Report n. 137

Some Results on Partial Differential Equations and Asian Options

Emilio Barucci, Sergio Polidoro, Vincenzo Vespri

Pisa, Marzo 1999

Some Results on Partial Differential Equations and Asian Options

Emilio Barucci

Dipartimento di Statistica e Matematica Applicata all'Economia
Università degli Studi di Pisa

Sergio Polidoro

Dipartimento di Matematica
Università degli Studi di Bologna

Vincenzo Vespri

Dipartimento di Matematica "Ulisse Dini"
Università degli Studi di Firenze

Abstract

We analyze partial differential equations arising in the evaluation of Asian Options. The equations are strongly degenerate partial differential equations in three dimensions. We show that the solution of the no arbitrage partial differential equation is sufficiently regular and standard numerical methods can be employed to approximate it.

Keywords: 35K70, 35H05, 65M06, 90A09.
Classification: Ultraparabolic partial differential equations, Hypoellitic equations, Finite difference methods, Contingent claims valuation.

1 Introduction

The valuation of path dependent (Asian) European contingent claims, i.e., claims whose final payoff depends on the history of the underlying asset price, is a difficult task in mathematical finance. Only in some simple cases the no-arbitrage price of a path dependent contingent claim computed as the expected final payoff under the risk neutral probability measure is obtained in closed form, for some results see [26]. For example, considering a lognormal stochastic process for the underlying asset price, we have that its geometric average is still characterized by a lognormal probability density, but things change dramatically considering the arithmetic average, only recently its moments have been determined in [14]. In some cases, introducing path dependent variables the state space can be augmented in such a way that the no-arbitrage price of the claim is obtained as the solution of a Cauchy problem.

When a closed form solution for the price of the contingent claim is not available, we have to rely upon numerical methods. Different methodologies have been applied in the literature to address the problem. The approaches can be split in two groups: methods aiming at estimating the conditional expectation of the final payoff and methods aiming at solving numerically the Cauchy problem associated with the no-arbitrage Partial Differential Equation (PDE).

Looking at the methods focusing on the estimation of the expectation of the final payoff, we have the following: Monte Carlo simulations ([18, 27]), fast Fourier transform to calculate the density of the sum of random variables as the convolution of the density functions ([8]). In [14] a closed form solution of the no-arbitrage price of the Arithmetic average fixed strike price is obtained through the inversion of a Laplace transform. Considering a lognormal diffusion process for the underlying asset, the pricing density can be approximated by substituting the arithmetic average of the price with the geometric average which preserves the lognormality for the average. The approximation is obtained by equating the moments of the true distribution and of the approximating distribution, see [19] for an approximation up to the second moment and [29] for an approximation up to fourth moment, or by adjusting the strike price to correct for the misspricing, see [30]. In [7] an upper-bound to the approximation error occurred using this procedure is proposed.

The standard approximation methods based on partial differential equations require some regularity conditions of the solution of the no arbitrage PDE. Considering path dependent contingent claims in a general setting, the PDE is a strongly degenerate parabolic equation in three dimensions (time, the underlying asset price and the path dependent variable). In this setting, the needed regularity seemed out of reach. To avoid this difficulty, many authors considered a two dimensional second order partial differential equation which is obtained from the original one through a change of variable (*similarity reduction method*) when the contingent claim final payoff has a particular form (see [17, 1, 12, 31, 25]). This method covers a large set of contingent claim contracts, including arithmetic Asian options, but not a contingent claim characterised by a general final payoff.

A general payoff and therefore a PDE in three dimensions is considered in [5, 4, 3]. In [5] it is proposed a numerical method based on some analysis arguments and on some probabilistic remarks. Convergence of the numerical scheme is proved by means of the central limit theorem. In [4, 3] a method based on the notion of viscosity solution is developed. Convergence of the numerical scheme is proved through the Dini's theorem. The advantage of these two methods, with respect to the classical ones, is that they don't need the regularity of the coefficients of the PDE, on the other hand they don't give any explicit convergence estimate.

This paper aims at showing that the solution of the no arbitrage partial differential equation is in fact sufficiently regular and that the standard numerical methods can be employed to approximate it. We first present theoretical results concerning existence, uniqueness and regularity of the solution, then the theoretical results are coupled with results about the capability of numerical methods to approximate the no-arbitrage price of the contingent claim. Differently from the above mentioned methods, a convergence estimate is explicitly given. The results obtained for the geometric average are built on the results obtained in [24]. The numerical method proposed in this paper is a classical finite-difference method. The main problem encountered applying this method to the arithmetic average case is the lack of suitable a-priori estimates near infinity for the partial derivatives of the solution.

We concentrate our attention on arithmetic and geometric average options, some insights are given also for more general path dependent options and assuming a stochastic process for the underlying asset price characterized by constant elasticity variance.

The paper is organized as follows. In Section 2 we present the path dependent contingent claims framework, then we consider in detail the arithmetic and the geometric average options and the other examples discussed above. In Section 3 we state the analytic and the numeric results for the Cauchy problem related to the geometric average options, while in Section 4 the arithmetic average options are considered. In both sections we first present the theoretical results and then in two subsections we describe the numerical methods. Finally, Section 5 contains some remarks about the general case.

2 Pricing Path Dependent Contingent Claims

We consider a standard complete markets economy. We take as given a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ which is right continuous and such that $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$.

There are two assets traded in two markets: a risk-free asset and a risky asset. The risky asset price follows the Itô stochastic differential equation:

$$dS_t = \mu_s(t,S)dt + \sigma_s(t,S)dW_t, \quad S_0 > 0, \tag{2.1}$$

where W_t is a Brownian motion in $\Re$, $\mathcal{F}_t$ is the natural filtration of W_t. Let the classical conditions ensuring existence and uniqueness of (2.1) be satisfied. The risk-free asset is a bond whose price evolves as follows:

$$dB_t = r(t)B_t dt, \quad B_0 > 0. \tag{2.2}$$

$r(t)$ is the risk-free rate at time t. Note that if we directly refer to the risk-neutral probability measure associated with the Brownian motion W then $\mu_s = r$. The analysis can be extended to a Brownian motion in $\mathbb{R}^n$ still with complete markets, for simplicity we concentrate our attention to the scalar case.

Following [31, 5] we consider the class of European path dependent contingent claims defined by the path dependent variable $A_t = \Psi([S_\tau]_{\tau \leq t})$ which is assumed to follow the stochastic differential equation

$$dA_t = \mu_a(t, S_t, A_t)dt + \sigma_a(t, S_t, A_t)dW_t \tag{2.3}$$

and by final payoff function $\Omega(S_T, A_T)$.

The no-arbitrage price of the contingent claim is given by the expected value of the final payoff under the risk neutral probability measure. The price can also be obtained as the solution $V(t, S_t, A_t)$ of the following Cauchy problem:

$$-rV + \frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma_s^2\frac{\partial^2 V}{\partial S^2} + \mu_a\frac{\partial V}{\partial A} + \frac{1}{2}\sigma_a^2\frac{\partial^2 V}{\partial A^2} + \sigma_s\sigma_a\frac{\partial^2 V}{\partial S \partial A} = 0 \tag{2.4}$$

with

$$V(T, S_T, A_T) = \Omega(S_T, A_T). \tag{2.5}$$

In what follows we restrict our attention to the class of path dependent options characterized by a payoff which is a function of the entire history of the underlying asset price $(S_t,\ 0 \leq t \leq T)$. We do not consider *forward start* options.

