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1 Introduction

In this paper we characterize the hedging strategy for an European contingent claim in an incomplete Markovian market.

The paper is related to the analysis developed in [Ocone and Karatzas, 1991, Colwell et al., 1991, Bensoussan, Elliott, 1995], where a hedging strategy for an European contingent claim is obtained in a general setting in a complete markets economy. In this paper, specializing the market model to be Markovian, we extend their analysis to an incomplete markets setting. A hedging strategy of an European contingent claim associated with a risk neutral martingale measure is computed by means of some techniques developed in the stochastic calculus of variations literature.

Our approach is different from the classical one. In a Markovian market model (either complete or incomplete), assuming that the contingent claim final payoff is written as a function of the underlying state variables (stock prices and other factors), the hedging strategy of a contingent claim is provided by means of the partial derivatives of its no arbitrage price which is given by the solution of a second order partial differential equation. This approach allows us to write the hedging strategy in terms of the partial derivatives of a function which is unknown and therefore we rely upon numerical methods to solve the partial differential equation and to compute the associated partial derivatives. Our aproach is different, it defines the hedging strategy as the conditional expectation of the Malliavin derivative of the final payoff.

Our analysis has also some feedbacks for the complete markets case. In a general setting the hedging strategy for an European contingent claim has been obtained through Malliavin calculus in [Ocone and Karatzas, 1991], the authors stress that their hedging formulae can be handled only in case of deterministic coefficients and they say that "it would be interesting to try to extract more useful information from these formulae in situations with random, possibly Markovian, coefficients", see [Ocone and Karatzas, 1991, pag.188]. In [Colwell et al., 1991] the task is accomplished. In what follows we refine the [Colwell et al., 1991] by allowing for lower regularity in the coefficients of the stochastic differential equations for the asset prices.

The paper is organized as follows. In Section 2 we introduce the Markovian model. In Section 3, given a generic risk neutral martingale measure, we provide a hedging strategy for a contingent claim together with the associated cost process.

2 A Markovian Model

We consider the following Stochastic Differential Equation (SDE) in $\Re^n$:

$$dX^i(t) = X^i(t)[b^i(t, X(t))dt + \sum_{j=1}^{n} \sigma_j^i(t, X(t))dW_j(t)], \quad X^i(0) \in (0, +\infty); \; i = 0, \dots, n, \quad (1)$$

where $W = (W_1, \dots, W_n)^\top$ is a standard Brownian motion in $\Re^n$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathrm{P})$. We shall denote by $\{\mathcal{F}_t\}$ the P-augmentation of the filtration $\mathcal{F}_t^W = \sigma(W(s); 0 \le s \le t)$ generated by W. The coefficients $b = (b^1, \dots, b^n)$ and $\sigma =$

$\{\sigma^i_j,\ i=1,\dots n,\ j=1,\dots n\}$ are progressively measurable functions $[0,T]\times\Re^{n+1}\to\Re$. X^0 represents the price of the riskless asset ($\sigma^0_j=0,\ j=1,\dots,n$) and $b^0(t,X(t))=r(t,X(t))$ (the risk-free rate).

The coefficients b^i and σ^i_j satisfy appropriate growth and Lipschitz conditions so that the solution of (1) is a Markov process. $[0,T]$ is a fixed finite time horizon, T will denote the expiration date of the European contingent claims. We assume that $\int_0^T|r(s,X(s))|ds\leq L<\infty$ a.s. for some $L>0$. Assume $T=1$ for simplicity.

We observe that under suitable continuity conditions on the coefficients ensuring that the system of SDE has a pathwise unique, strong solution, then the filtration $\mathcal{F}$ coincides with the P-augmentation of the filtration generated by X (see [Hofmann et al., 1992]).

