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ABSTRACT

The dynamical consequences of the endogeneization of the supply of labour within the basic neoclassical Solow's model in continuous time are investigated. The rate of change of the supply of labour is modelled via a malthusian relation between fertility and income, whereas the process of entry into the labour market is modelled via a suitable time-delay representation. Strongly persistent oscillations appears via the mechanism of the Hopf bifurcation which suggest a simple alternative explanation of the demoeconomic fluctuation. Hence the present work shows that the basic neoclassical growth paradigm, once endowed with a "realistic" formulation of the labour supply, becomes capable to endogenously explain the main stylised fact of economic growth, namely the generation of globally stable oscillations around a path of balanced growth.

1 Introduction

The aim of the present paper is to investigate the effects of endogenous labour supply dynamics within the standard neoclassical Solow's growth model (Solow 1956) in continuous time. This endogeneisation is based on a simple coupling of an age structure argument (absent in the original Solow's) plus a "classical" relation between fertility and income, which was well recognised by Solow itself. A realistic formulation of the process of labour supply recruitment must take into account past demographic behaviours in that the new entries into the labour force at time t are the outcome of the fertility behaviour of past generations. This process is "filtered" by the age structure mechanism, which is embedded in our Solow-type model by resorting to time-lags. As largely recognised, time-delays represent a simple and clever way to embed age structure within complex models (see for instance the classical work by McDonald (1978,1989)). Moreover fertility is assumed to positively depend on the stage of economic growth, synthetised by an index of the level of percapita income. This dates back to the classical malthusian view (Malthus 1798[1]). Positive relations between fertility and income, although often confuted in the rich countries, still keep relevance in the developing world. Moreover very recent modelling efforts are based on such assumption. For instance Prskawetz and Feichtinger (1995, p. 61) notice : "... *the wealth of the industrial countries may distract attention from malthusian forces that nevertheless are quite visible in many developing countries.*"

Other early works on the interactions between population growth and the economy are for instance Day (1983), Day and Walther (1989), Day et al. (1989), Feichtinger and Sorger (1990), Feichtinger and Dockner (1990), Prskawetz and Feichtinger (1995). In particular Day (1983), Day and Walther (1989) and Prskawetz and Feichtinger (1995) postulate a nonlinear interaction between population growth and the economy in a discrete-time framework. Although all these contributions are capable to obtain complex behaviours, they pay sometimes the price to resort to ad-hoc or complicated assumptions. Moreover they exploit the "complex behaviours potential" embedded within discrete nonlinear maps. The aim of the present paper is somewhat different: we aim to verify whether highly general assumptions are nonetheless capable to preserve nontrivial dynamical behaviours, i.e., first of all, persistent oscillations.

For this purpose we adopt the simplest possible relation between fertility and income (i.e.: a linear relation), and resort to a continuous time-framework. As the present investigation demonstrates, stable oscillatory behaviours may be generated within the Solow's model of balanced growth, in a fully endogenous manner (contrary to other neoclassical models such as the Real Business Cycle scheme, where oscillations are induced by a stochastic forcing), by resorting even to fairly simple assumptions. More precisely, persistent oscillations may occur in the Solow's model when the rate of change of the labour supply is correctly assumed to depend (even in the simplest manner) on past demographic behaviours. From this point of view our results appear of some interest in the area of the neoclassical theory of growth. As well known, a main contribution of the 1956 paper by Solow has been to prove that in one-good economies, provided the production function satisfies the standard neoclassical conditions, then it always exists a long term globally stable balanced growth. This central result has permitted to overtake the "unhappy" view of the Harrod and Domar's knife-edge, by showing that growth can be a rule for the economy. The present work shows that the basic neoclassical growth paradigm not only explains the stylised fact of balanced growth, but, once endowed with a correctly demographically founded formulation of the labour supply, becomes capable to endogenously explain the

[1] As known Malthus was first to develop, in the Essay, a general scheme of demo-economic dynamics with fully endogenous population depending on economic growth.

other main stylised fact of economic growth, namely the generation of globally stable oscillations around a path of balanced growth. Altough other modeling efforts (several among the aforementioned authors) have found growth with cycle within the descriptive (i.e.: non optimal) neoclassical growth model, the present work appears to be appealing in that it is based on a minimal set of extra nonlinear ingredients.

The present paper is organised as follows. In the second section a basic Solow-type model with endogenous population depending on the current fertility is introduced and its properties are studied. In the third section we introduce a more general model embedding a time-lag in the reaction of rate of change of the supply of labour to past levels of income, and its properties are investigated by means of local stability analysis plus Hopf bifurcation. In the fourth section we complete the analysis of the model by investigating its global properties via numerical simulations and we discuss the economic meaning of its main results. The core of our results is summarised in the conclusions.

2 A basic model with endogenous population

The standard textbook form of the classical Solow growth model is defined by the ordinary differential equation (ODE):

$$\dot{k} = sf(k) - (\delta + n)\,k \tag{1}$$

where $k = K/L$ denotes the capital-labour ratio, $f(k)$ the production per unit of labour, s the saving rate ($0 \leq s \leq 1$), $\delta > 0$ the rate of capital depreciation and $n > 0$ the rate of growth of the supply of labour, which is assumed fully exogenous. In particular, when a Cobb-Douglas production function is chosen, the model collapses in the following Bernoulli ODE:

$$\dot{k} = sk^{\alpha} - (\delta + n)\,k \tag{2}$$

where α denotes the elasticity of substitution of the capital ($0 \leq \alpha \leq 1$).

The problem of the endogeneisation of the supply of labour was already broadly considered by Solow (1956) itself, who specified the rate of change of the supply of labour n as a function of the current level of percapita income $f(k)$, by writing:

$$n = n(f(k)) = n(k^{\alpha}) \tag{3}$$

Solow made a quick qualitative analysis of the effects of a very general (non monotonic) form for the n function, and evidenced the conditions under which labour supply effects may gives rise to instability with respect to his balanced path of economic growth.