Many path dependent contingent claims are traded in financial markets, for a survey see [26]. Among them we remember the following:

- *Arithmetic average floating strike call option:*

$$\Omega(S_T, A_T) = \max(S_T - \frac{A_T}{T}, 0),\ A_t = \int_0^t S_\tau d\tau,\ \mu_a = S,\ \sigma_a = 0;$$

- *Arithmetic average fixed strike call option:*

$$\Omega(S_T, A_T) = \max(\frac{A_T}{T} - E, 0),\ A_t = \int_0^t S_\tau d\tau,\ \mu_a = S,\ \sigma_a = 0;$$

- *Geometric average floating strike call option:*

$$\Omega(S_T, A_T) = \max(S_T - e^{\frac{A_T}{T}}, 0),\ A_t = \int_0^t \log(S_\tau)d\tau,\ \mu_a = \log(S),\ \sigma_a = 0;$$

- *Geometric average fixed strike call option:*

$$\Omega(S_T, A_T) = \max(e^{\frac{A_T}{T}} - E, 0), \; A_t = \int_0^t \log(S_\tau) d\tau, \; \mu_a = \log(S), \; \sigma_a = 0.$$

In the first two contracts the path dependent variable is given by the arithmetic average of the underlying asset price, in the last two contracts the path dependent variable is given by the geometric average of the underlying asset price. In practice only contracts written on the arithmetic average are exchanged in financial markets. However, it is useful to evaluate contracts written on the geometric average for two main reasons: the problem is easier and it provides, under some conditions, an approximation to the no arbitrage price of a claim written on the arithmetic average.

3 Geometric Average Options

First of all we consider a lognormal stochastic process for the underlying asset price, i.e., $\mu_s = \mu S$ and $\sigma_s = \sigma S$, where μ and σ are two non zero constants. Then the Cauchy problems for the price of the Geometric average floating strike call option and of the Geometric average fixed strike call option become

$$-rV + \frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \log(S)\frac{\partial V}{\partial A} = 0 \tag{3.1}$$

with

$$V(T, S, A) = \max(S - e^{\frac{A}{T}}, 0), \tag{3.2}$$

or

$$V(T, S, A) = \max(e^{\frac{A}{T}} - E, 0). \tag{3.3}$$

Letting $x = \frac{\sqrt{2}}{\sigma}\log(S), y = \frac{\sqrt{2}}{\sigma}A$ and

$$u(x, y, t) = e^{\frac{2r-\sigma^2}{2\sqrt{2}\sigma}x + \left(\frac{2r+\sigma^2}{2\sqrt{2}\sigma}\right)^2 t} V\left(T - t, e^{\frac{\sigma x}{\sqrt{2}}}, \frac{\sigma y}{\sqrt{2}}\right),$$

we see that the differential equation (3.1) is equivalent to

$$\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t}, \tag{3.4}$$

and that the *final* conditions (3.2) and (3.3) become the *initial* conditions

$$u(x, y, 0) = e^{\frac{2r-\sigma^2}{2\sqrt{2}\sigma}x} \max\left\{e^{\frac{\sigma x}{\sqrt{2}}} - e^{\frac{\sigma y}{\sqrt{2}T}}, 0\right\} \tag{3.5}$$

and

$$u(x,y,0) = e^{\frac{2r-\sigma^2}{2\sqrt{2}\sigma}x} \max\left\{ e^{\frac{\sigma y}{\sqrt{2T}}} - E, 0 \right\}. \tag{3.6}$$

Equation (3.4) has been extensively studied in the literature. Let's recall some known results. Equation (3.4) was first considered by Kolmogorov (see [28, p.167]), then in the introduction of [16] concerning hypoelliptic operators. See [20] and its bibliography for an exhaustive survey of results about equations like (3.4).

In [28] it is shown that the function Γ defined by

$$\Gamma(x,y,t;\xi,\eta,\tau) = \frac{\sqrt{3}}{2\pi(t-\tau)^2} \exp\left(-\frac{(x-\xi)^2}{4(t-\tau)} - \frac{3}{(t-\tau)^3}\left(y - \eta - \frac{t-\tau}{2}(x+\xi) \right)^2 \right) \tag{3.7}$$

if $t > \tau, \Gamma(x,y,t;\xi,\eta,\tau) = 0$ if $t \leq \tau$, is a fundamental solution of (3.4). We stress that Γ has some important properties that are distinctive of the heat kernel: for every fixed $(\xi,\eta,\tau) \in \mathbb{R}^3$, Γ is a C^∞ function in the variables $(x,y,t) \in (\mathbb{R}^3 \setminus \{(\xi,\eta,\tau)\})$ and, for every $t > \tau$, it is a Gauss kernel in the variables (x,y).

In order to simplify the notations, in the following we shall denote by L the differential operator

$$\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial y} - \frac{\partial}{\partial t}$$

and by $\mathcal{S}_T$ the set $\mathbb{R} \times \mathbb{R} \times]0,T[$. For a given function φ, that is continuous on $\mathbb{R}^2$, we shall say that u is a solution to the Cauchy problem

$$\begin{cases} Lu = 0 & \text{in } (x,y,t) \in \mathcal{S}_T, \\ u(x,y,0) = \varphi(x,y) & \text{in } (x,y) \in \mathbb{R}^2, \end{cases} \tag{3.8}$$

if the equation $Lu = 0$ is satisfied in every point of $\mathcal{S}_T$ and $u(x,y,t) \to \varphi(x_0,y_0)$ as $(x,y,t) \to (x_0,y_0,0)$, for every $(x_0,y_0) \in \mathbb{R}^2$. The following result holds true (see [23]).

THEOREM 3.1 *Let $\varphi \in C(\mathbb{R}^2)$, such that*

$$\int_{\mathbb{R}^2} e^{-c(x^2+y^2)} \varphi(x,y)\, dx\, dy < \infty \tag{3.9}$$

for some positive constant c. Then the function u defined as

$$u(x,y,t) = \int_{\mathbb{R}^2} \Gamma(x,y,t;\xi,\eta,0)\varphi(\xi,\eta)\, d\xi\, d\eta \tag{3.10}$$

is a solution to the Cauchy problem (3.8). Moreover, if u and v are two solutions of (3.8) and satisfy the following condition

$$\int_{\mathcal{S}_T} e^{-c(x^2+y^2)} |u(x,y,t) - v(x,y,t)|\, dx\, dy\, dt < \infty,$$

for some constant c, then $u \equiv v$.