Differently from other incomplete markets models, see e.g. [Bensoussan, Elliott, 1995, El Karoui and Quenez, 1995], where the state variables only represent the prices of the securities traded in the market and the number of Wiener processes is larger than the security prices, in our setting we have extended the state variables to include not only security prices, but also other factors, e.g., volatility factors, etc.. The first $k+1$, $(1<k<n)$ variables X^i $(i=0,\dots,k)$ denote the prices of the assets traded in the economy (including the risk-free asset). Instead, X^i $(i=k+1,\dots,n)$ denote the risk factors which are not traded in the markets.

We consider an agent who can trade all the $k+1$ assets in the economy, we denote by $\pi^i(t)$ the amount he invests in the i-th stock at time t $(i=0,\dots,k)$. The agent's wealth at time t is denoted by $V(t)$. The resulting portfolio of risky assets $\{\pi(t)=(\pi^1(t),\dots,\pi^k(t))^\top, 0\leq t\leq 1\}$ is assumed to be predictable with respect to $\mathcal{F}_t$, to take values in $\Re^k$, and to satisfy the integrability conditions $\int_0^1\|\sigma^{k\top}(t,X(t))\pi(t)\|^2dt<\infty$, *a.s.* and $\int_0^1|\pi(t)^\top(b(t,X(t))-r(t,X(t))\mathbf{1})|dt<\infty$ *a.s.* Given π, the wealth invested in the riskless asset (π^0) is determined by the budget constraint and the wealth process $V(t)$ satisfies the following equation

$$dV(t)=r(t,X(t))V(t)dt+\pi^\top(t)[(\overline{b}(t,X(t))-r(t,X(t))\mathbf{1})dt+\overline{\sigma}(t,X(t))dW(t)] \quad (2)$$

where $\mathbf{1}$ is an $\Re^k$ vector with entries all equal to 1, $\overline{b}(t,X(t))$ and $\overline{\sigma}(t,X(t))$ represent the first k elements of b and the first k rows of σ . Let $\overline{X}$ denote the prices of the k traded securities. We assume that the $k\times n$ matrix $\overline{\sigma}(t,X(t))$ has full rank for any t, so that the matrix $\overline{\sigma}(t,X(t))\overline{\sigma}^\top(t,X(t))$ is invertible.

An European contingent claim is defined as follows.

Definition 2.1 *An European contingent claim is a non-negative $\mathcal{F}_1$-measurable random variable H such that $E[\beta(1)H]<\infty$.*

$E[\cdot]$ denotes the expectation operator under the measure P and $\beta(t)=e^{-\int_0^t r(s,X(s))ds}$. Below we will often specialize the contingent claim to be of the form $H=g(X(1))$ for a "regular" function g. Note that the contingent claim payoff is allowed to be a function not only of the prices of the traded factors but also of the non traded factors.

A portfolio strategy $\pi(t)$ provides *perfect hedging* of the contingent claim H if given an initial wealth $x>0$ we have $V(1)=H$ almost surely. Perfect hedging of a contingent claim H is obtained through a self-financed portfolio, i.e., given the initial wealth $x>0$

the contingent claim H is replicated running in continuous time the portfolio strategy $\pi(t)$ without adding-withdrawing money for any t. Perfect hedging with a self-financed strategy for every contingent claim is a peculiarity of complete markets, in incomplete markets it is not allowed.

In an incomplete markets setting, given the original probability measure P there is a continuum of probability measures equivalent to the original one such that the asset prices are martingale under this new measure (equivalent martingale measures). The set of the equivalent martingale measures can be characterized as follows. Let $\hat{P}$ one of such measures, then we have that

$$\hat{\mathrm{P}}(A) = \int_A Z(1)dP, \quad \forall A \in \mathcal{F}_1,$$

where $Z(t)$ is defined as the exponential martingale

$$Z(t) = exp\{-\int_0^t \lambda^\top(s, X(s))dW(s) - \frac{1}{2}\int_0^t \|\lambda(s, X(s))\|^2 ds\}.$$

and $\lambda(t, X(t))$ is an n dimensional *relative risk process* such that

$$\overline{\sigma}(t, X(t))\lambda(t, X(t)) = \overline{b}(t, X(t)) - r(t, X(t))\mathbf{1}, \quad 0 \leq t \leq 1. \tag{3}$$