As in this paper we are essentially interested in the effects of forces of "fundamental" nature, in what follows we will assume, for simplicity, that the map n be linear. We therefore have the model:

$$\dot{k} = sk^{\alpha} - \delta k - nk^{1+\alpha} \tag{4}$$

or:

$$\dot{k} = k^{\alpha}\left(s - \delta k^{1-\alpha} - nk\right) \tag{5}$$

where $n > 0$ now is a parameter tuning the reaction of the rate of change of supply of labour (and hence of the employment) to changes in percapita income. The system (5) always admits, as in the basic Solow model, the zero equilibrium (E_0) and a unique positive equilibrium (E_1) which is globally asymptotically stable for every positive initial condition. The positive equilibrium is found as the solution k_1 of the equation:

$$s - nk = \delta k^{1-\alpha} \tag{6}$$

It is easy to see that k_1 is a smooth function of the structural parameters: $k_1 = k_1(s, n, \delta, \alpha)$ satisfying:

$$\frac{\partial k_1}{\partial s} > 0\,; \ \frac{\partial k_1}{\partial n} < 0\,; \ \frac{\partial k_1}{\partial \delta} < 0\,; \ \frac{\partial k_1}{\partial \alpha} > 0 \tag{7}$$

We notice that the previous conclusions are unhaltered if we replace the assumption of linearity of the $n(.)$ with a general increasing function, saturating or not.

The model (4) is unrealistic: the assumption that the rate of change of the supply of labour is a function of the current income (i.e. possibly due to an underlying participation effect) is hardly defendable in the neoclassical framework. Despite this lack of realism it represents an adequate modelling frame within which to cast the investigation of the consequences of fully endogenous population dynamics, as will be done in the next section by resorting to time-lags. As the introduction of time delays does not modify the equilibria of a previously unlagged model, this makes it of some interest to compare the main steady-state properties at the positive equilibrium E_1 of model (4) with the corresponding features of the basic Solow's model. This comparison is made in Table 1 below which reports the equilibrium level of the percapita income $Y/L = k^{\alpha}$, and the long term rate of growth of the absolute level of the output $\dot{Y}/Y$. For simplicity in table 1 we have assumed $\delta = 0$ in both models. The assumption $\delta = 0$ will be mantained in all the subsequent sections of the present paper, as it permits more clearcut results while preserving all the dynamical features of the more general model with nonzero capital depreciation rate. We notice that the assumption of no capital depreciation was systematically employed by Solow in his 1956 paper.

Table 1. Long-term per-capita income and rate of growth of the output in the Solow's Model (SM) and in our extended Solow Model with Population dynamics (SMP)

	SM	SMP
$\frac{Y}{L}$	$\left(\frac{s}{n_s}\right)^{1/(1-\alpha)}$	$\frac{s}{n}$
$\frac{\dot{Y}}{Y}$	n_s^{α}	$s^{\alpha}n^{1-\alpha}$

As known the normative implication of the Solow's model is that there is a trade-off in the society's choice between a higher per capita income with a lower population growth and a higher population growth with a lower per capita income (a feature actually observed in developing countries). In the SMP model high levels of growth of the output are joined with relatively high (relatively low) level of population growth depending on whether is high (respectively low) the elasticity of capital when is high (respectively low) the saving ratio and is low (resp. high) the income reaction of the fertility. In other words the undesirable trade-off between economic growth and population growth of the Solow's model is strongly mitigated in the SMP model when, the fertility behaviour being equal, the households are far-sighted (in the sense that they are high saving ratios) in the case of low elasticity of capital, or, the saving ratio being fixed, the households have an intense "malthusian" behaviour when there is a high elasticity of capital. In sum: the trade-off is mitigated if in an economy characterised by low a, the households are very far-sighted and scarcely malthusian (and the converse in an economy with a high a).

3 The effects of time delays

3.1 Time delays as a synthetic representation of the overall demographic process

As already pointed out the representation of the overall demographic mechanism of age structure is quite involved (see the discussion in Manfredi and Fanti 1999). As pointed out in the literature time-delays represent good approximations in that they permit a more economical, or more simple, representation while often preserving, at the same time, the same richness of results.

The intuitive idea is simply that the rate of change of the supply of labour is related to past fertility following a prescribed pattern of time-delay. There are two main alternatives: fixed delays and distributed delays. The former is better suited when there is no variability in the process of transmission of the past into the future: for instance when we assume that all individuals are recruited in the labour force more or less at the same fixed age. Viceversa when recruitment may occur at different ages, i.e. with different delays (for instance because the time needed to complete formal education is heterogeneous within the population) distributed delays appear more suited[2]. The introduction of a distributed delay in the population term leads to the integro-differential equation (IDE for simplicity):

$$\dot{k} = sk^{\alpha} - \delta k - \left(\int_{-\infty}^{t} n\left(k^{\alpha}(\tau)\right) G(t-\tau) d\tau \right) k \tag{8}$$

where the term $n\left(k^{\alpha}(\tau)\right)$ may be interpreted as the past (rather than current), income-related fertility, and $G(t-\tau)$ is the corresponding delaying kernel, usually taken as a probability density function. Still for simplicity, and coherently with the past section we still assume that the map n is linear. In this case n may be interpreted, without ambiguity, as a "malthusian"

[2] See Invernizzi and Medio (1992) for an economically oriented discussion on the role played by distributed delays as models for heterogeneous behaviours.

parameter relating (past) fertility to past income.Of course, a positive relationship between fertility and income is fully compatible with the standard neoclassical household model (Becker 1981), postulating a trade-off between the size of the family and the level of consumption.

In what follows we will treat (8) by reducing it to a dynamical system of higher order. This is possible if we postulate that G be a reducible kernel belonging to the so-called erlangian family (McDonald 1989). We say that a density function $f(x)$ is erlangian with parameters (r, β) when it has the form:

$$f(x; r, \beta) = \frac{\beta^r}{(r-1)!} x^{r-1} e^{-\beta x} \quad x > 0;\ r = 1, 2, ..;\ \beta > 0 \tag{9}$$

As well known (McDonald 1989) in presence of erlangian kernels, IDE as (8) may be reduced to higher order systems of ordinary differential equations.