REMARK 3.2 *Note that the condition* (3.9) *is satisfied whenever the payoff is a continuous function such that* $0 \leq V(T,S,A) \leq S$. *Theorem 3.1 ensures that, among all the solutions of* (3.8), *the function defined by* (3.10) *is the unique solution corresponding to a price* V *such that* $0 \leq V(t,S,A) \leq S$. *This condition is satisfied by most of the contingent claim contracts.*

REMARK 3.3 *The results stated above can be generalised to the equation*

$$a(x,y,t)\frac{\partial^2 u}{\partial x^2} + b(x,y,t)\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t}, \tag{3.11}$$

where a *and* b *are bounded and Hölder continuous functions and* $a(x,y,t) \geq a_0 > 0$ *for every* (x,y,t) *(see [22]; of course in* (3.10) *the function* Γ *has to be replaced by the fundamental solution of* (3.11)*). Thus we may consider equations* (3.1) *with coefficients* r *and* σ *depending on* t *and* S.

3.1 Numerical approximation

In order to provide a numerical approximation of the solution of (3.8), we will adapt some results in [24], where a boundary value problem for a bounded subset of $\mathbb{R}^3$ was considered. Let's define the "discrete" operator

$$L_G u\,(x,y,t) = -\frac{u(x,y,t) - u(x,y+\delta x,t-\delta)}{\delta} + \frac{u(x-h,y+\delta x,t-\delta) - 2u(x,y+\delta x,t-\delta) + u(x+h,y+\delta x,t-\delta)}{h^2},$$

that is an approximation of the operator L in the sense that

$$L_G u = Lu + (h^2+\delta)O(h,\delta)$$

(here $O(h,\delta)$ denotes a bounded function). The operator L_G is well defined in the grid

$$G = \left\{(j\Delta_x, k\Delta_y, n\Delta_t) \in \mathbb{R}^3 : j,k,n \in \mathbb{Z}\right\},$$

with $\Delta_x = h$, $\Delta_t = \delta$ and $\Delta_y = t\delta$ and the following approximation result holds.

THEOREM 3.4 *Let* u *be a solution of (3.8),* u_G *be a solution of* $L_G u_G = 0$ *in* $G \cap S_T, u_G = \varphi$ *in* $G \cap \{t=0\}$. *Then, for every* $\varepsilon_1, \varepsilon_2 > 0$ *and for every* H *compact subset of* S_T *there exists a grid* G, *verifying the following stability condition*

$$\frac{\Delta_t}{(\Delta_x)^2} \leq \frac{1}{2}, \tag{3.12}$$

such that

$$\max_{(x,y,t)\in G\cap H} |u(x,y,t) - u_G(x,y,t)| \leq \varepsilon_1,$$

$$\max_{(x,y,t)\in G\cap H}\left|\frac{\partial u}{\partial x}(x,y,t)-\frac{u_G(x+\Delta_x,y,t)-u_G(x,y,t)}{\Delta_x}\right|\leq\varepsilon_2.$$

Moreover $\varepsilon_1 = O\big((\Delta_x)^2\big)$ *and* $\varepsilon_2 = O\big(\Delta_x\big)$.

REMARK 3.5 *The techniques employed in the proof can be easily adapted to the study of equation* (3.11). *In that case the stability condition becomes*

$$\frac{\Delta_t}{(\Delta_x)^2}\leq\frac{1}{2a_0},\quad \big(\text{where } a_0=\sup a(x,y,t)\big).$$

It is convenient to assume that the coefficients a and b are differentiable functions and that their derivatives are locally Hölder continuous.

Proof of Theorem 3.4. We basically repeat the proof of Theorem 3.3 in [24]. The new difficulties are due to the fact that the domain of the solution is unbounded, so we shall briefly indicate the needed changes. The proof is divided in three Steps: we first show that it is sufficient to approximate the solution of a Cauchy problem corresponding to an initial datum $\widetilde{\varphi}$ with compact support, then we will prove the convergence result assuming some further regularity on the solution, finally we shall remove that additional conditions.

Step 1 For every $R>0$ we consider a function $\widetilde{\varphi}\in C(\mathbb{R}^2)$ such that $\widetilde{\varphi}(x,y)=\varphi(x,y)$ when $|(x,y)|\leq R$, $\widetilde{\varphi}(x,y)=0$ when $|(x,y)|\geq 2R$. Let H be a compact subset of $\mathbb{R}^2\times]0,\infty[$. Then, for every $\varepsilon>0$, there exists $R>0$ such that, if $\widetilde{u}$ denotes the solution to the problem

$$\begin{cases} Lu=0 & \text{in } \{t>0\}\\ u=\widetilde{\varphi} & \text{on } \{t=0\},\end{cases}$$

then we find

$$\max_{z\in H}|u(z)-\widetilde{u}(z)|\leq\varepsilon/4. \tag{3.13}$$

The proof of this assertion follows from the fact that $|\varphi(\xi,\eta)|\leq e^{c|(\xi,\eta)|}$, while

$$\Gamma(z;\xi,\eta,0)\leq c_0e^{-c_1|(\xi,\eta)|^2}$$

for every $z\in H$ and for $(\xi,\eta)\in\mathbb{R}^2, |(\xi,\eta)|\geq R$, (where c_0, c_1 and R are suitable positive constants. For a detailed proof of the last inequality see Lemma 3.1 in [24]).

Step 2 Suppose that

$$\sup_{\{t>0\}}\left|\frac{\partial^4\widetilde{u}}{\partial x^4}\right|\leq M,\quad \sup_{\{t>0\}}\left|\left(x\frac{\partial}{\partial y}-\frac{\partial}{\partial t}\right)^2\widetilde{u}\right|\leq M \tag{3.14}$$

for some $M \in \mathbb{R}$ and denote by $\widetilde{u}_G$ the solution to the discrete problem

$$\begin{cases} L_G v = 0 & \text{in } G \cap \{t > 0\} \\ v = \widetilde{\varphi} & \text{in } G \cap \{t = 0\}. \end{cases}$$

Then, since

$$L_G \widetilde{u} = L\widetilde{u} + \frac{\Delta_t}{2}\left(x\frac{\partial}{\partial y} - \frac{\partial}{\partial t}\right)^2 \widetilde{u}(x_1, y_1, t_1) + \frac{\Delta_x^2}{24}\frac{\partial^4 \widetilde{u}}{\partial x^4}(x_2, y_2, t_2) + \frac{\Delta_x^2}{24}\frac{\partial^4 \widetilde{u}}{\partial x^4}(x_3, y_3, t_3),$$

for every $(x, y, t) \in G \cap \{t > 0\}$ and for suitable $(x_1, y_1, t_1), (x_2, y_2, t_2), (x_3, y_3, t_3) \in \{t > 0\}$, we have

$$|L_G \widetilde{u}| \le \left(\frac{\Delta_t}{2} + \frac{\Delta_x^2}{12}\right) M.$$

From this inequality, by standard techniques, we find that, if Δ_x and Δ_t satisfy the *stability condition* (3.12), then

$$|\widetilde{u}_G(x, y, t) - \widetilde{u}(x, y, t)| \le \frac{tM}{3}\Delta_x^2, \tag{3.15}$$

for any $(x, y, t) \in G \cap H$. As a consequence, if we choose Δ_x and Δ_t small enough, we obtain

$$\sup_{G \cap \{t \ge 0\}} |\widetilde{u}_G - \widetilde{u}| \le \varepsilon/4.$$

