Report n. 142

Competitive equilibria with money and restricted participation

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Pisa, June 1999

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June 21, 1999

Abstract

We analyze an economy with inside financial assets and outside money. Households have different access on both types of assets and money. The paper does not try to explain why outside money exists, but, using a well known approach in terms of needs of money to pay taxes, it studies the basic properties of the model. We provide a proof of existence of equilibrium in this economy which is based on Degree Theorem. Since households' demand functions are not differentiable everywhere standard techniques of differential topology cannot be applied. Due to this, we show that there exists an open and dense set of economies where households' demand functions are C^1.

1 Introduction

In a standard general equilibrium framework with incomplete markets, consumers face the same opportunities to transfer wealth across spot markets. In real life, we can find many cases where the participation constraints on financial markets varies from a class of consumers to another. For example, we can think of collateral securities in American real estate market or of a credit line which is secured by financial assets.

In the recent literature we can find several models (see for example [Balasko, Y., D. Cass and P.Siconolfi (1990)], [Cass, D. P.Siconolfi, and A.Villanacci (1992)], [Polemarchakis, H. M. and P. Siconolfi (1997)], [Siconolfi, P. (1989)]) which present a wide range of diverse restriction on financial market participation. These kind of general equilibrium models are called "restricted participation model" and they can be seen as generalization of the incomplete market case. While [Cass, D. P.Siconolfi, and A.Villanacci (1992)] propose a model where the individual participation constraint is described by a differentiable strictly quasi-concave function of consumer's assets a_h, our model in enriched by the presence of the outside money whose exchange is restricted too. We assume a_h is a function of both consumer's assets and outside money demands ,i.e. $a_h(b_h, m_h^0) \geq 0$.

We recall that by the term *Outside money* we refer to money which is a direct debt of the public sector, e.g. circulating currency, or is based on such debt, e.g. commercial bank deposits matched by bank holdings of public sector debt. Examples are fiat money, gold and foreign exchange reserves. On the other hand, *Inside money* is a form of money which is based on private sector debt; the prime modern example being commercial bank deposits to the extent that they are matched by bank lending to private sector borrowers [Pearce, D. W. (1992)].

From now on, unless it otherwise specified, money means outside money.

We can find many contributions in order to understand why money exists in a General Equilibrium Model. (The reader can see for example [Lerner, A. (1947)], [Starr, R.M. (1974)] and [Starr, R.M. (1989)]). Even when money exists in a general equilibrium context that does not imply the existence of a positive price for money. [Hahn, F. (1965)]. For that reason, there are several additional assumptions in order to overcome the well-known hot potato problem [Cass D. and K Shell (1980)], [Dubey, P. and J. Geanakoplos (1992)], [Grandmont, J.M. and Y. Younes (1990)], , [Magill, M. and M. Quinzii (1988)], [Magill, M. and M. Quinzii (1996)], [Starr, R.M. (1974)], [Starr, R.M. (1989)].

We do not try to explain why money exists, but, using a well known approach in terms of needs of money to pay taxes, we first study the basic properties of the model.

We assume that households have to use money to pay taxes at the end of the second period. This idea was first introduced by Lerner [Lerner, A. (1947)] and then developed by other economists [Starr, R.M. (1974)], [Starr, R.M. (1989)], [Villanacci, A. (1991)], [Villanacci, A. (1993)]. As in [Villanacci, A. (1993)], in our case, taxes are linear function of households' wealths.

We present the set up of the model in section two while section three deals with Household demand function. Due to participations constraints and money constraint, demand function is not differentiable everywhere. Consequently we prove that differentiability is a generic property of demand function. That allows us to get the existence result which is proved in section five.

2 Set up of the model

We consider an exchange economy with two periods; today and tomorrow. The state of the world today is known, and it is called state 0. Tomorrow is called period 1 and S states of the world are possible. The set of possible states of the world is $\{0, 1, ..., S\}$ with generic element s. The time structure is the following: in state 0, households receive endowments of goods and money, they exchange goods and assets and consume the goods they acquired. Households are not allowed to buy and sell assets freely, but they must take into account their own participation constraints.

Tomorrow uncertainty is resolved, one of the S states occurs and households receive their endowments of goods and money. They exchange goods and fulfill the obligations underwritten in state 0. Finally households consume the goods they acquired and they use money to pay taxes.