3.2 The case of the exponentially fading memory

Under the simplest assumption on the delaying kernel, that of an exponentially fading memory (this amounts to put $r = 1$ in (9)) with mean delay $T = 1/\beta$, the IDE (8) may be reduced, by posing:

$$X = \int_{-\infty}^{t} k^{\alpha}(\tau) G(t-\tau) d\tau \tag{10}$$

to the 2-dimensional ODE system:

$$\begin{aligned} \dot{k} &= s k^{\alpha} - nkX \\ \dot{X} &= \beta(k^{\alpha} - X) \end{aligned} \tag{11}$$

Obviously (8), and hence (11) as well, preserve the equilibria of the basic unlagged system (4). It is convenient to make the change of variable: $k^{\alpha} = Z$ which gives to the second equation the classical adaptive form. In fact:

$$\dot{Z} = \alpha k^{\alpha-1} \dot{k} = \alpha Z \left(s Z^{\frac{\alpha-1}{\alpha}} - nX \right)$$

Definitively, under the assumption of an exponentially fading memory, the system takes the form:

$$\begin{aligned} \dot{Z} &= \alpha \left(s Z^{\frac{2\alpha-1}{\alpha}} - nZX \right) \\ \dot{X} &= \beta(Z - X) \end{aligned} \tag{12}$$

It is easy to show that the main qualitative features of the basic Solow's model are preserved by the delayed versions (11) or (12). The system has again the zero equilibrium $E_1 = (0, 0)$ and the nonzero equilibrium $E_1 = (Z_1, X_1)$ with coordinates:

$$Z_1 = \left(\frac{s}{n}\right)^{\alpha} = X_1 \tag{13}$$

implying:

$$k_1 = \frac{s}{n} \tag{14}$$

The local stability analysis about E_1 leads to the jacobian matrix:

$$J(E_1) = \begin{pmatrix} -nZ_1(1-\alpha) & -\alpha n Z_1 \\ \beta & -\beta \end{pmatrix}$$

It holds:

$$Tr(J(E_1)) = -(-nZ_1(1-\alpha)+\beta) < 0$$

and:

$$Det(J(E_1)) = \beta\left(nZ_1(1-\alpha)+\alpha n Z_1\right) = \beta n Z_1 > 0$$

The last relations show that the system (12) is always locally stable, independently on the size of the delay. Moreover a straightforward analysis of the directions of motion in the (Z, X) plane shows that E_1 is globally asymptotically stable[3]: E_1 may be embedded in a sequence of rectangular regions in all of which the directions of motion point inward entering next rectangle.[4]

3.3 The case of the humped delay

The assumption of an exponentially fading memory, although often used as the basic memory model for a very wide spectrum of phenomena, is not a satisfactory representation of the demographic process of fertility plus aging with transition into the labour force. From this point of view a "humped" distribution is a more faithful and consistent distribution. Let us therefore make the assumption that the memory "mechanism" be of the "humped" type, by choosing as the delaying kernel the second member of the erlangian family ($r = 2$):

$$G(u) = \beta^2 u e^{-\beta u} \tag{15}$$

The density (15) is the simplest type of humped erlangian density. Under (15) the basic IDE (8) may be reduced to the form:

$$\begin{aligned} \dot{Z} &= \alpha Z\left(sZ^{\frac{\alpha-1}{\alpha}} - nX\right) \\ \dot{X} &= \beta(R-X) \\ \dot{R} &= \beta(Z-R) \end{aligned} \tag{16}$$

Notice that in this case the delay has been represented via a pair of adaptive mechanisms. As before we have the zero equilibrium E_0 and the positive equilibrium $E_1 = (Z_1, X_1, R_1)$ where:

$$Z_1 = X_1 = R_1 = \left(\frac{s}{n}\right)^{\alpha} \tag{17}$$

The local stability analysis gives the jacobian:

$$J(E_1) = \begin{pmatrix} -(1-\alpha)nZ_1 & -\alpha n Z_1 & 0 \\ 0 & -\beta & \beta \\ \beta & & -\beta \end{pmatrix} \tag{18}$$

[3]This actually holds for all initial conditions belonging to the interior of the first orthant.

[4]Notice that the system inherits the lack of uniqueness typical of Solow's model at the origin. This does not modify the main results.

The characteristic polynomial:

$$P(X) = X^3 + a_1 X^2 + a_2 X + a_3$$

has the coefficients:

$$a_1 = 2\beta + (1-\alpha)\, nZ_1 \; ; \; a_2 = 2\beta\,(1-\alpha)\, nZ_1 + \beta^2 \; ; \; a_3 = \beta^2 nZ_1$$

which are strictly positive. Hence, by applying the usual Routh-Hurwicz stability test, E_1 will be locally stable provided:

$$\Delta_2 = a_1 a_2 - a_3 > 0$$

This leads to the stability condition:

$$2\beta^2 + nZ_1\,(4-5\alpha)\,\beta + 2\left((1-\alpha)\, nZ_1\right)^2 > 0 \tag{19}$$

The parabola $f(\beta)$ defined by (19) is convex and it has a strictly positive intercept. The abscissa of its vertex is positive or negative depending on whether its second coefficient is negative or positive. This happens when: $4 - 5\alpha > 0$ i.e. for $\alpha < \frac{4}{5} = \alpha_1$, $f(\beta)$ is always greater than zero. Viceversa, when $\alpha > \frac{4}{5}$ the abscissa of the vertex will be positive. In this case as long as the discriminant D is negative $f(\beta)$ will remain strictly positive and no loss of stability is possible in this case as well. As:

$$\begin{aligned} D &= \left[nZ_1\,(4\,(1-\alpha)-\alpha)\right]^2 - 16\left((1-\alpha)\, nZ_1\right)^2 = \\ &= (nZ_1)^2\,[9\alpha - 8]\,\alpha \end{aligned}$$

this happens for $\alpha < 8/9 = \alpha_2$ (where $\alpha_2 > \alpha_1$). In sum, for $\alpha < \alpha_2$ no loss of stability is possible (α_1 is not relevant as a stability threshold). Viceversa, for $\alpha > \alpha_2$, $f(\beta)$ always has two strictly positive real roots, let us denote them again as β_1, β_2, $\beta_1 < \beta_2$ implying that losses of stability may occurs. In particular both the values β_1, β_2 represent feasible Hopf bifurcation values for the β parameter. They are given by:

$$\beta_{1,2} = \frac{nZ_1}{4}\left[5\alpha - 4 \pm \sqrt{[9\alpha - 8]\,\alpha}\right] \tag{20}$$

Hence (fig. 1) for $\alpha > \alpha_2$ the E_1 equilibrium is locally stable for "very large" (in relative terms) values of β, i.e. for $\beta > \beta_2$, or else for very small values of β, i.e. for $\beta < \beta_1$.

Fig. 1 Form of the "stability" parabola $f(\beta)$.