Step 3 We next remove the additional hypothesis (3.14). First of all, we note that, as a consequence of the representation formula (3.10), the function $\widetilde{u}$ is uniformly continuous; then there exists $t_0 > 0$ such that

$$\sup_{\{t \ge 0\}} |\widetilde{u}(x, y, t) - \widetilde{u}(x, y, t + t_0)| \le \varepsilon/4. \tag{3.16}$$

Moreover, again by (3.10), we get the following estimates

$$\sup_{\{t \ge t_0\}} \left|\frac{\partial^4 \widetilde{u}}{\partial x^4}\right| \le c(t_0) \max|\widetilde{\varphi}|, \quad \sup_{\{t \ge t_0\}} \left|\left(x\frac{\partial}{\partial y} - \frac{\partial}{\partial t}\right)^2 \widetilde{u}\right| \le c(t_0) \max|\widetilde{\varphi}|.$$

Consider now the function $v(x, y, t) = \widetilde{u}(x, y, t + t_0)$: it is a solution to the problem

$$\begin{cases} Lv = 0 & \text{in } \{t > 0\} \\ v(x, y, 0) = \widetilde{u}(x, y, t_0) & \text{for } (x, y) \in \mathbb{R}^2, \end{cases}$$

its derivatives are bounded, then, by Step 2,

$$\sup_{G \cap \{t \ge 0\}} |v_G - v| \le \varepsilon/4, \tag{3.17}$$

where (obviously) v_G denotes the solution of the problem

$$\begin{cases} L_G v = 0 & \text{in } G \cap \{t > 0\}, \\ v(x,y,0) = \widetilde{u}(x,y,t_0) & \text{for } (x,y,0) \in G. \end{cases}$$

Again by (3.16) we have

$$\sup_{(x,y,0)\in G} |v_G(x,y,0) - \widetilde{u}_G(x,y,0)| \le \frac{\varepsilon}{4},$$

hence

$$\sup_{(x,y,t)\in G\cap\{t\ge 0\}} |v_G(x,y,0) - \widetilde{u}_G(x,y,0)| \le \frac{\varepsilon}{4}.$$

From this inequality, from (3.13), (3.17) and from (3.16) we obtain the first claim. The second claim can be proved in the same manner. From (3.13) and from the representation formula (3.10) we readily obtain

$$\sup_{H\cap\{t\ge t_0\}} \left| \frac{\partial u}{\partial x} - \frac{\partial \widetilde{u}}{\partial x} \right| \le \frac{\varepsilon}{4},$$

while from (3.15) it follows that

$$\left| \frac{\widetilde{u}(x+\Delta_x,y,t) - \widetilde{u}(x,y,t)}{\Delta_x} - \frac{\widetilde{u}_G(x+\Delta_x,y,t) - \widetilde{u}_G(x,y,t)}{\Delta_x} \right| \le$$

$$\left| \frac{\widetilde{u}(x,y,t) - \widetilde{u}_G(x,y,t)}{\Delta_x} \right| + \left| \frac{\widetilde{u}(x+\Delta_x,y,t) - \widetilde{u}_G(x+\Delta_x,y,t)}{\Delta_x} \right| \le \frac{2Mt}{3}\Delta_x.$$

Arguing as above, we get from this inequality

$$\sup_{(x,y,t)\in H\cap\{t\ge t_0\}} \left| \frac{\partial u}{\partial x}(x,y,t) - \frac{u_G(x+\Delta_x,y,t) - u_G(x,y,t)}{\Delta_x} \right| \le \frac{\varepsilon}{2} + \widetilde{M}\Delta_x,$$

with $\widetilde{M}$ depending on M and on H and this completes the proof of Theorem 3.4.

4 Arithmetic Average Options

Assuming that the underlying asset price follows a lognormal stochastic process, i.e., $\mu_s = \mu S$ and $\sigma_s = \sigma S$, then the Cauchy problems for the price of the Arithmetic average floating strike call option and of the Arithmetic average fixed strike call option become

$$-rV + \frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + S\frac{\partial V}{\partial A} = 0 \tag{4.1}$$

with

$$V(T,S,A) = \max\left(S - \frac{A}{T}, 0\right), \tag{4.2}$$

or

$$V(T,S,A) = \max\left(\frac{A}{T} - E, 0\right). \tag{4.3}$$

Before starting the study of the above Cauchy problem, we observe that the change of variable $x = \log(S)$, performed in the previous Section, transforms the differential equation (4.1) into a PDE like

$$\frac{\partial^2 u}{\partial x^2} + e^x \frac{\partial u}{\partial y} = \frac{\partial u}{\partial t}, \tag{4.4}$$

that looks more difficult than (3.4). Indeed, many "local" properties of the Kolmogorov equation (3.4) also hold for the equation (4.4), such as the C^∞ regularity of the solutions (since (4.4) satisfies the Hörmander's condition). Due to the exponential growth of the coefficient of the derivative $\frac{\partial u}{\partial y}$, we cannot hope to get, for the solutions of (4.4), the same "global" properties proved for the solutions of (3.4). In the sequel we shall consider the differential equation in its original form (4.1) rather than in the form (4.4). We will follow the approach proposed by Gleit in [15] which makes use of the results by [2].

Letting

$$u(x,y,t) = x^m e^{qt} V\left(T - \frac{2t}{\sigma^2}, x, \frac{2y}{\sigma^2}\right), \quad m = \frac{r}{\sigma^2}, \; q = m^2 + m,$$

we see that the differential equation (4.1) is equivalent to

$$x^2 \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial y} = \frac{\partial u}{\partial t};$$

and the Cauchy problem (4.1) with final condition (4.2) or (4.3) becomes

$$\begin{cases} x^2 \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial y} = \frac{\partial u}{\partial t} & \text{for } (x,y,t) \in \Omega_T \\ u(x,y,0) = \varphi(x,y) & \text{for } (x,y) \in \mathbb{R}^+ \times \mathbb{R}, \end{cases} \tag{4.5}$$

where $\Omega_T = \mathbb{R}^+ \times \mathbb{R} \times]0, \frac{\sigma^2}{2}T[$ and

$$\varphi(x,y) = x^m \max\left(x - \frac{2|y|}{\sigma^2 T}, 0\right) \tag{4.6}$$

in the case (4.2) (we are interested in positive values for y),

$$\varphi(x,y) = x^m \max\left(\frac{2y}{\sigma^2 T} - E, 0\right) \tag{4.7}$$

in the case (4.3).

We next state an existence result for solutions to problem (4.5), by using the notion of super- and sub-solution. We shall say that u is *supersolution* to the problem (4.5) if

$$\begin{cases} x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial y} \le \frac{\partial u}{\partial t} & \text{for } (x,y,t) \in \Omega_T \\ u(x,y,0) \ge \varphi(x,y) & \text{for } (x,y) \in \mathbb{R}^+ \times \mathbb{R} \end{cases} \tag{4.8}$$

and that u is *subsolution* to (4.5) if the condition (4.8) holds with the reverse inequalities.