We will use the following notations:

- e_h^{sc} and x_h^{sc} are respectively, the endowment and demand of good c in state s, of household h.
- $e_h^s \equiv (e_h^{sc})_{c=1}^C$, $e_h = (e_h^s)_{s=0}^S$, $e = (e_h)_{h=1}^H$.
- $x_h^s \equiv (x_h^{sc})_{c=1}^C$, $x_h = (x_h^s)_{s=0}^S$, $x = (x_h)_{h=1}^H$.
- e_h^{sm} is the endowment of money in state s, owned by household h.
- $e_h^m = (e_h^{sm})_{s=0}^S$, $e^m = (e_h^m)_{h=1}^H$.
- b_h^i is the demand of asset i, of household h. $b_h \equiv (b_h^i)_{i=1}^I$.
- q^{sm} is the price of money in state s.
- $q^m = (q^{sm})_{s=0}^S$ is money price vector.
- m_h^s is the demand of money in state s of household h.
- $m_h = (m_h^s)_{s=0}^S$, $m = (m_h)_{h=1}^H$.

Households' utility functions have the following properties.

Assumption 1 *i) u_h is a smooth function, i.e., a C^∞ function.*
ii) u_h is differentiably strictly increasing, i.e., $Du_h(x_h) \gg 0$.
iii) the Hessian matrix D^2u_h is negative definite
iv) For any $\underline{u} \in \mathbb{R}$, $Cl\left\{x \in \mathbb{R}_{++}^C : u_h(x) \geq \underline{u}\right\} \subseteq \mathbb{R}_{++}^C$.

We assume consumers cannot issue outside money.

Assumption 2 $m_h^s \geq 0$ *for all s and all h.*

Prices of goods, money and assets are expressed in units of account. We assume that prices of goods are strictly positive.

Assumption 3 $p^{sc} \in \mathbb{R}_{++}$ *for all s and all c, where* p^{sc} *is the price of good c in state s.*

We denote the matrix of assets yields by $Y = \begin{bmatrix} y^{11} & .. & y^{1I} \\ & & \\ y^{S1} & & y^{SI} \end{bmatrix}$; Y is a $S \times I$ matrix . Moreover $Y^M = \begin{bmatrix} Y & \mathbf{1} \end{bmatrix}$ is a $S \times (I+1)$ matrix.

It greatly simplifies our analysis to assume that

Assumption 4 *We assume* $S > I + 1$ *and* $RankY = I$. *Moreover* Y^M *is such that* $rankY^M = I + 1$

Remark 5 *The previous Assumption means there are no redundant assets in the economy. As D.Cass, P. Siconolfi and A. Villanacci [Cass, D. P.Siconolfi, and A.Villanacci (1992)] say, "..In this context, Assumption 4 is not at all innocuous. When their portfolio holdings are constrained, households may very well benefit from the opportunity afforded by the availability of additional bonds whose yields are not linearly independent".*

Households deal with two different kinds of constraints in the assets market. On one hand they must take into account the incompleteness of the asset market (i.e. $rankY = I < S$) and on the other hand, they must consider their own participation constraint. The latter is expressed by the following function:

$$\begin{aligned} a_h^{\#J_h} &: \mathbb{R}^I \times \mathbb{R} \to \mathbb{R}^{\#J_h} \\ a_h^j &: \left(b_h, m_h^0\right) \mapsto a_h^j\left(b_h, m_h^0\right) \quad j = 1, ..., \#J_h \end{aligned} \tag{1}$$

where $a_h^{\#J_h} = \left[a_h^j\left(b_h, m_h^0\right)\right]_{j=1}^{\#J_h}$, J_h is a set of indexes such that $J_h \subseteq I$. a_h^j verifies the following Assumption.

Assumption 6 a_h^j *is a* C^2, *differentiably strictly quasi-concave function, i.e. for every* $(b_h, m_h^0) \in \mathbb{R}^{I+1}$ *and every* $\Delta \in \mathbb{R}^{I+1}$

$$Da_h^j\left(b_h, m_h^0\right)\Delta = 0 \Rightarrow \Delta^T D^2 a_h^j\left(b_h, m_h^0\right)\Delta < 0.$$

If the set J_h is clearly specified and there is no possibility of misunderstanding about it, we can drop J_h from a_h. Moreover this function a_h verifies the following further conditions:

Assumption 7 *i)* $a_h(0,0) > 0$.

ii) For every $(b_h, m_h^0) \in \mathbb{R}^{I+1}$ *such that* $a_h^{J_h'}(b_h, m_h^0) = 0$,
rank $\left(Da_h^{J_h'}(b_h, m_h^0)\right) = \#J_h'$, *for every index subset* $J_h' \subseteq J_h$.

iii) For every asset i, there exists at least one consumer h' such that for every $(b_{h'}, m_{h'}^0) \in \mathbb{R}^{I+1}$ *the following condition holds :*
$D_{b_{h'}^i} a_{h'}(b_{h'}, m_{h'}^0) = 0$

iv) there exists at least one consumer h' such that :
$D_{m_{h'}^0} a_{h'}(b_{h'}, m_{h'}^0) = 0$.