At the points $\beta = \beta_1, \beta = \beta_2$ stability is lost. In the window $\beta_1 < \beta < \beta_2$ the E_1 equilibrium is locally unstable. At the points $\beta = \beta_1, \beta = \beta_2$ a Hopf bifurcation occurs. To formally prove this fact we need to show that: i)purely imaginay eigenvalues exist for the linearised system at $\beta = \beta_1, \beta = \beta_2$ due to a "continuous" movement of a pair of complex eigenvalues; ii)the crossing of the imaginary axis by the involved complex pair occurs with nonzero speed. The first part of the proof is evident, see for instance Liu (1994). To check that the crossing occurs with nonzero speed, we have to consider (Asada and Semmler 1992, Liu 1994) the sign of the derivative $d\Delta_2/d\beta$ evaluated at the bifurcation point, which differs from the derivative of the real part of the bifurcating complex pair only by a nonzero constant. It holds:

$$\frac{d\Delta_2}{d\beta} = 6\beta^2 + 2nZ_1(4-5\alpha)\beta + 2((1-\alpha)nZ_1)^2$$

At both the bifurcation points $\beta_{1,2}$ it holds:

$$\left(\frac{d\Delta_2}{d\beta}\right)_{\beta_{1,2}} = \beta(4\beta + (4-5\alpha)nZ)$$

Now we have to specifically consider the behaviour of $d\Delta_2/d\beta$ in the two distinct bifurcation points. For instance at $\beta = \beta_2$ we have:

$$\begin{aligned}\left(\frac{d\Delta_2}{d\beta}\right)_{\beta=\beta_2} &= \beta_2\left(\left[5\alpha - 4 + \sqrt{[9\alpha-8]\alpha}\right]nZ + (4-5\alpha)nZ\right) = \\ &= \left(\sqrt{[9\alpha-8]\alpha}\right)nZ\beta_2 > 0\end{aligned}$$

This confirm that crossing at β_2 always occurs with nonzero speed. It's similar to prove that crossing at β_1 always occurs with nonzer speed. This fully proves that the Hopf theorem (Guckenheimer and Holmes 1983) holds, confirming the existence of a Hopf bifurcation at both the points β_1, β_2. Hence, the smooth functions of the structural parameters: $\beta_1 = \beta_1(s,n,\alpha)$; $\beta_2 = \beta_2(s,n,\alpha)$ where:

$$\begin{aligned}\beta_1(s,n,\alpha) &= \frac{n}{4}\left(\frac{s}{n}\right)^\alpha\left[5\alpha - 4 - \sqrt{[9\alpha-8]\alpha}\right] \\ \beta_2(s,n,\alpha) &= \frac{n}{4}\left(\frac{s}{n}\right)^\alpha\left[5\alpha - 4 + \sqrt{[9\alpha-8]\alpha}\right]\end{aligned}$$

respectively represent bifurcation surfaces in the parameter space, for which it holds:

$$\beta_1(s,n,\alpha) \leq \beta_2(s,n,\alpha) \tag{21}$$

We may summarise our main findings by the following:

PROPOSITION 1. *When the profit share α is below a prescribed threshold ($\alpha < \alpha_2$) the system (16) replies the traditional Solow's behaviour, with convergence to the unique globally stable equilibrium E_1. When $\alpha > \alpha_2$ then the system (16) continues to converge to the globally stable equilibrium E_1 only when β is sufficiently large or sufficiently small, i.e. for $\beta > \beta_2$ and $\beta < \beta_1$. In the whole window $\beta_1 < \beta < \beta_2$ the equilibrium E_1 is locally unstable. At the points $\beta = \beta_1, \beta = \beta_2$ Hopf bifurcations occur.*

4 Simulative evidence and working of the system

The fact to know that a Hopf bifurcation exists nothing says about the stability properties of the involved periodic orbits, i.e. it does not say whether the bifurcation is supercritical or subcritical (i.e. whether the periodic orbit is locally stable or unstable). Unfortunately the investigation of the stability properties of periodic orbits emerged via Hopf bifurcation at dimensions greater than dimension two is a quite hard task (Marsden and MacCracken 1976). Moreover the predictions of the Hopf theorem are local in nature: they nothing say about global behaviours. We therefore resorted to numerical simulation to clarify the stability nature of the Hopf bifurcations occurred at the points $\beta = \beta_1, \beta = \beta_2$, and more generally to investigate the global properties of our model. The simulative evidence shows two remarkable facts: i)both the points $\beta = \beta_1, \beta = \beta_2$ generate supercritical bifurcations (i.e. locally stable oscillations). In particular the whole window $\beta_1 < \beta < \beta_2$ is a region of stable oscillations. ii)all the the properties of the model seem[5] to hold globally: when the E_1 equilibrium is locally stable, then this stability seems to be global and not only local; when E_1 looses its stability due to the switch occurring at the bifurcation points, then the emerging limit cycle seems not only locally stable but also globally stable.

We can now summarise our main dynamics findings, by illustrating the working of the model. There is a region defined by $\alpha < \alpha_2 = 0.88$ in which the traditional behaviour of the Solow's model is confirmed and the economy converges to a long term (globally stable) steady state. Viceversa, for very large α i.e. $\alpha > \alpha_2$ the economy may be destabilised by the action of the delay. More in detail, as long as β is very large (in relative terms), i.e. for $\beta > \beta_2$, which corresponds to "very small" values of the mean delay $T = 2/\beta$, the E_1 equilibrium preserves its stability. But as β is decreased (this happens for increasing mean delays) stability may be lost. This happens for $\beta = \beta_2$ where a first Hopf bifurcation occurs and E_1 exchanges its stability with a stable limit cycle. The whole window $\beta_1 < \beta < \beta_2$ (characterised by intermediate values of the mean delay) is characterised by (globally) stable oscillations. Finally, by furtherly decreasing β, a further bifurcation occurs at $\beta = \beta_1$ where the E_1 equilibrium is restored: hence for very large mean delays the Solow's traditional behaviour is recovered once again.

There is an important remark to do concerning the degree of plausibility of our work. Persistent oscillations seem to require an unplausibly large value of the elasticity of the capital-labour

[5] We could not produce a formal proof of this fact.

ratio α. Although this is indisputable it is important to recall that the purpose of our work is simply that of evidencing the forces which may lead to persistent oscillations in the Solow model when a realistic distributed-delayed mechanism of formation of the supply of labour is assumed jointly with endogeneous population according to a general malthusian view. Indeed, by resorting to higher order distributions of the delaying kernel (such as gamma densities of higher order, i.e. $Gamma(n, \beta)$ with $n = 3, 4, 5...$) which should more realistically capture the phenomenon of the delayed entry into the labour force, we are able to obtain the same type of qualitative behaviour found, i.e. persistent oscillations, with more and more realistic values of the elasticity of the capital labour ratio. The obvious drawback of these more realistic variants is that the power of the analytical treatment is lost (Manfredi and Fanti,1999), and the system can only be analysed via simulation.