THEOREM 4.1 *Let $\varphi \in C(\mathbb{R}^+ \times \mathbb{R}), \overline{u}, \underline{u}$ be supersolution and subsolution, respectively, to problem (4.5) and suppose that $\underline{u} \le \overline{u}$ in Ω_T. Then there exists a solution u to the problem (4.5) such that*

$$\underline{u} \le u \le \overline{u} \quad \text{in } \Omega_T. \tag{4.9}$$

The proof of the Theorem is postponed in Subsection 4.1, while some results about the numerical approximation of the solution of (4.5) are presented in Subsection 4.2.

REMARK 4.2 *Choosing $\underline{u} \equiv 0$ and $\overline{u}(x,y,t) = x^{m+1}e^{qt}$ for the problem corresponding to the initial condition (4.6), $\underline{u} \equiv 0$ and $\overline{u}(x,y,t) = \frac{\sigma^2}{2T}e^{kt}x^m\sqrt{x^2+y^2}$, $k = m^2+2m+2$ for the problem corresponding to (4.7), we readily obtain the existence of a solution to problem (4.5).*

In both cases we have $\underline{u}(0,y,t) = \overline{u}(0,y,t) = 0$, then we implicitly assumed that $u(0,y,t) = 0$ for every y,t; thus, by Theorem B in [2], u is the unique solution that satisfies (4.9). Indeed, the hypothesis $u(0,y,t) = 0$ is not necessary, as shown in Remark 4.6.

We conclude this Section with a remark due to Ingersoll [17], that in some cases allows to reduce the Cauchy problem (4.5) in another form, where the PDE is a nondegenerate parabolic equation in only one space variable.

Suppose that the final condition is a homogeneous function of degree 0 in the variables S, A:

$$V(T,S,A) = S \cdot V\left(T, 1, \frac{A}{S}\right)$$

(this condition is satisfied, for example, when we consider the datum (4.2), but it's not by (4.3)). Denote by W the solution to the Cauchy problem

$$\begin{cases} x^2\frac{\partial^2 W}{\partial x^2} + (1-x)\frac{\partial W}{\partial x} + \frac{\partial W}{\partial t} = 0 & \text{for } (x,t) \in \mathbb{R}^+ \times]0,T[, \\ W(T,x) = V(T,1,x), & \text{for } x > 0, \end{cases}$$

then the function $V(t,S,A) = S \cdot W\left(t, \frac{A}{S}\right)$ is a solution to the problem corresponding to the PDE (4.1).

4.1 The Cauchy problem

In order to give the proof of Teorem 4.1 we first recall a result by ([21], Theorem 1.5), about the Dirichlet problem in bounded sets. In the sequel we will denote by L the operator

$$L = x^2 \frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial y} - \frac{\partial}{\partial t}. \tag{4.10}$$

THEOREM 4.3 *Let Ω be a bounded open subset of $\mathbb{R}^3$ such that $\overline{\Omega} \subset \{x \neq 0\}$ and let $f \in C(\partial\Omega)$. Then the problem*

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ u = f & \text{in } \partial\Omega, \end{cases} \tag{4.11}$$

has a generalized solution, in the sense of Perron-Wiener.

Note that the hypothesis $\overline{\Omega} \subset \{x \neq 0\}$ does not explicitly appear in [21], but it is assumed the condition (that actually is equivalent) that the coefficient of the derivative $\frac{\partial^2}{\partial x^2}$ is bigger than some positive constant.

Some remarks on the above generalized solution are in order. The solution u given by the Perron-Wiener method turns out to be a $C^\infty(\Omega)$ function, that solves $Lu = 0$ in the classical sense. Moreover the following interior regularity result holds (see Theorem 1.4 in [21]).

PROPOSITION 4.4 *Let Ω be a bounded open set such that $\overline{\Omega} \subset \{x \neq 0\}$ and let u be a solution of $Lu = 0$ in Ω. Then, for every open set Ω' such that $\overline{\Omega}' \subset \Omega$ there exist two positive constants c and α, that do not depend on u, such that*

$$|u(z) - u(w)| \leq c|z - w|^\alpha \sup_\Omega |u|,$$

for every $z, w \in \Omega'$.

Concerning the boundary condition $u = f$ on $\partial\Omega$, it may happens in general that it is not satisfied. We shall say that a point $z_0 \in \partial\Omega$ is a *regular* point if the generalised solution u to the Dirichlet problem (4.11) satisfies $u(z) \longrightarrow f(z_0)$ as $z \longrightarrow z_0$ $(z \in \Omega)$, for every given $f \in C(\partial\Omega)$.

Sufficient conditions for the regularity of a boundary point are given in [21] (see Theorems 6.1 and 6.3). In the case of the operator (4.10) we can state such conditions as follows. We shall say that the vector $n \in \mathbb{R}^3$ is an *outer normal vector to the open set Ω at the point $z_0 \in \partial\Omega$* if $B_{|n|}(z_0 + n) \subset \mathbb{R}^3 \backslash \Omega$ (here $B_r(z)$ denotes the euclidean ball with center z and radius r).

PROPOSITION 4.5 *Let Ω be an open subset of $\mathbb{R}^3$ such that $\overline{\Omega} \subset \{x \neq 0\}$ and let $(x_0, y_0, t_0) \in \partial\Omega$. If there exists an outer normal vector $n = (n_x, n_y, n_t)$ such that $n_x \neq 0$, then (x_0, y_0, t_0) is a regular point.*

Moreover, if $n_x = 0$, but $x_0 n_y - n_t > 0$ and there exists a positive constant δ such that $x_0^2\delta^2 \leq x_0 n_y - n_t$ and that

$$\left\{(x,y,t) \in \mathbb{R}^3 : \delta^2(x-x_0)^2 + (y - y_0 - \delta^2 n_y)^2 + (t - t_0 - \delta^2 n_t)^2 \leq \delta^4\right\} \subset \mathbb{R}^3 \backslash \Omega,$$

then (x_0, y_0, t_0) is a regular point.

We can now use the above results, concerning the bounded open sets, to prove Theorem 4.1.

Proof of Theorem 4.1. We shall define a sequence of bounded open sets that fills Ω_T and we shall solve a suitable boundary value problem on every such domain. We start by providing a continuous function $\widetilde{\varphi}$ that extends φ to the domain $\overline{\Omega}_T$ in such a way that $\underline{u} \leq \widetilde{\varphi} \leq \overline{u}$ (we may define the extension as $\widetilde{\varphi}(x,y,t) = \max\{\underline{u}(x,y,t), \varphi(x,y)\}$).