Being $k < n$, equation (3) gives us multiple solutions. Given $\lambda(t, X(t))$ solution of (3), then define the process

$$\hat{W}(t) \equiv W(t) + \int_0^t \lambda(s)ds, \quad 0 \leq t \leq 1,$$

which is an $\Re^n$-Brownian motion under $\hat{P}$. The SDE (1) becomes

$$dX^i(t) = X^i(t)[\hat{b}^i(t, X(t))dt + \sum_{j=1}^{n} \sigma^i_j(t, X(t))d\hat{W}_j(t)], \quad X^i(0) \in (0, +\infty); \ i = 1, \ldots, n, \tag{4}$$

where $\hat{b}(t, X(t))$ is an n dimensional process, $\hat{b}^i(t, X(t)) = r(t, X(t))$ for $i = 1, \ldots, k$, i.e., the drift of the traded assets is the risk free rate under the equivalent measure. The drift of the other stochastic differential equations representing non traded factors are not the risk-free rate.

Given the initial wealth $x > 0$, the resulting wealth V at time t satisfies the following equations:

$$\beta(t)V(t) = x + \int_0^t \pi(s)\beta(s)\overline{\sigma}(s, X(s))d\hat{W}(t) \tag{5}$$

$$\beta(t)V(t) = x + \int_0^t \pi(s)(\beta(s)\overline{X}(s))^{-1}d(\beta\overline{X})(t). \tag{6}$$

Note that we restrict our attention to Markovian risk premia and therefore the $X(t)$ turns out to be Markovian under the equivalent measures.

An equivalent martingale measure of particular interest is the *minimal equivalent martingale measure* defined by the relative risk process

$$\lambda(t, X(t)) = \overline{\sigma}^\top(t, X(t))(\overline{\sigma}(t, X(t)\overline{\sigma}^\top(t, X(t)))^{-1}(\overline{b}(t, X(t)) - r(t, X(t))\mathbf{1}).$$

3 Hedging in an incomplete markets setting

In an incomplete markets setting, in general, we can not perform perfect hedging of an European contingent claim through a self-financed strategy. However, some contingent claims can be replicated through a self-financed strategy. In a Markovian setting, assuming that X only represents the prices of the assets (and therefore they are martingales under an equivalent martingale measure) and that there are more Brownian motions than traded assets then it has been shown that every contingent claim of the form $g(X(1))$ can be perfectly hedged yielding a unique no-arbitrage price for the contingent claim (the wealth process associated with the hedging strategy), see [Bensoussan, Elliott, 1995]. In this setting the expectation of a contingent claim is constant for every equivalent martingale measure, see [El Karoui and Quenez, 1995, Proposition 1.7.1]. This is not the case of our setting. Denoting by X both security prices and non traded factors, then in general we do not have perfect hedging through a self-financed strategy and we have a continuum of equivalent martingale measures yielding a continuum of no arbitrage prices defined as the expectations of the contingent claim payoff under an equivalent martingale measure.

In case of a continuum of no arbitrage prices, an additional criterium is employed to select the price of a contingent claim (e.g., risk minimizing hedging, mean variance hedging, etc.). As a first step, before discussing some of these criteria, we want to characterize a *generalized hedging strategy* for an European contingent claim associated with a given martingale measure.

