator for data sampled at fixed time intervals. The results, for the SR process is reported in Table 3; those for the O-U process are very similar and are therefore omitted.

Table 3. Quadratic Variation Estimate of σ in the Square Root process
True values: k $= 0.8$, $\vartheta = 0.10$, $\sigma = 0.06$, $\delta = 0.01$
(standard errors in brackets)

T	$\Delta = 0.01$	$\Delta = 0.05$	$\Delta = 0.10$	$\Delta = 0.25$	$\Delta = 0.50$	$\Delta = 1.00$
100	.5988E-1	.5941E-1	.5885E-1	.5710E-1	.5455E-1	.4994E-1
	(.4230E-3)	(.9802E-3)	(.1351E-2)	(.2129E-2)	(.2869E-2)	(.4017E-2)
2000	.5989E-1	.5943E-1	.5886E-1	.5722E-1	.5469E-1	.5010E-1
	(.9451E-4)	(.2178E-3)	(.9451E-4)	(.4687E-3)	(.6534E-3)	(.9149E-3)

The length of the generated process does not appear to influence the sample bias of the estimator which is not significant only when the sample step $\Delta = \delta = 0.01$. The results of a few experiments conducted with $\delta = 0.001$ confirmed the need for a sample step of at least 0.01 in order to have a quadratic variation estimate of σ which is not significantly biased.

As could be envisaged, the substitution of a badly biased estimate of σ will, consequently, have notable repercussions on the estimates of k and ϑ.

Tables 4 and 5 report the estimates of k and ϑ calculated from the expressions in Cleur and Manfredi (1999) with σ substituted by its quadratic variation estimate reported in Table 3. The final estimate of σ based on the residuals from the reparametrized Euler scheme defined in equation (6) is reported in the same Tables. These results are very close to those in Tables 3 and 4 when the estimate of σ is close to the corresponding true value. An increasing underestimation in $\hat{\sigma}$ leads to an increasing underestimation in $\hat{k}$. The fact that the quadrature variation estimator of σ performs much better than the final estimator reported in Table 4 and 5 should come as no surprise; infact, a strongly biased initial value for σ should have a negative effect on the estimates of k and ϑ which, in turn, leads to a poor final estimate of σ.

Table 4. DMLE of the Square Root process. T $= 100$
True Values: k $= 0.8$, $\vartheta = 0.10$, $\sigma = 0.06$, $\delta = 0.01$
(standard errors in brackets)

	$\Delta = 0.01$	$\Delta = 0.05$	$\Delta = 0.10$	$\Delta = 0.25$	$\Delta = 0.50$	$\Delta = 1.00$
k	.8371	.8241	.8089	.7616	.6934	.5795
	(.1314)	(.1293)	(.1289)	(.1199)	(.1090)	(.9329E-1)
ϑ	.1000	.1000	.1000	.1000	.1000	.9998E-1
	(.2360E-2)	(.2360E-2)	(.2393E-2)	(.2383E-2)	(.2391E-2)	(.2459E-2)
σ	.5976E-1	.5880E-1	.5762E-1	.5436E-1	.4962E-1	.4199E-1
	(.4256E-3)	(.9318E-3)	(.1292E-2)	(.1945E-2)	(.2537E-2)	(.3101E-2)

Table 5. DMLE of the Square Root process. T = 2000
True Values: k = 0.8, $\vartheta = 0.10$, $\sigma = 0.06$, $\delta = 0.01$
(standard errors in brackets)

	Δ = 0.01	Δ = 0.05	Δ = 0.10	Δ = 0.25	Δ = 0.50	Δ = 1.00
k	.7988	.7868	.7718	.7297	.6658	.5588
	(.2844E-1)	(.2807E-1)	(.2719E-1)	(.2570E-1)	(.2373E-1)	(.2056E-1)
ϑ	.1000	.1000	.1000	.1000	.1000	.1000
	(.5441E-3)	(.5440E-3)	(.5429E-3)	(.5451E-3)	(.5483E-3)	(.5556E-3)
σ	.5976E-1	.5883E-1	.5770E-1	.5452E-1	.4989E-1	.4254E-1
	(.9388E-4)	(.2081E-3)	(.2918E-3)	(.4400E-3)	(.5646E-3)	(.6890E-3)

In Cleur and Manfredi (1999), it was observed that when the discretization step is sufficiently small, a value of $\delta = 0.01$ was used in that paper, the Euler scheme gives a satisfactory approximation to the underlying continuous process observed at discrete time points in the sense that results obtained from data generated by the exact solution and by the Euler scheme substantionally coincided. This is also confirmed in the context of the present paper so that, for the sake of economy, the corresponding tables of results will be omitted.