For every $k \in \mathbb{N}$ we set $\Omega_k =]1/k, k+1[\times]-k, k[\times]0, T[$. By Theorem 4.3, there exists a generalised solution u_k to the boundary value problem

$$\begin{cases} Lu = 0 & \text{for } (x,y,t) \in \Omega_k \\ u = \widetilde{\varphi} & \text{for } (x,y,t) \in \partial\Omega_k. \end{cases} \tag{4.12}$$

Moreover Proposition 4.5 ensures that every point in the following set

$$\big\{(x,y,t) \in \partial\Omega : t = 0\big\} \cup \big\{(x,y,t) \in \partial\Omega : x = 1/k\big\} \cup \\ \big\{(x,y,t) \in \partial\Omega : x = k+1\big\} \cup \big\{(x,y,t) \in \partial\Omega : y = k\big\}$$

is regular. From this fact and from the maximum principle for degenerate operators ([6, Proposition 3.1]) it follows that

$$\underline{u} \leq u_k \leq \overline{u} \quad \text{in} \quad \Omega_k. \tag{4.13}$$

We can now provide a solution to (4.5) by using the sequence $\big(u_k\big)_{k\in\mathbb{N}}$. For every $k \in \mathbb{N}$ we let

$$A_k = \big\{(x,y,t) \in \Omega_k : T/3k < t < T(1 - 1/3k)\big\}$$

and we note that

$$\Omega_T = \bigcup_{k\in\mathbb{N}} A_k. \tag{4.14}$$

From (4.13) it follows that the sequence $\big(u_k\big)_{k\geq 2}$ is bounded in $\overline{A}_1$ and, from Proposition 4.4, we see that it is also equicontinuous, thus there exists a subsequence $\big(u_{k_{1,j}}\big)_{j\in\mathbb{N}}$ that uniformly converges to some function $v_1 \in C(\overline{A}_1)$. We note that, as a consequence, v_1 is a *weak* solution to $Lu = 0$ in A_1 (then, by Hörmander's results [16], it is also a $C^\infty(A_1)$ function and classical solution to $Lu = 0$) moreover

$$\underline{u} \leq v_1 \leq \overline{u} \quad \text{in} \quad A_1.$$

We next apply the same argument to the sequence $\left(u_{k_{1,j}}\right)_{j\in\mathbb{N}}$ on the set $\overline{A}_2$ and we obtain a subsequence $\left(u_{k_{2,j}}\right)_{j\in\mathbb{N}}$ that converges in $C(\overline{A}_2)$ to some function v_2, that is classical solution to $Lu = 0$ and such that $\underline{u} \leq v_2 \leq \overline{u}$ in A_2. Note that, since v_2 is the limit of a subsequence of $\left(u_{k_{1,j}}\right)_{j\in\mathbb{N}}$, it musts coincide with v_1 in A_1.

We next proceed by induction: for every $m \in \mathbb{N}$ we consider the sequence $\left(u_{k_{m-1,j}}\right)_{j\in\mathbb{N}}$ on the set $\overline{A}_m$ and we extract from it a subsequence $\left(u_{k_{m,j}}\right)_{j\in\mathbb{N}}$ converging in $C(\overline{A}_m)$ to some function v_m, classical solution of $Lu = 0$, such that $\underline{u} \leq v_m \leq \overline{u}$ in A_m and that it equals v_{m-1} on the set A_{m-1}.

We then define a function u in the following way: for every i $(x, y, t) \in \Omega_T$ we choose $m \in \mathbb{N}$ such that $(x, y, t) \in A_m$ and we set $u(x, y, t) = v_m(x, y, t)$. Note that the definition is well posed, since, if $(x, y, t) \in A_n$, then $v_n(x, y, t) = v_m(x, y, t)$. Moreover u is classical solution to $Lu = 0$ in Ω_T and satisfies $\underline{u} \leq u \leq \overline{u}$ in Ω_T.

To conclude the proof of Theorem 4.1 we have to verify that, for any $(x_0, y_0) \in \mathbb{R}^+ \times \mathbb{R}$, we have

$$\lim_{(x,y,t)\to(x_0,y_0,0)} u(x, y, t) = \varphi(x_0, y_0). \tag{4.15}$$

We already observed that, if $k \in \mathbb{N}$ is such that $(x_0, y_0, 0) \in \partial\Omega_k$, then Proposition 4.5 ensures that every solution u_k assumes the boundary datum φ in $(x_0, y_0, 0)$. This fact is not sufficient to guarantee that also the function u has the same property, nevertheless the proof of Proposition 4.5 is based on the use of "barriers", that give an estimate of the rate of convergence as (x, y, t) goes to $(x_0, y_0, 0)$, that is *uniform* with respect to k. From that uniform estimates we can get (4.15). See [21] and [9] for a more complete treatment of the regularity of the boundary.

We next state a comparison result for the problem (4.5) that improves Theorem B in [2] (in the sense that the assumption on the behaviour of the solution near the boundary is weaker than the one made in [2]). We first define the following function in $\mathbb{R}^+ \times \mathbb{R}$

$$\psi(x, y) = \log\left(x^2 + y^2 + 1\right) - \log(x) \tag{4.16}$$

and we note that

$$\psi(x, y) \xrightarrow[\|(x,y)\|\to\infty]{} \infty, \quad \psi(x, y) \xrightarrow[(x,y)\to(0,y_0)]{} \infty,$$

for every $y_0 \in \mathbb{R}$.

REMARK 4.6 *Let $u \in C(\overline{\Omega}_T)$, such that $Lu \leq 0$ in Ω_T, that $u(x,y,0) \geq 0$ for every $(x,y) \in \mathbb{R}^+ \times \mathbb{R}$ and assume that there exist two positive constants M and k such that*

$$u(x,y,t) \geq -Me^{k\psi^2(x,y)},$$

for every $(x,y,t) \in \Omega_T$ (ψ denotes the function defined in (4.16)). Then $u \geq 0$ in Ω_T.

We do not give the proof of the above statement since it follows the same lines as the proof of Theorem B in [2], making use of the function defined in (4.16) instead of the one proposed in [2].

4.2 Numerical approximation

In this Section we shall consider an approximation method for the Cauchy problem (4.5). The main technical difficulties are due to the lack of suitable a priori estimates "near infinity" for the derivatives of the solution.

In order to avoid such difficulties, it is convenient to approximate one of the functions u_k defined in (4.12), by using the same technique employed in the study of geometric average options.

In the proof of Theorem 4.1 we proved that, for every compact $H \subset \Omega$ and for any $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$\max_H |u_k - u| \leq \frac{\varepsilon}{2}. \tag{4.17}$$

Moreover, by Theorem 2.1 in [24], there exists a grid G and a function $\left(u_k\right)_G$, defined on $\Omega_k \cap G$, such that

$$\max_{\Omega_k \cap G} |u_k - \left(u_k\right)_G| \leq \frac{\varepsilon}{2}, \tag{4.18}$$

thus

$$\max_{H \cap G} |u - (u_k)_G| \leq \varepsilon.$$

(actually, Theorem 2.1 in [24] applies to the differential equation with constant coefficients but, as already observed in Remark 3.5, it can be easily adapted to any differential equation with Hölder continuous coefficients.)

We stress that the above idea is not completely satisfactory, since the inequality (4.17) relies on some compactness argument and does not give an explicit dependence of the approximation error ε, in terms of k (while inequality (4.18) actually does).