To build a hedging strategy for an European contingent claim we relax the self-financing requirement by considering a portfolio strategy $\pi(t)$ in $\Re^k$ and a *cost process* $C(t)$ in $\Re$, which is supposed to be progressively measurable with respect to $\mathcal{F}_t$. Given the initial wealth $x > 0$ then the wealth at time t $(U(t))$ associated with the portfolio process $\pi(t)$ and the cost process $C(t)$ is

$$U(t) = V(t) + C(t) = \frac{x}{\beta(t)} + C(t) + \frac{1}{\beta(t)} \int_0^t \pi(s)^\top \beta(s) \overline{\sigma}(s, X(s)) d\hat{W}(s). \tag{7}$$

where $C(0) = 0$. $V(t)$ is the wealth associated with the amount of money x at time $t = 0$. Considering a self-financing portfolio we have that $C(t) = 0$, $\forall\ t \geq 0$: given the initial amount of money x, then the wealth process $U(t)$ is obtained by running in continuous time the portfolio strategy $\pi(t)$ without inserting or withdrawing money at any time t. $C(t)$ describes the amount of money which is required to have the wealth process $U(t)$, in particular $dC(t)$ denotes the amount of money that the investor should add or withdraw in order to implement the portfolio $\pi(t)$ and to obtain the wealth $U(t)$. Note that the discounted wealth satisfies the following equation

$$\beta(t)U(t) = x + \beta(t)C(t) + \int_0^t \pi(s)^\top \beta(s) \overline{\sigma}(s, X(s)) d\hat{W}(s). \tag{8}$$

Note that if $r(t, X(t)) = 0$, $\forall 0 \leq t \leq 1$, then $\beta(t) = 1\ \forall 0 \leq t \leq 1$, $dX^i = X^i \sum_{j=1}^n \overline{\sigma}_j^i(t, X(t)) d\hat{W}_j$ $i = 1, \ldots, k$, and therefore

$$U(t) = x + C(t) + \int_0^t \pi(s)^\top \overline{\sigma}(s, X(s)) d\hat{W}(s) = x + C(t) + \int_0^t \pi(s)^\top \overline{X}^{-1}(t) d\overline{X}(t). \tag{9}$$

Let a martingale measure $\hat{P}$ and the associated relative risk process $\hat{\lambda}(t, X(t))$ be fixed. The couple $(\pi(t), C(t))$ is a generalized hedging strategy for the contingent claim H under the equivalent martingale measure $\hat{P}$ if $U(1) = H$ and

$$\beta(t)U(t) = \hat{E}[\beta(1)H|\mathcal{F}_t], \quad 0 \leq t \leq 1. \tag{10}$$

$\hat{E}[\cdot]$ denotes the expectation operator under the equivalent martingale measure $\hat{P}$. $U(t)$ defines the price of the contingent claim according to $\hat{P}$.

In what follows we briefly recall the stochastic analysis tools that are needed. Our analysis builds on the results obtained in a complete markets setting in [Ocone and Karatzas, 1991], where the hedging strategy is given by the conditional expectation of the Malliavin derivative of the final payoff through the Clark-Ocone formula. In the general case these formulae are obscure unless the coefficients of the SDE (1) are deterministic, in a Markovian setting more explicit formulae have been obtained in [Colwell et al., 1991]. In what follows we extend their analysis in three directions. First, we consider the incomplete markets setting. Second, we require less regularity on the coefficients of the SDE than in [Colwell et al., 1991].

We begin by developing the representation of the hedging strategy through Clark-Ocone formula in the incomplete market setting.

First of all let us introduce the Malliavin derivative on the Wiener space. Let $\mathcal{S}$ be the class of smooth functionals of the Brownian motion W, i.e., the random variables $F(\omega)$ of the form

$$F(\omega) = f(W(t_1, \omega), \ldots, W(t_d, \omega)),$$

where $f : \Re^{d\times n} \to \Re$ is bounded with bounded derivatives, i.e., $f \in C_b^\infty(\Re^{d\times n})$. Observe that $\mathcal{S}$ is a subspace of $L^2(\Omega, \mathcal{F}, P)$. The gradient DF of F is defined as follows

$$D^i F(\omega)(t) = \sum_{j=1}^{d} \frac{\partial}{\partial x^{ij}} f(W(t_1, \omega), \ldots, W(t_d, \omega)) 1_{[0,t_j]}(t), \quad i = 1, \ldots, n.$$