Finally, of particular interest seems to be the process of switching and reswitching of stability between the E_1 equilibrium and the limit cycle emerging via Hopf. At the points $\beta = \beta_1, \beta = \beta_2$ distinct Hopf bifurcations occurs giving rise to periodic behaviours in suitable neighborhood of these points. A prediction of Hopf theorem is that the ray of the emerging periodic orbit depends linearly on the distance between the actual value of the bifurcation parameter and its value at the bifurcation point. In simple words: let be given a dynamical system depending on a bifurcation parameter μ and undergoing a Hopf bifurcation at μ_0. Let us suppose that the bifurcation is supercritical in a right neighborhood $(\mu_0, \mu_0 + \sigma)$. This means that for every $\mu \in (\mu_0, \mu_0 + \sigma)$ a stable limit cycle exists with ray proportional to $\|\mu - \mu_0\|$. Now, as simulations show, in our systems these neighborhoods are of the type $(\beta_1, \beta_1 + \sigma), (\beta_2, \beta_2 - \rho)$. But simulations also show that the ray of the periodic orbits emerging at $\beta = \beta_2$ is strictly increasing as β decreases from β_2 to a threshold value β^* and then decreasing as β is furtherly decreased from β^* to β_2 where the fluctuations are reabsorbed (and the ray converges to zero). This seems to denote that the process of switching between the two regimes of bifurcation is a smooth one. This is coherent with the fact that, although $\beta = \beta_1, \beta = \beta_2$ are distinct Hopf bifurcation point, the whole bifurcation process is due to the "activity" of a unique complex pair of eigenvalues which has negative real parts for large β, crosses (with nonzero speed) the imaginary axis a first time at $\beta = \beta_2$, keeps positive real part as long as $\beta_1 < \beta < \beta_2$, and crosses anew (always with nonzero speed) the imaginary axis at $\beta = \beta_1$. The conjectured form of this process is represented in fig. 2.

Fig. 2. Schematic view of the conjectured form of the process of stability switching and reswitching between the E_1 equilibrium and the limit cycles appearing at the

Hopf bifurcation points $\beta = \beta_1, \beta = \beta_2$

The economic interpretation of our findings is the following. When $\alpha > \alpha_2$ increasing delays may destabilise the traditional Solow's behaviour, by generating persistent oscillations. These fluctuations appear to be the outcome of the demographic memory operating through the age structure delay.

Notably, these fluctuations persist in a very wide range of the mean delay. The fact that periodic behaviours may persist also for very long time scales of the delay seems to suggest the existence of possible "supergenerational" echoes, deriving from patterns ascribed to birth generations different from the "last" ones, a fact which seems to be of a certain interest.

We illustrate the actual working of our model by resorting to a concrete example in which, just to reduce complexity, we sterilize the effects of α, s and n and concentrate only on the dynamical effects of the delay parameter β. In the following experiments we set $a = 0.92$, $s = 0.3$, $n = 0.01$.

The simulations show that by decreasing the delay parameter β the phase portrait of the system undergoes the following transformations: convergence to a globally stable node→convergenge to a (globally) stable focus→convergence to a (globally) stable limit cycle→ reswitch with convergenge to a (globally) stable focus. More in detail: i)the equilibrium point E_1 is a stable node as long $\beta > \beta_3 \cong 0.1$ (i.e. a mean delay about 20 years); ii) E_1 is a stable focus for $\beta_3 > b > \beta_2 \cong 0.064$ (a mean delay about 31 years) where β_2 is the largest bifurcation point; ii)at β_2 the first stable limit cycle appears. Fig. 3a and 3b report a two-dimensional view of the involved cycle. The motion along the cycle is counterclockwise. The amplitude of these cycles increase, by decreasing β, up to the point $\beta^* \cong 0.02$ (a mean delay about 50 years); iv) by furtherly decreasing β from $\beta^* \cong 0.02$ the amplitude of the limit cycle starts decrease, up to the smaller bifurcation point $\beta_1 = 0.0056$, where limit cycles disappear; v)in the range $\beta_1 > \beta > 0$ E_1 is a stable focus again.

Fig. 3. a)A stable limit cycle appeared at $\beta = \beta_2$;

b) *a stable limit cycle appeared at $\beta = \beta_1$*

Table 2 reports in a synoptical view the process of phase transition in our model when $\alpha > \alpha_2$.

Table 2. Windows of the delay parameter and

relative behaviour of the SMP model

Windows of β	$(0, \beta_1)$	(β_1, β_2)	(β_2, β_3)	(β_3, ∞)
	E_1 stable focus	Stable limit cycle	Stable focus	Stable node

The shape of the bifurcation curves helps in understanding the feature of the bifurcation process. Although, as previously pointed out, the overall bifurcation loci are surfaces in the four-dimensional parameter space (with the restriction $0 < \alpha < 1$), i.e quite complex to represent, some insight comes from the exploration of the 2-dimensional bifurcation relation between the delay parameter β and the remaining parameters taken one at time. The figures 4,5 below report the shape of the "restricted" (i.e. by keeping fixed the values of the remaining parameters) bifurcation functions in the planes (β, n) and (β, s).

Fig. 4. Form of the bifurcation loci in the (β, n) plane

Fig. 5. Form of the bifurcation loci in the(β, s) plane

From the economic point of view, the curves represented in fig. 4,5 describe the sensitivity on the bifurcation values β_1, β_2 of the delay parameter β to changes in, respectively: i)the n parameter, embedding the rection of fertility to changes in income, and ii)the saving ratio.

As clear from the two figures, both the bifurcation values β_1, β_2 of the delay parameter are monotonically increasing functions of respectively n and s. This implies that the entire window of periodic behaviours translates upward meaning that the mean delays causing the appearance of periodic behaviours tend to decrease. In substantive economic terms this fact seems to suggest that in societies where households are far-sighted and/or "strongly malthusian", oscillations appear when the average age of entry into the labour force is relatively anticipated compared to the oppsite case of myopic and "weakly malthusian" households.