To avoid this problem, instead of approximating a function u_k, we approximate a supersolution and a subsolution of the problem (4.12). To this end we consider a function $\chi \in C_0^\infty]1/k, k+1[\times]-k,k[$, such that $\chi \equiv 1$ in $]2/k,k[\times]-(k-1),k-1[$ and we set

$$\underline{\varphi}(x,y,t) = \chi(x,y)\varphi(x,y) + \big(1-\chi(x,y)\big)\underline{u}(x,y,t),$$
$$\overline{\varphi}(x,y,t) = \chi(x,y)\varphi(x,y) + \big(1-\chi(x,y)\big)\overline{u}(x,y,t).$$

Let $\underline{u}_k$ and $\overline{u}_k$ be the solutions of the following Cauchy problems in $\Omega_k(=]1/k, k+1[\times]-k, k[\times]0, T[)$

$$\begin{cases} Lu = 0 & \text{per } (x,y,t) \in \Omega_k, \\ u = \underline{\varphi} & \text{per } (x,y,t) \in \partial\Omega_k, \end{cases} \qquad \begin{cases} Lu = 0 & \text{per } (x,y,t) \in \Omega_k, \\ u = \overline{\varphi} & \text{per } (x,y,t) \in \partial\Omega_k. \end{cases} \tag{4.19}$$

Being $\underline{u} \le u \le \overline{u}$ in Ω, we have

$$\underline{\varphi} \le u \le \overline{\varphi} \quad \text{in } \partial\Omega_k \cap \{t < T\}$$

and therefore $\underline{u}_k \le u \le \overline{u}_k$ in Ω_k. Thanks to these facts we can state the following result.

THEOREM 4.7 *Let u be a solution of (4.5), $\underline{u}_k$ and $\overline{u}_k$ solutions of* (4.19). *Then for every $\varepsilon > 0$ there exists a grid G, satisfying the stability condition*

$$\frac{\Delta_t}{(\Delta_x)^2} \le \frac{1}{2a_0}, \quad \big(\text{where } a_0 = \sup a(x,y,t)\big),$$

such that

$$\left(\underline{u}_k\right)_G - \varepsilon \le \underline{u}_k \le u \le \overline{u}_k \le \left(\overline{u}_k\right)_G + \varepsilon \quad \textit{in } G \cap \Omega_k.$$

Note that, given the numerical solutions $\left(\underline{u}_k\right)_G$ and $\left(\overline{u}_k\right)_G$, Theorem 4.7 allows us to estimate the approximation error ε with respect to to the grid G, k and to the data $\underline{\varphi}$ and $\overline{\varphi}$.

5 The General Case

Let us consider now the Cauchy problem associated with the partial differential equation (2.4) in its general form. In this context there is no hope to prove the regularity properties of the solution that allow us to demonstrate the approximation results obtained above. In fact, being the matrix of the coefficients of the second derivatives in (2.4)

$$\frac{1}{2}\begin{pmatrix} \sigma_a^2 & \sigma_a\sigma_s \\ \sigma_a\sigma_s & \sigma_s^2 \end{pmatrix}$$

positive semidefinite with an eigenvalue equal to zero, it may be possible to convert the PDE in another nondegenerate PDE in a submanifold. In that case the Cauchy problem should be studied with different techniques. As observed in [5], it is possible to characterize such equation through Lie Algebra. To this end we rewrite equation (2.4) as follows

$$Lu = \frac{1}{2}X^2u + Yu, \tag{5.1}$$

where X and Y are the following directional derivatives:

$$X = \sigma_a \frac{\partial}{\partial x} + \sigma_s \frac{\partial}{\partial y}, \quad Y = \widetilde{\mu}_a \frac{\partial}{\partial x} + \widetilde{\mu}_s \frac{\partial}{\partial y} - \frac{\partial}{\partial t},$$

($\widetilde{\mu}_a$ e $\widetilde{\mu}_s$ are coefficients that depend on $\sigma_a, \sigma_s, \mu_a$ e μ_s).

We are now able to recognise equations which admit regular solutions. In fact, when the commutator $[X, Y](= XY - YX)$ is zero at every point of some open subset of $\mathbb{R}^3$, then it follows from the Frobenuis theorem that, up to a change of coordinates, the operator only acts on two variables, while the last one may be considered as a constraint. In that case the solutions of (5.1) are not necessarily smooth; on the other hand, when the change of variables is performed, the differential equation becomes nondegenerate and the classical numerical methods can be applied.

If otherwise the commutator $[X, Y]$ never vanishes, by the results in [16], every solution of (5.1) is a C^∞ function. In that case it seems possible to employ the numerical techniques used above. Among operators of this kind there are the ones corresponding to *Path Dependent Options in a Constant Elasticity Variance Environment* which will be considered in next subsection.

5.1 Path Dependent Options in a Constant Elasticity Variance Environment

In the above sections we have considered a geometrical brownian motion for the underlying asset price. This type of diffusion process is heavily employed in mathematical finance because it allows for a great tractability. The main drawback is that observed data are not well described by this process. Starting from this observation in [10, 11] the following diffusion process for the underlying asset price was proposed:

$$dS = \mu S dt + \sigma S^{\varrho} dz,$$

with $0 \leq \varrho \leq 1$. In this setting the Cauchy problem (2.4)-(2.5) becomes

$$-rV + \frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^{\alpha} \frac{\partial^2 V}{\partial S^2} + \mu_a \frac{\partial V}{\partial A} = 0, \tag{5.2}$$

with

$$V(T, S_T, A_T) = \Omega(S_T, A_T), \tag{5.3}$$

where $\alpha = 2\varrho$ and $\mu_a = S$ (arithmetic average) or $\mu_a = \log(S)$ (geometrc average).

The technique used in the study of the problem corresponding to the arithmetic average options can be employed in the study of the Cauchy problem (5.2)-(5.3).

Theorem 5.1 *Let $\varphi \in C(\mathbb{R}^+ \times \mathbb{R}), a, b$ and c be locally Hölder continuous functions in $\Omega_T (\equiv \mathbb{R}^+ \times \mathbb{R} \times]0, T[)$, with $a > 0$. Moreover, let $\overline{u}, \underline{u}$ be, respectively, supersolution and subsolution to the problem*

$$\begin{cases} a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} + cu = \frac{\partial u}{\partial t} & \text{for } (x,y,t) \in \Omega_T, \\ u(x,y,0) = \varphi(x,y) & \text{for } (x,y) \in \mathbb{R}^+ \times \mathbb{R}, \end{cases} \tag{5.4}$$

such that $\underline{u} \leq \overline{u}$ in Ω_T. Then there exists a solution u to (5.4) such that

$$\underline{u} \leq u \leq \overline{u} \quad \text{in } \Omega_T.$$

Consider $\Omega_k(=]1/k, k+1[\times]-k, k[\times]0, T[)$, let $\chi \in C_0^\infty]1/k, k+1[\times]-k, k[$ be a function such that $\chi \equiv 1$ in $]2/k, k[\times]-(k-1), k-1[$, put

$$\underline{\varphi}(x,y,t) = \chi(x,y)\varphi(x,y) + \big(1 - \chi(x,y)\big)\underline{u}(x,y,t),$$
$$\overline{\varphi}(x,y,t) = \chi(x,y)\varphi(x,y) + \big(1 - \chi(x,y)\big)\overline{u}(x,y,t)$$

and denote by $\underline{u}_k$ and $\overline{u}_k$ the solutions to the boundary value problem

$$\begin{cases} a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} + cu = \frac{\partial u}{\partial t} & \text{for } (x,y,t) \in \Omega_k, \\ u = \psi & \text{for } (x,y,t) \in \partial\Omega_k, \end{cases} \tag{5.5}$$

corresponding to $\psi = \underline{\varphi}$ and $\psi = \overline{\varphi}$, respectively.