For each $p \geq 1$ we introduce the norm

$$\|F\|_{1,p} = (E\{|F|^p + (\sum_{i=1}^{d} \|D_i F\|^2)^{\frac{p}{2}}\})^{\frac{1}{p}}. \tag{11}$$

The closure of $\mathcal{S}$ under the norm $\|\cdot\|_{1,p}$ is the Banach space of the random variables for which the Malliavin derivative is defined, we denote it by $\mathbf{D}_1^p$.

The first tool used below is the Clark-Ocone representation formula of a square integrable random variable (see [?]). Let $F \in \mathbf{D}_1^2$ then we have

$$F = E[F] + \int_0^1 E[(D_t F)^* | \mathcal{F}_t] dW(t).$$

We observe that in our Markovian setting, thanks to the hypothesis of strong solution and pathwise uniqueness for the stochastic system (4), the augmentation of the filtration generated by $\hat{W}$ coincides with $\mathcal{F}$ (and also with the augmentation of the filtration generated by the process $(X^0, X^1, \ldots, X^n)$, see [Hofmann et al., 1992]). This fact enables us to refer directly to the $\hat{P}$-augmentation of the filtration generated by X in the Clark-Ocone formula without going back to the original probability measure P.

Actually we will use in our analysis a generalization of Clark-Ocone formula (see [Airault, Malliavin 1995]) which is well suited for the Markovian case. We remark that the following result also answers the [Colwell et al., 1991] claim about the possibility of deriving their result as a corollary of the general result of [Ocone, 1984].

Theorem 3.1 *Let $u : [0,1] \times \Re^n \to \Re$ be a differentiable function such that*

$$\frac{\partial u}{\partial t} + \frac{1}{2}\Delta u = 0$$

$$u(1,x) = f(x)$$

$$du(1,x) = df(x)$$

where f is a C^1-function on $\Re^n$, Δ is the Laplace operator. Then, for any $0 \le t \le s \le 1$

$$u(t, Y(t)) = E[u(s, Y(s))|\mathcal{F}_t]$$

and

$$u(s, Y(s)) = E[u(s, Y(s))|\mathcal{F}_t] + \int_t^s E[du(s, Y(s))|\mathcal{F}_\tau]^\top \cdot dY(\tau) \tag{12}$$

where $\cdot$ denotes Ito's contraction. In particular, from (12) with $s = 1$ and from the two boundary conditions, for any C^1-function f, we have, for $0 \le t \le 1$

$$f(Y(1)) = E[f(Y(1))|\mathcal{F}_t] + \int_t^1 E[df(Y(1))|\mathcal{F}_\tau]^\top \cdot dY(\tau). \tag{13}$$

In order to get an explicit formula for the hedging strategy using the formula (13) we have to explicitly write the first Malliavin derivative. Since our process is Markovian, then we can employ the chain rule and the Malliavin derivative can be written in terms of the derivative of the underlying process.

The second basic tool is a result concerning the existence, uniqueness and smoothness (in the sense of Malliavin derivative) of solutions of stochastic differential equations. With very little hypothesis (Lipschitz and growth conditions) on the coefficients of the equation of $X(t)$ it is shown that there exists a unique continuous solution to the equation and for any fixed t the random variable $X(t)$ belongs to the space $\mathbf{D}_1^p$ for any p, see [Malliavin, 1997, Nualart, 1995]. Moreover, the linear stochastic differential equation for the derivative process DX is explicitly written. We observe that this result generalizes the diffeomorphism theorem (see [Kunita, Malliavin]) where the coefficients are supposed more regular, a result implicitly used in [Colwell et al., 1991].

We state now the two results.