5 Conclusions

This paper represents a contribution to the recent literature on endogenous economic cycles. Traditional explanations in continuous time are essentially based on the "easterlinian" mechanism (Samuelson (1976), Feichtinger and Dockner (1990), Feichtinger and Sorger (1989, 1990)). Some of these efforts are based on clever modelling tricks, in that they avoid the direct use of economic variables by collapsing in purely demographic models plus some nonlinearity embedding the interaction with the economic subsystem (Samuelson (1976) and Feichtinger and Sorger (1989)), or are based on somewhat ad hoc assumptions, such as Feichtinger and Sorger (1990).

In this paper an alternative explanation of the generation of demoeconomic fluctuations in (continuous time) Solow's type growth models is proposed, which is based on two very simple but highly realistic assumptions, i.e. a malthusian relation between income and fertility, and the existence of the (indisputable) delay of transition into the labour force, due to the age structure process. The ensuing model exhibits a very simple and resilient mechanism for the generation of persistent oscillations emerging from a full recognition of the demoeconomic interaction within the true core of the most traditional neoclassical growth theory, i.e. the Solow's model in continuous time.

This fact seems to be of some interest in the area of the neoclassical theory of growth, as it shows that the neoclassical growth paradigm not only explains the stylised fact of balanced growth, but, once endowed with a correctly demographically founded formulation of the labour supply, becomes capable to endogenously explain the other main stylised fact of economic growth, namely the generation of globally stable oscillations around a path of balanced growth.

A last point concerns economic policies. The present work provides clearcut results on the interaction between levels of production, rates of population growth and the generational lag in the entry in the labor force which could be useful for a better design of the economic policy, for instance for what concerns the labour market and its interaction with the eductional system.

6 REFERENCES

[1] T. Asada, W. Semmler (1992), Growth and Finance: an Intertemporal Model, *Journal of Macroeconomics*, 17, 4, 623-649.

[2] Day R.H. (1983), The emergence of chaos from classical economic growth, *Quarterly Journal of Economics*, 201-213.

[3] Day R.H., Walter J. (1989), Economic growth in the very long run: on the multiple phase of interaction of population, technology, and social infrastructure, in Barnett W.A., Geweke J., Shell K. (eds.), Economic complexity. Chaos, sunspots, bubbles and nonlinearities, Cambridge Univ. Press, 253-289.

[4] Day R.H., Kim K.H., Macunovich D. (1989), Complex demoeconomic dynamics, *Journal of Population Economics*, 1989, 2, 139-159.

[5] Feichtinger G., Dockner (1990), Capital accumulation, endogenous population growth, and Easterlin cycles, *Journal of Population Economics*, 73-87.

[6] Feichtinger G., Sorger G. (1989), Self-generating fertility waves in a non-linear continuous overlapping generations model, *Journal of Population Economics*, 2, 267-280.

[7] Feichtinger G., Sorger (1990), Capital accumulation, aspiration adjustment, and population growth: limit cycles in an Easterlin-type model, *Math. Population Studies*, 2, 93-103.

[8] Guckenheimer J., Holmes P. (1983), *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, Springer Verlag.

[9] Invernizzi S., Medio A. (1992), On lags and chaos in economic dynamic models, *Journal Math. Economics*, 20, 521-550.

[10] Liu W.M. (1994), Criterion of Hopf bifurcation without using eigenvalues, *Journal Math. Analysis and Applications*, 182, 250-256.

[11] McDonald (1978), Time lags in biological systems, Lecture Notes Biomath, Springer Verlag.

[12] MacDonald N. (1989), *Biological delay systems: linear stability theory*, Cambridge University Press.

[13] Malthus T.R. (1798), An essay on the principle of population as it affects the future improvements of society, London.

[14] Manfredi P., Fanti L. (1999), Age structure and persistent oscillations in Solow-type growth models, WP Dip. Statistica e Matematica Applicata, Università di Pisa.

[15] Marsden J., MacCracken M. (1976), *The Hopf bifurcation and its applications*, Springer Verlag.

[16] Samuelson P.A. (1976), An economists nonlinear model of self-generated fertility waves, *Population Studies*, 30, 243-247.

[17] Solow R. (1956), A model of balanced growth, *Quarterly J. of Economics*, 70, 65-94

[18] Prskawetz A., Feichtinger G. (1995), Endogenous population growth may imply chaos, *Journal Population Economics*, .