Theorem 5.2 *Let u be a solution to (4.5), $\underline{u}_k$ and $\overline{u}_k$ be the solutions to (5.5) corresponding to $\psi = \underline{\varphi}$ and $\psi = \overline{\varphi}$, respectively. Then, for every $\varepsilon > 0$ there exists a grid G, satisfying the stability condition*

$$\frac{\Delta_t}{(\Delta_x)^2} \leq \frac{1}{2a_0}, \quad \big(\text{where } a_0 = \sup a(x,y,t)\big),$$

such that

$$\big(\underline{u}_k\big)_G - \varepsilon \leq \underline{u}_k \leq u \leq \overline{u}_k \leq \big(\overline{u}_k\big)_G + \varepsilon \quad \text{in } G \cap \Omega_k.$$

Remark 5.3 *When $\mu_a(S) = S$, we can write the Cauchy problem corresponding to Path Dependent Options in a Constant Elasticity Variance Environment in the form (5.4) by choosing the coefficients $a = \frac{1}{2}\sigma^2 x^\alpha, b = xr, c = -r$. In the case $\mu_a(S) = \log(S)$, in order to rewrite equation (5.2) in the form (5.4), it is convenient to perform the chenge of variables $x = \log(S)$.*

We do not give the proof of Theorems 5.1 and 5.2, since it is essentially the same as the proof of Theorems 4.1 and 4.7, respectively.

6 Conclusions

In this paper we have analyzed the Cauchy problem associated with the no arbitrage price of an Asian option. We have shown that the solution of the Cauchy problem and therefore the no arbitrage price of an Asian contingent claim has enough regularity so that classical finite difference methods can be applied to approximate the solution. An estimate of the approximation error is obtained.

These results are useful in an applied perspective because a closed form solution for the no arbitrage price of Asian options is not available in general. Numerical methods aiming at approximating the Cauchy problem solution have been employed in the mathematical finance literature without an adequate analysis of their convergence and of their approximation error. This is the void filled in our paper.

References

[1] B. Alziary, J. P. Décamps, P.F. Koehl *A P.D.E. approach to Asian options: analytical and numerical evidence*, Journal of Banking anf Finance **21** (1997), 613-640.

[2] D. G. Aronson, P. Besala, *Uniqueness of positive solution of parabolic equations with unbounded coefficients,* Colloquium Mathematicum **18** (1967), 125–135.

[3] G. Barles *Convergence of Numerical schemes for degenerate parabolic equations arising in finance theory,* Numerical Methods in Finance, L. Rogers and Talay, D. eds., Cambridge University Press, 1996.

[4] G. Barles, Ch. Daher, M. Romano *Convergence of numerical schemes for problems arising in finance theory,* Math. Models and Meth. in Appl. Sciences, **5** (1995), 125–143.

[5] J. Barraquand, T. Pudet *Pricing of American Path-Dependent Contingent Claims,* Mathematical Finance (1996), 17-51.

[6] J. -M. Bony, *Principe de maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés,* Ann. Inst. Fourier, Grenoble **19,1** (1969), 277–304.

[7] L. Bouaziz, E. Briys and M. M. Crouhy *The pricing of forward-starting asian options,* Journal of Banking anf Finance **18** (1994), 823-839.

[8] A. Caverhill and L. Clewlow *valuing average rate Asian options,* RISK **3**, 25-29.

[9] C. Constantinescu, A. Cornea, *Potential theory on harmonic spaces,* Springer-Verlag, Berlin, (1972).

[10] J. Cox, S. Ross, *The valuation of options for alternative stochastic processes,* Journal of Financial Economics, 3 (1976), 145-166.

[11] J. Cox, M. Rubinstein, *Options Markets,* (1985) Englewood Cliffs.

[12] J. Dewynne, P. Wilmott, *Asian options as linear complementarity problems,* Advances in Futures and Options Research, **8**, 145-173.

[13] A. Eydeland and H. Geman, *Asian options revisited: inverting the Laplace transform,* RISK, 1995.

[14] H. Geman and M. Yor, *Bessel Processes, Asian Options, and Perpetuities,* Mathematical Finance **3**, 349-375.

[15] A. GLEIT *Valuation of General Contingent Claims*, Journal of Financial Economics **6** (1978), 71-87.

[16] L. HÖRMANDER, *Hypoelliptic second order differential equations,* Acta Math. **119** (1967), 147–171.

[17] J. INGERSOLL *Theory of Financial Decision Making* Rowman & Littlefield (1987)

[18] A. KENNA AND A. VORST *A pricing method for options based on average asset values*, Journal of Banking and Finance **14** (1990) 113-129.

[19] E. LEVY *Pricing Europan average rate currency options*, Journal of International Money and Finance, **11**, 474-491.

[20] E. LANCONELLI, S. POLIDORO, *On a class of hypoelliptic evolution operators,* Rend. Sem. Mat. Univ. Pol. Torino **52,1** (1994), 29–63.

[21] M. MANFREDINI, *The Dirichlet problem for a class of ultraparabolic equations,* Adv. Differential Equations, **2,5** (1997), 831–866.

[22] S. POLIDORO, *On a class of ultraparabolic operators of Kolmogorov-Fokker-Planck type,* Le Matematiche, **49** (1994), 53–105.

[23] S. POLIDORO, *Uniqueness and representation theorems for solutions of Kolmogorov-Fokker-Planck equations,* Rendiconti di Matematica, VII, **15** (1995), 535–560.

[24] S. POLIDORO, C. MOGAVERO, *A finite difference method for a boundary value problem related to the Kolmogorov equation,* Calcolo **32,3-4** (1995), 193–205.

[25] C. ROGERS AND Z. SHI *The Value of an Asian Option*, Journal of Applied Probability **32** (1995), 1077-1088.

[26] M. RUBINSTEIN *Exotic Options*, (1991) University of California at Berkeley.

[27] A. RUTTIENS *Average-rate Options, Classical Replica*, RISK, **3**, 33-36.

[28] A. N. SHIRYAYEV (ed.), *Selected works of A. N. Kolmogorov, vol II, Probability theory and mathematical statistics,* Kluver Academic Publishers, Dordrecht-Boston-London (1991).

[29] S. TURNBULL AND L. WAKEMAN *A Quick Algorithm for Pricing European Average Options* Journal of Financial and Quantitative Analysis **26**, 377-389.

[30] T. VORST *Analytica boundaries and approximations of the prices and hedge ratios of average exchange rate options*, working paper, Erasmus University, Rotterdam.

[31] P. Wilmott, J. Dewynne and S. Howison *Option Pricing*, Oxford Financial Press.