Theorem 3.2 *Consider the SDE in $\Re^m$*

$$dY(t) = \sum_{j=1}^{d} A^j(t, Y(t))dW_j(t) + A^0(t, Y(t))dt, \quad Y(0) = y_0 \tag{14}$$

Let the coefficients $A^0,\ A^j : [0,1] \times \Re^m \longrightarrow \Re^m$ *be measurable functions, globally Lipschitz with linear growth. Then there exists a unique continuous solution* Y *to (14) such that for any* t $Y^i(t)$ $(i = 1, \ldots, m)$ *belongs to* $\mathbf{D}_1^\infty$ *where* $\mathbf{D}_1^\infty = \cap_{p\geq 1}\mathbf{D}_1^p$. *Moreover the derivative* $D_r^j Y(t)$ *satisfies the following linear equation*

$$D_r^j Y(t) = A^j(r, Y(r)) + \int_r^t \sum_\alpha \overline{A}_{k\alpha}(s) D_r^j(Y^k(s)) dW^\alpha(s) + \int_r^t \overline{A}_{0k}(s) D_r^j(Y^k(s)) ds \quad (15)$$

for $r \leq t$ *and*

$$D_r^j Y(t) = 0$$

for $r > t$, *and where* $\overline{A}_{k\alpha}(s)$ *and* $\overline{A}_{0k}(s)$ *are uniformly bounded adapted* m*-dimensional processes.*

Remark 3.3 *If the coefficients of the equation (14) are continuously differentiable then we have that*

$$\overline{A}^i_{kl}(s) = (\partial_k A^i_l)(s, Y(s))$$
$$\overline{A}^i_{0k}(s) = (\partial_k A^i_0)(s, Y(s))$$

A similar result, with stronger hypothesis, (which is also used in [Colwell et. al., 1991]) can be obtained when the coefficients of equation (14) are of class $C^{1+\alpha}$, $\alpha > 0$. This is refered as the *Diffeomorphism Theorem.*

Theorem 3.4 *Consider* $(d+1)$ *vector fields* $A_0, \ldots, A_d$ *on* $\Re^m$ *and the following system of stochastic differential equations:*

$$dY(t) = \sum_{q=1}^{d} A_q(t, Y(t)) dW^q + A_0(t, Y(t)) dt \quad Y(0) = y_0 \quad (16)$$

where $W = (W_1, \ldots, W_d)$ *is a Brownian motion on* $\Re^d$. *Fix the starting point at* $Y(t_0)$ *and denote by* $U_{t\leftarrow t_0}$ *the stochastic flow of diffeomorphisms of* $\Re^d$ *associated to (16), that is*

$$U_{t\leftarrow t_0}(Y(t_0)) = Y(t) \quad t > t_0$$

where Y *satisfies (16) and the initial condition* $Y(t_0)$. *Then the map* $Y(t_0) \mapsto U_{t\leftarrow t_0}(Y(t_0))$ *is* C^1 *and its Jacobian* $J_{t\leftarrow t_0}$ *satisfies the following* $(d \times d)$ *matrix SDE*

$$d(J_{t\leftarrow t_0}) = [\ \sum_{q=1}^{n} M_q\ dW^q + M_0\ dt\ ] J_{t\leftarrow t_0} \quad (17)$$

where $(M_k)^\beta_\alpha = \partial_\alpha A^\beta_k$ *for* $k = 1, \ldots, n$ *and* $(M_0)^\beta_\alpha = \partial_\alpha A^\beta_0$.

Remark 3.5 *The Jacobian process* J_t *(where for brevity we write* $J_{t\leftarrow 0} = J_t$*) is connected to the Malliavin derivative of* Y *in the following way*

$$D_r^j Y^i(t) = J^i_l(t) J^{-1}(r)^l_k A^k_j(r, Y(r)). \quad (18)$$

To show this fact it is enough to verify that this process satisfies (15).