The results for the CLSE are presented in Tables 6 and 7 and, as can be seen, are very different from those for the DMLE. Most importantly, there is only a small and insignificant bias in the estimate of σ. For a small T (i.e. T = 100), the estimate of k increases slightly over the range of sampling steps and does not drastically decrease as with the DMLE; in any case, the bias in $\hat{k}$ cannot be ignored. For a large T (i. e. T = 2000) instead, the CLSE converges to the true values.

Table 6. CLSE of the Square Root process. T = 100
True Values: k = 0.8, $\vartheta = 0.10$, $\sigma = 0.06$, $\delta = 0.01$
(standard errors in brackets)

	Δ = 0.01	Δ = 0.05	Δ = 0.10	Δ = 0.25	Δ = 0.50	Δ = 1.00
k	.8436	.8409	.8437	.8472	.8536	.8815
	(.1365)	(.1389)	(.1409)	(.1503)	(.1730)	(.2410)
ϑ	.9997E-1	.9997E-1	.9999E-1	.1000	.1000	.1000
	(.2352E-2)	(.2353E-2)	(.2378E-2)	(.2348E-2)	(.2368E-2)	(.2431E-2)
σ	.6000E-1	.6002E-1	.6003E-1	.6014E-1	.6032E-1	.6092E-1
	(.4437E-3)	(.1010E-2)	(.1436E-2)	(.2417E-2)	(.3783E-2)	(.6578E-2)

Table 7.CLSE of the Square Root process. T = 2000
True Values: k = 0.8, $\vartheta = 0.10$, $\sigma = 0.06$, $\delta = 0.01$
(standard errors in brackets)

	$\Delta = 0.01$	$\Delta = 0.05$	$\Delta = 0.10$	$\Delta = 0.25$	$\Delta = 0.50$	$\Delta = 1.00$
k	.8020	.8020	.8021	.8020	.8024	.8035
	(.2906E-1)	(.2954E-1)	(.3002E-1)	(.3192E-1)	(.3569E-1)	(..4561E-1)
ϑ	.9999E-1	.9999E-1	.9999E-1	.1000	.9999E-1	.9999E-1
	(.5285E-3)	(.5285E-3)	(.5287E-3)	(.5288E-3)	(.5351E-3)	(.5414E-3)
σ	.6000E-1	.6000E-1	.6000E-1	.6000E-1	.6001E-1	.6005E-1
	(.9858E-4)	(.2236E-3)	(.2918E-3)	(.5404E-3)	(.8285E-3)	(.1386E-2)

The fact that the CLSE provides a very good estimate of σ for the whole range of sampling steps suggests that this estimator and not the quadratic variation estimator applied above could be used for obtaining the initial value of σ in the DMLE. If this is followed, for a small T the resulting estimates will, in any case, be biased as is evident from Table 1, but for a large T results similar to those in Table 8 will be obtained.

Table 8. DMLE of the Square Root process. T = 2000
True Values: k = 0.8, $\vartheta = 0.10$, $\sigma = 0.06$, $\delta = 0.01$
initial value of σ= CLSE of σ(standard errors in brackets)

	$\Delta = 0.05$	$\Delta = 0.10$	$\Delta = 0.25$	$\Delta = 0.50$	$\Delta = 1.00$
k	.8015	.8020	.8019	.8017	.8034
	(.2964E-1)	(.3011E-1)	(.3149E-1)	(.3440E-1)	(.4380E-1)
ϑ	.9997E-1	.1000	.1000	.9997E-1	.1000
	(.5317E-3)	(.5429E-3)	(.5327E-3)	(.5306E-3)	(.5371E-3)
σ	.6000E-1	.6000E-1	.5999E-1	.5998E-1	.5997E-1
	(.2234E-3)	(.3240E-3)	(.5520E-3)	(.8186E-3)	(.1390E-2)

No apparent gain can be seen in Table 8 with respect to Table 7 so that the CLSE might probably be the best choise when T is large.