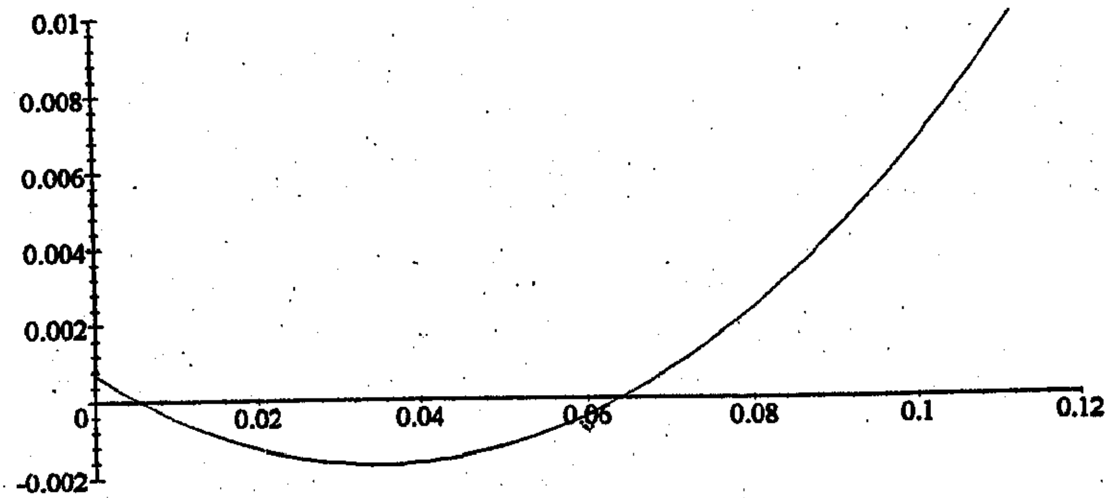
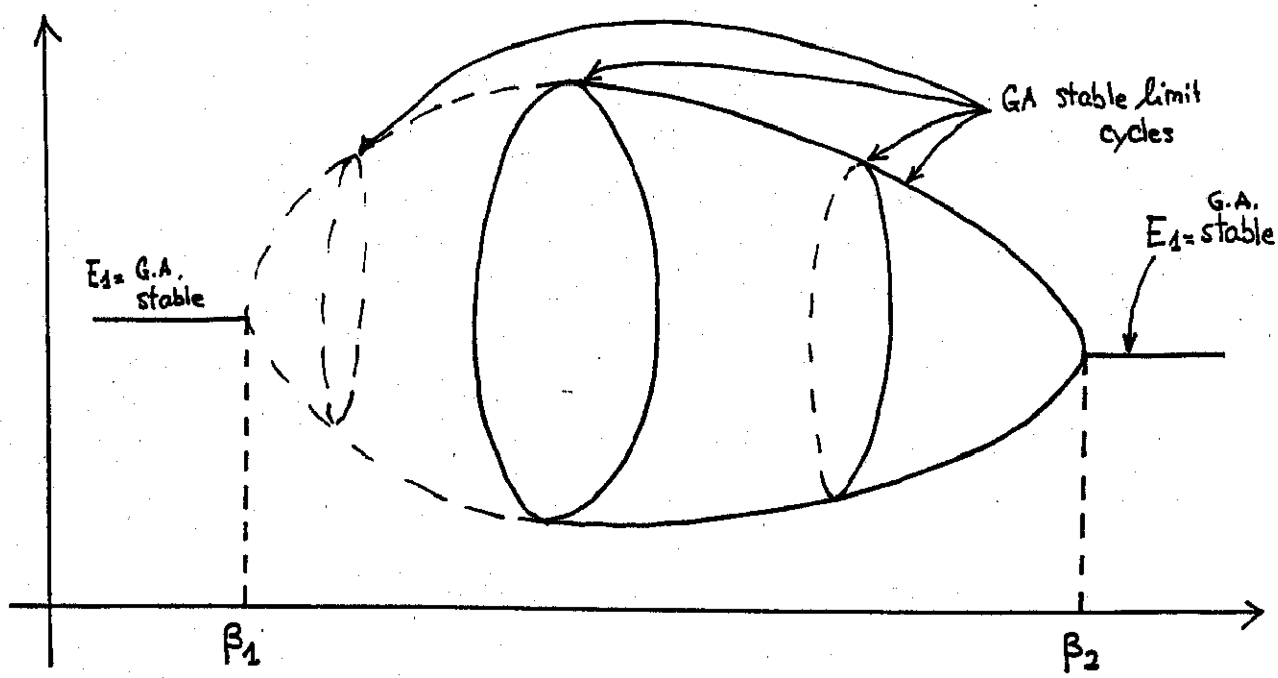


Fig. 1 Form of the "stability" parabola $f(\beta)$.