Let us go back to our problem. Consider the solution $X = (X^1, \dots, X^n)$ of (4). To simplify the notation we assume in the following that $r = 0$, see Remark 3.6 for the case $r \neq 0$. Given an equivalent martingale measure $\hat{P}$ we have that X evolves according to the following equations:

$$dX^i(t) = X^i(t) \sum_{j=1}^{n} \sigma_{ij}(t, X(t)) d\hat{W}_j(t) \quad i = 1, \dots, k$$

$$dX^i(t) = X^i(t)[\hat{b}^i(t, X(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t, X(t)) d\hat{W}_j(t)] \quad i = k+1, \dots, n.$$

Under $\hat{P}$ we have that $(X^1, \dots, X^k)$ is a martingale and $(X^{k+1}, \dots, X^n)$ is a semimartingale.

In order to write the equation as in Theorem 3.4, we consider the coefficients $\Sigma(t, X(t))$ which is the $n \times n$-matrix obtained as the product between the $n \times n$ matrix having the diagonal equal to the X^i's and the $n \times n$ matrix σ and $B(t, X(t))$ which is the vector whose components are $X^i \hat{b}^i(t, X(t))$. With this notation we can rewrite the above system of SDE as:

$$dX^i(t) = \sum_{j=1}^{n} \Sigma_{ij}(t, X(t)) d\hat{W}_j(t) \quad i = 1, \dots, k$$

$$dX^i(t) = [\hat{B}^i(t, X(t))dt + \sum_{j} \Sigma_{ij}(t, X(t)) d\hat{W}_j] \quad i = k+1, \dots, n.$$

Let $\mathcal{L}$ the infinitesimal generator associated to X

$$\mathcal{L} = \frac{1}{2} \sum_{ij} (\Sigma \Sigma^\top)_{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=k+1}^{n} \hat{B}^i(t, x) \frac{\partial}{\partial x_i}$$

Let us consider the function $g(X(1))$ and apply Theorem 3.1, then we get the following representation

$$\hat{E}[g(X(1))|\mathcal{F}_t] = \hat{E}[g(X(1))] + \int_0^t \hat{E}[dg(X(1))|\mathcal{F}_s] \cdot dX(s)$$

$$= \hat{E}[g(X(1)] + \int_0^t \hat{E}[\frac{\partial g}{\partial x}(X(1))J(1)|\mathcal{F}_s] J(s)^{-1} \Sigma(s) d\hat{W}(s).$$

The no arbitrage price of the contingent claim at time t is given by

$$U(t) = \hat{E}[g(X(1))|\mathcal{F}_t]$$

$$= \hat{E}[g(X(1))] + \sum_{h=k+1}^{n} \int_0^t \hat{E}[\sum_i \sum_j \frac{\partial \tilde{g}}{\partial x_i}(X(1)) J_{ij}(1)|\mathcal{F}_s] J^{-1}_{jh}(s) X^h(s) \sigma_{hk}(s) d\hat{W}^k(s) +$$

$$+ \sum_{h=1}^{k} \int_0^t \hat{E}[\sum_i \sum_j \frac{\partial g}{\partial x_i}(X(1)) J_{ij}(1)|\mathcal{F}_s] J^{-1}_{jh}(s) dX^h(s)$$

where J^{-1} indicates the inverse matrix of J.