3.3 An Indirect Estimate of the SR process

In recent years, indirect estimation procedures via simulation have been increasingly applied. Such procedures, based on repeated approximations to the underlying model to be estimated, are computationally intensive, but have provided very promising results in the estimation of SDE processes (see, for example, Bianchi and Cleur (1996), Broze et al. (1995), Calzolari et al. (1998), Cleur and Manfredi (1999) and Gourieroux et al (1994)). In the rest of this paper, the capability of the indirect estimation procedure defined in Gourieroux et al. (1994) in correcting for the heavy bias, due to the sampling of the underlying process, will be examined; it was observed above that as the sampling step increased, so to did the bias in the estimates of k and σ. Table 8 summarizes the results for the SR process when T=100. Calibration, which is an integral part of the procedure, was carried out on the DMLE analyzed above.

Table 9. Indirect Estimates of the Square Root process. T = 100
True Values: k = 0.8, $\vartheta = 0.10$, $\sigma = 0.06$, $\delta = 0.01$.
(standard errors in brackets)

	$\Delta = 0.05$	$\Delta = 0.10$	$\Delta = 0.25$	$\Delta = 0.50$	$\Delta = 1.00$
k	.8038	.8046	.8062	.8078	.8072
	(.1874)	(.1938)	(.2019)	(.2224)	(.2616)
ϑ	.1000	.1000	.1000	.1000	.1000
	(.3585E-2)	(.3588E-2)	(.3596E-2)	(.3635E-2)	(.3648E-2)
σ	.6006E-1	.6012E-1	.6017E-1	.6026E-1	.6061E-1
	(.1336E-2)	(.2015E-2)	(.3362E-2)	(.5199E-2)	(.8336E-2)

The overall capability of the indirect estimate to correct for bias is clearly evidenced in Table 6. It may be noted that, as the sampling step increases, so to does the bias in the estimates of k and σ which however remains very small and insignificant; the bias in the estimates of σ is present only at the fourth decimal point. The corresponding standard errors of the estimates, which are notably higher than their counterparts in Tables 4 and 6, increase more markedly than the bias in the estimates themselves and this, perhaps, constitutes the only observable defect in the indirect estimation procedure.

These results enforce the conclusion in Cleur and Manfredi (1999) where the indirect estimation procedure was proposed as a general strategy in its own rights every time the data does not satisfy the optimal conditions, among which a small sample step, necessary for obtaining good estimates of the parameters of the underlying continuous process.

3.4 Estimation of The O-U process

Estimates of the Ornstein-Uhlenbeck process have much the same properties of the estimates of the SR process reported above. Results for T = 100 are summarized in Tables 9-12.

Table 10. DMLE of the Ornstein-Uhlenbeck process. T = 100
True Values: $k = 0.8$, $\vartheta = 0.10$. $\sigma = 0.06$ taken as known, $\delta = 0.01$
(standard errors in brackets)

	Δ = 0.01	Δ = 0.05	Δ = 0.10	Δ = 0.25	Δ = 0.50	Δ = 1.00
$\hat{k}$	.8407	.8396	.8418	.8410	.8401	.8487
	(.1324)	(.1332)	(.1347)	(.1337)	(.1353)	(.1503)
$\hat{\vartheta}$	.9999E-1	.9999E-1	.9995E-1	.1000	.9999E-1	.1000
	(.7478E-2)	(.7461E-2)	(.7523E-2)	(.7517E-2)	(.7582E-2)	(.7738E-2)

Table 11. DMLE of the Ornstein-Uhlenbeck process. T = 100
True Values: $k = 0.8$, $\vartheta = 0.10$, $\sigma = 0.06$, $\delta = 0.01$
(standard errors in brackets)

	Δ = 0.01	Δ = 0.05	Δ = 0.10	Δ = 0.25	Δ = 0.50	Δ = 1.00
k	.8373	.8224	.8077	.7590	.6875	.5719
	(.1319)	(.1304)	(.1289)	(.1200)	(.1079)	(.9162E-1)
ϑ	.9999E-1	.9999E-1	.9995E-1	.1000	.9999E-1	.1000
	(.7478E-2)	(.7462E-2)	(.7527E-2)	(.7527E-2)	(.7605E-2)	(.7790E-2)
σ	.5975E-1	.5879E-1	.5760E-1	.5429E-1	.4951E-1	.4183E-1
	(.4200E-3)	(.9296E-3)	(.1298E-2)	(.1916E-2)	(.2477E-2)	(.2997E-2)

Table 11. CLSE of the Square Root process. T = 100
True Values: $k = 0.8$, $\vartheta = 0.10$, $\sigma = 0.06$, $\delta = 0.01$
(standard errors in brackets)