Fig. 1

Fig. 2



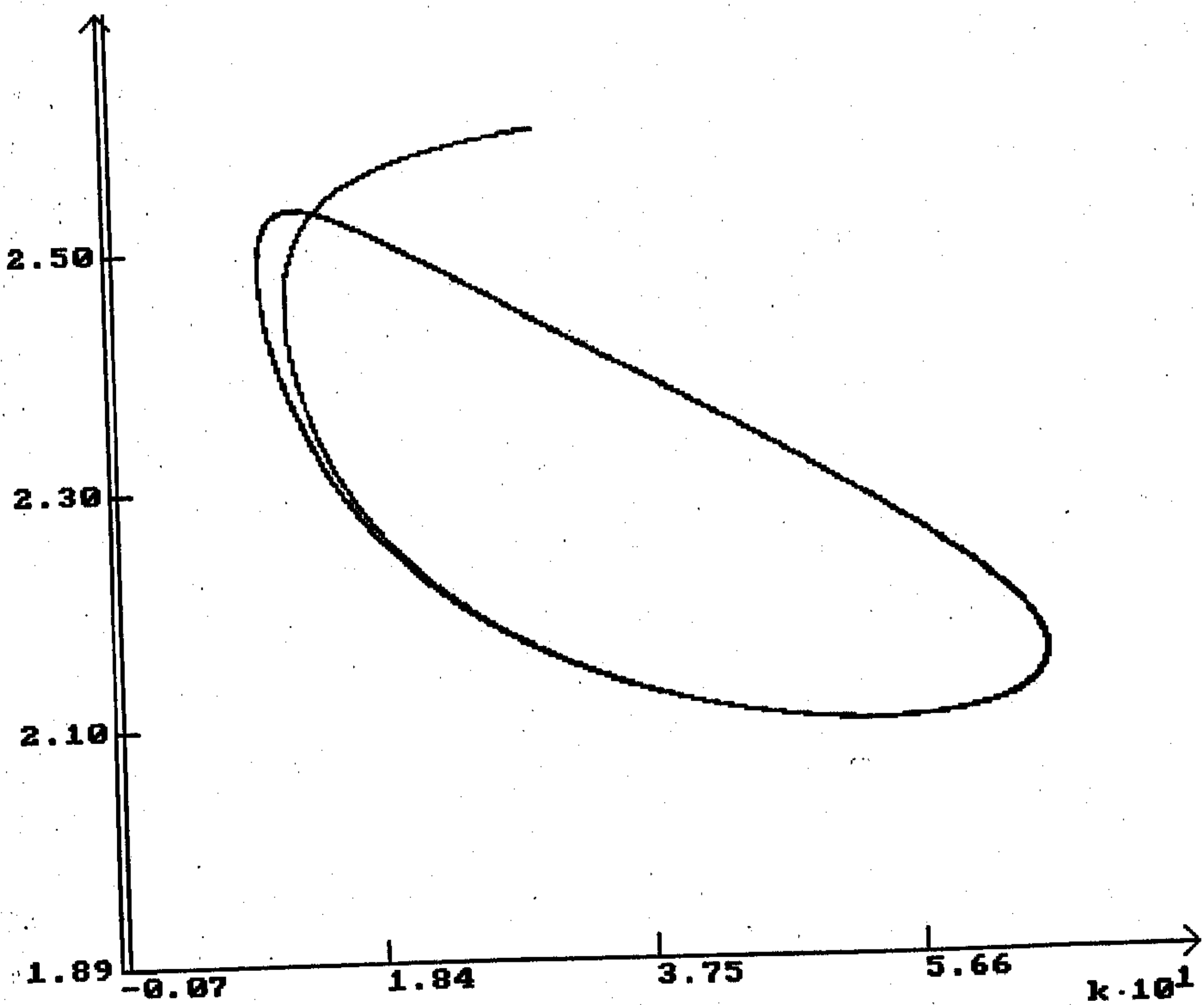


Fig 3a

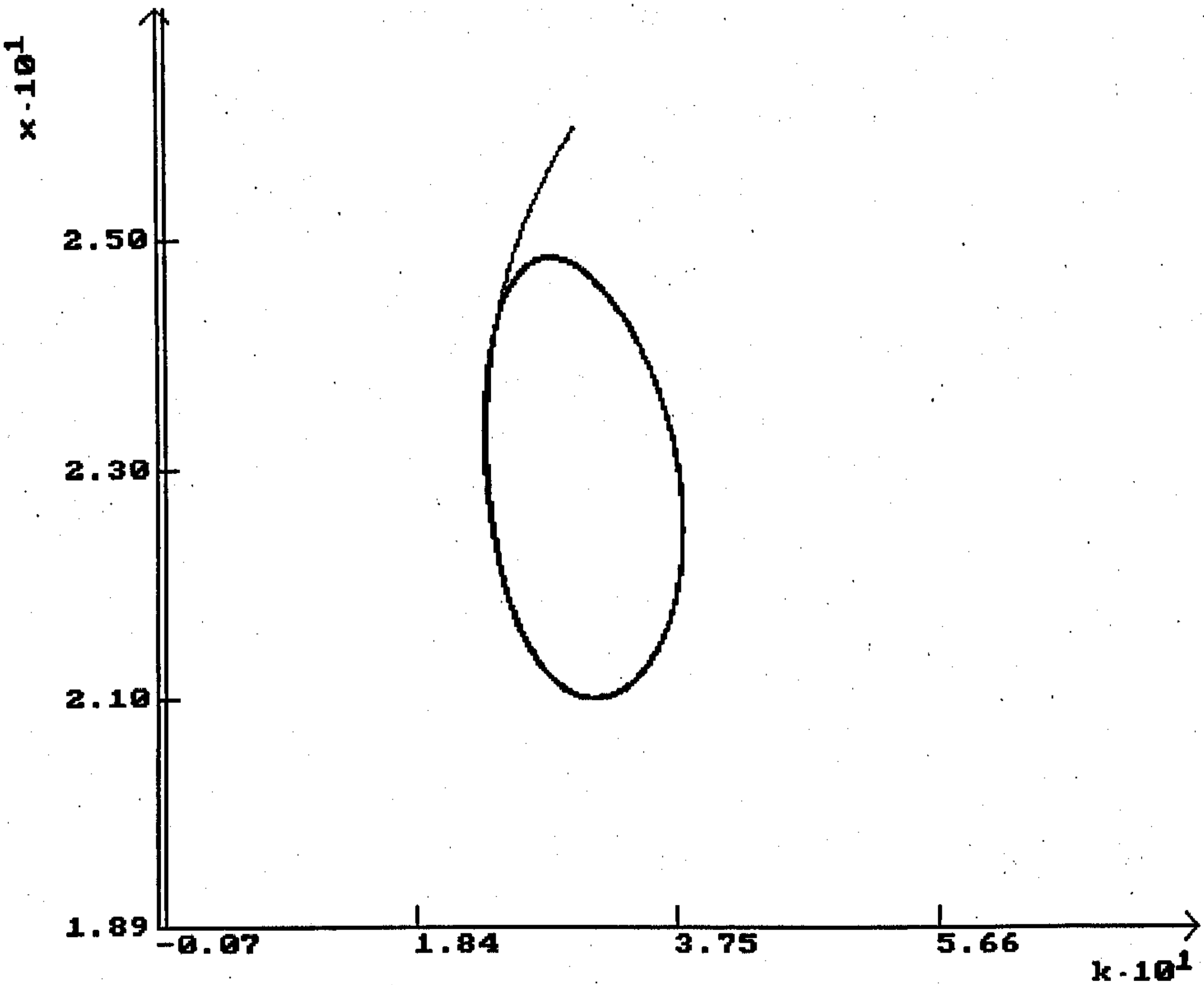


Fig 3b

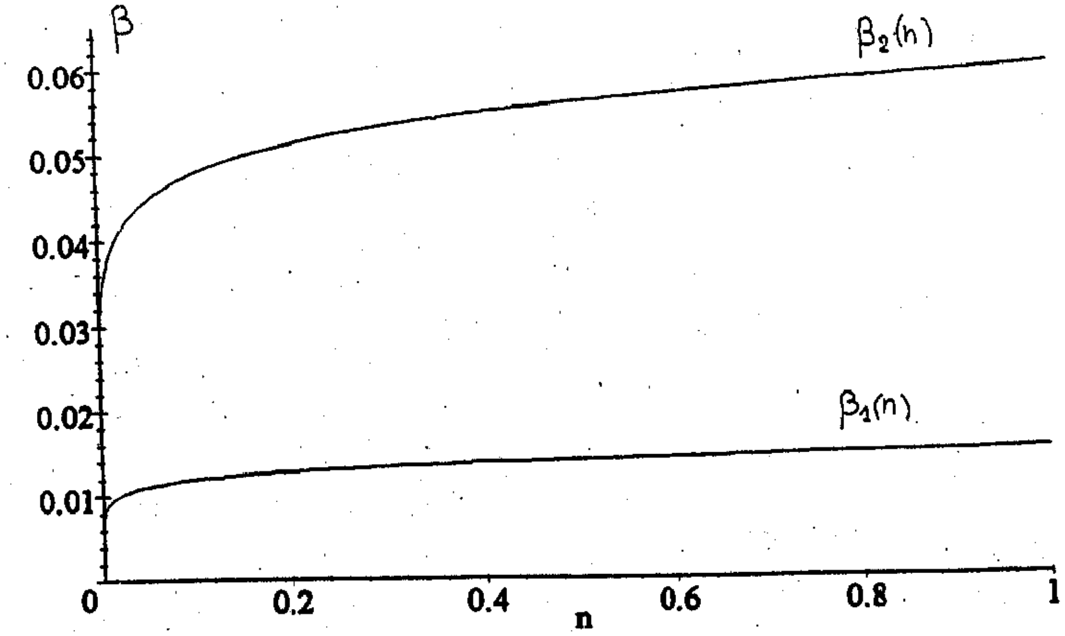


Fig. 4. Form of the bifurcation loci in the (β, n) plane

Fig. 4

$k \cdot 10^1$