When the coefficients are not smooth then we can still obtain a representation result using Theorem 3.2. Consider the $n \times n$-matrices valued process

$$Y^i_j(t) = \delta^i_j + \int_0^t [\overline{A}^i_{kl}(s)Y^k_j(s)dW^l(s) + \overline{A}^i_{k0}(s)Y^k_j(s)ds] \tag{19}$$

then the Malliavin derivative of the process $X(t)$ can be written

$$D^j_r X^i(t) = Y^i_l(t){Y^{-1}}^l_k(r)A^k_j(r, X(r)).$$

Therefore with the same notation as above the following representation holds:

$$\hat{E}[g(X(1))|\mathcal{F}_t] = \hat{E}[g(X(1))] + \int_0^t \hat{E}[D_s g(X(1))|\mathcal{F}_s]d\hat{W}(s)$$

$$= \hat{E}[g(X(1))] + \sum_{h=k+1}^{n} \int_0^t \hat{E}[\sum_i \sum_j \frac{\partial g}{\partial x_i}(X(1))Y_{ij}(1)|\mathcal{F}_s]Y^{-1}_{jh}(s)X^h(s)\sigma_{hk}(s)d\hat{W}^k(s)+$$

$$+\sum_{h=1}^{k} \int_0^t \hat{E}[\sum_i \sum_j \frac{\partial g}{\partial x_i}(X(1))Y_{ij}(1)|\mathcal{F}_s]Y^{-1}_{jh}(s)dX^h(s)$$

where Y^{-1} indicates the inverse matrix of Y.

In both cases we have three components. The first component represents x, the initial amount of money needed to develop the generalized hedging strategy. The second component comes from the traded assets, the third component comes from non traded factors. Considering the case of a smooth payoff function, then the generalized hedging strategy $(\pi(t), C(t))$ associated with the contingent claim $g(X)$ and the equivalent martingale measure $\hat{P}$ is as follows

$$x = \hat{E}[g(X(1))],$$

$$C(t) = \sum_{h=k+1}^{n} \int_0^t \hat{E}[\sum_i \sum_j \frac{\partial g}{\partial x_i}(X(1))J_{ij}(1)|\mathcal{F}_s]J^{-1}_{jh}(s)X^h(s)\sigma_{hk}(s)d\hat{W}^k(s),$$

$$\pi^h = \frac{1}{X^h(t)}E[\sum_i \sum_j \frac{\partial g}{\partial x_i}(X(1))J_{ij}(1)|\mathcal{F}_s]J^{-1}_{jh}(s) \quad h = 1, \ldots, k.$$

Both the cost process and the portfolio providing the generalized hedging strategy are experessed through the Malliavin derivative of the final payoff. Note that without the tools we have used in our anlysis it would not be possible to explictly determine the two components.

Remark 3.6 *When the risk free rate is not zero $r(t, X(t)) \neq 0$, the above analysis can be developed in terms of the deflated process $\beta(t)X(t)$ where $\beta(t) = exp\{-\int_0^t r(s, X(s))ds\}$.*

Remark 3.7 *In the case of complete markets then the results established in this section gives us the results established in [Colwell et al., 1991].*

4 Risk minimizing Hedging Strategy

We consider the couple $(\pi(t), C(t))$ replicating the contingent claim $g(X(1))$, i.e., $U(1) = g(X(1))$. We define the *remaining risk* at time t associated with the portfolio strategy π replicating the contingent claim as $R(\pi)_t = E[\{C(1)-C(t)\}^2|\mathcal{F}_t]$. In [Follmer and Schweizer, 1990] the hedging strategy minimizing the risk is determined assuming that the stochastic process for the asset price is already a martingale; in [Follmer and Schweizer, 1990] the risk minimizing hedging strategy is extended for a general asset price stochastic process minimizing the risk in a local sense. It has been shown that if a hedging strategy $(\pi(t), C(t))$ is locally risk-minimizing then $C(t)$ is ortoghonal to $W(t)$.

A martingale measure particularly important in an incomplete markets setting is the *minimal martingale measure* P which is defined by the following risk process:

$$\tilde{\lambda}(t) = \sigma(t, X(t))^{k\top}(\sigma(t, X(t))^k \sigma(t, X(t))^{k\top})^{-1}[b^k(t, X(t)) - r(t, X(t))1^k].$$

Under this martingale measure, performing the analysis developed above, we can verify the orthogonality of $C(t)$ and $W(t)$.

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