	Δ = 0.01	Δ = 0.05	Δ = 0.10	Δ = 0.25	Δ = 0.50	Δ = 1.00
k	.8409	.8405	.8437	.8456	.8522	.8822
	(.1331)	(.1363)	(.1408)	(.1495)	(.1690)	(.2419)
ϑ	.9999E-1	.9999E-1	.9995E-1	.1000	.9998E-1	.1000
	(.7416E-2)	(.7395E-2)	(.7465E-2)	(.7447E-2)	(.7525E-2)	(.7688E-2)
σ	.6000E-1	.6002E-1	.6005E-1	.6012E-1	.6036E-1	.6098E-1
	(.4237E-3)	(.9690E-3)	(.1410E-2)	(.2359E-2)	(.3695E-2)	(.6651E-2)

Table 12. Indirect Estimates of the Ornstein-Uhlenbeck process. T = 100
True Values: k = 0.8, $\vartheta = 0.10$, $\sigma = 0.06$, $\delta = 0.01$.
(standard errors in brackets)

	$\Delta = 0.05$	$\Delta = 0.10$	$\Delta = 0.25$	$\Delta = 0.50$	$\Delta = 1.00$
k	.8045	.8044	.8044	.8045	.8048
	(.1869)	(.1938)	(.2009)	(.2224)	(.2616)
ϑ	.9967E-1	.9969E-1	.9961E-1	.9957E-1	.9962E-1
	(.1122E-1)	(.1127E-1)	(.1137E-1)	(.1154E-1)	(.1168E-1)
σ	.6006E-1	.6012E-1	.6017E-1	.6028E-1	.6055E-1
	(.1335E-2)	(.2021E-2)	(.3352E-2)	(.5187E-2)	(.8345E-2)

4 Conclusions

The estimates of diffusion models from discrete data using standard methods when T is small are significantly biased even when the diffusion coefficient, σ, is known. All maximum likelihood estimators tend to behave like the DMLE considered in this paper so that the possibility of obtaining a good initial value for σ becomes crucial. Simulation based estimators like the one defined Gourieroux et al (1994) instead appear to resolve this problem very well. For a large T, instead, a conditional least squares estimator defined in Overbeck and Ryden (1997) should be preferred to the maximum likelihood estimators of the type considered in this paper.

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References

[1] Ball, C. A. and Torous, W. N. (1996), Unit roots and the estimation of interest rate dynamics, Journal of Empirical Finance, 3, 215-238.

[2] Bianchi, C., Cesari, R. and Panattoni, L. (1994), Alternative estimators of a diffusion model of the term structure of interest rates. A Monte Carlo comparison, Temi di discussione della Banca d'Italia

[3] Bianchi, C. and Cleur, E. M. (1996), Indirect estimation of stochastic differential models: some computational experiments, Computational Economics, 9, 257-274.

[4] Broze, L., Scaillet, O. and Zakoian, J-M. (1995), Quasi Indirect Inference for Diffusion Processes,Discussion Paper n. 9095, CORE, Louvain.

[5] Calzolari, G., Di Orio, F. and Fiorentini, G. (1998), Control variates for variance reduction in indirect inference: interest rate models in continuous time, Econometric Journal, Vol.1, C100-C112.

[6] Cleur, E.M.and Manfredi, P. (1999), One dimensional SDE models, low order numerical methods and simulation based estimation: a comparison of alternative estimators, forthcoming in Computational Economics

[7] Gard, T. (1987), Stochastic differential equations, Marcel Dekker, New York.

[8] Johnson, N. and Kotz, T. (1992), Continuous Univariate Distributions, Wiley, New York.

[9] Kloeden, P.E. and Platen, E. (1994), Numerical Solution of SDE, Springer Verlag, New York.

[10] Kloeden P.E., Platen, E. and Schurz, H. (1995), Numerical Solution of SDE through Computer Experiments, Springer, New York.

[11] Liptser, L.S. and Shiryayev, A. N. (1981), Statistics of random Processes, Springer Verlag, New York.

[12] Polson, N. G. and Roberts, G. O. (1994), Bayes factors for discrete observations from diffusion processes, Biometrika, Vol. 81,1, 11-26.

[13] Shoji, I. and Ozaki, T. (1997), Comparative Study of Estimation Methods for Continuous Time Stochastic Prosesses, Journal of Time Series Analysis, Vol.18, n.5, 485-506.