Remark 8 *Assumption 7 has important economic meanings.*

i) people are not obliged to operate in the assets and/or money markets. Moreover, there is a small neighborhood of $(0,0)$ *where every consumer can freely operate.*

iii) for every asset there exists at least one household who is unrestricted on that asset market.

iv) there exists at least one consumer who can arbitrary vary his money demand.

Remark 9 *Assumption 7i) are used in order to prove the existence of the competitive equilibrium. They allow to show that the test economy belongs to a full measure and open set in which consumers' demand functions are differentiable.*

Households are not able to create wealth by acting on the assets and money markets. Hence we obtain the following NO ARBITRAGE Condition that allows us to define the set of no arbitrage assets and money prices

Definition 10 *Let us define the no-arbitrage asset and money price set as:*
$\widehat{Q}_h = \{\hat{q} = (q, q^m) \in \mathbb{R}^I \times \mathbb{R}^{S+1} : \nexists (b_h, m_h^0)$, *such that* $a_h(b_h, m_h^0) \geq 0$ *and*
$\begin{bmatrix} -q & -q^{m0} \\ Y & q^{m1} \end{bmatrix} \begin{bmatrix} b_h \\ m_h^0 \end{bmatrix} > 0 \}$.

where q^{m1} *is the vector* $q^{m1} = (q^{ms})_{s=1}^S = (q^{m1}, ..., q^{ms})$ *of dimension* $S \times 1$.

$\widehat{Q} = \bigcap_{h \in H} \widehat{Q}_h$ *is the set of no arbitrage.*

In period 1, Mr. h pays taxes using money; taxes are proportional to the value of his endowments.

- $\tau_h^{sc} \in [0,1] \subseteq \mathbb{R}$ is the percentage of taxes that Mr. h has to pay for good c, in state s.
- $\tau_h^s = (\tau_h^{sc})_{c=1}^C$ $\tau_h = (\tau_h^s)_{s=1}^S$,$\tau = (\tau_h)_{h=1}^H$.

An economy is described by a vector $\omega = (e, e^m, \tau)$ of endowments of goods and money and tax parameters.

Assumption 11 $\omega \in \widehat{\Omega} = \mathbb{R}_{++}^{GH} \times X^m \times T$ *where*

i) $X^m \equiv \left\{ e^m \in \mathbb{R}^{(S+1)H} : \sum_{h=1}^{H} e_h^{m0} > 0 \text{ and for } s \geq 1, \sum_{h=1}^{H} (e_h^{m0} + e_h^{ms}) > 0 \right\}$

ii) $T \equiv \left\{ \begin{array}{l} \tau \in [0,1]^{SCH} : a)\forall s \geq 1, \\ \exists h \text{ and } \exists c : \tau_h^{sc} > 0 \\ b)\exists h^* \text{ such that } \forall s \geq 1, \exists c \ : \tau_{h^*}^{sc} \neq 1 \end{array} \right\}$

iii) $\omega \in \Omega \equiv \{\omega \in \widehat{\Omega} : \exists p \in \mathbb{R}_{++}^{G}$ *such that* $\forall h,\ p^0 e_h^0 + q^{0m} e_h^0 > 0$ *and for* $s = 1, ..., S$ $p^s e_h^s + e_h^{ms} - \sum_{c=1}^{C} \tau_h^{sc} p^{sc} e_h^{sc} > 0\}$

Remark 12 *As Villanacci [Villanacci, A. (1993)] observes, condition i) implies that in each states of the world there exists a positive amount of money; moreover part a) of ii) means that taxes are a nontrivial function of wealth, while part b) says there exists at least one consumer, who, in every states, does not use all his wealth to pay taxes. Condition iii) says in every state of the world, households are able to pay their taxes using their initial endowments.*

Remark 13 *Assumption 11 is used in order to prove the existence of equilibria for the test economy (see Lemma 25)*

3 Household's Demand Function.

Each household maximizes his utility function subject to his budget constraints which depends on his endowments and taxes and on participation constraints in both assets and money market.

For $(\omega, \widehat{p}, \widehat{q}, \widehat{q}^m) \in \Omega \times \mathbb{R}^G_{++} \times \hat{Q}$,we have :

$$
\begin{array}{lll}
\textbf{(P1)} & \max\limits_{(x_h, b_h, m_h)} \; u(x_h) & s.t. \\
& \widehat{p}^0 x_h^0 + \widehat{q}^{0m} m_h^0 + \widehat{q} b_h & \leq \widehat{p}^0 e_h^0 + \widehat{q}^{0m} e_h^{m0} \\
& m_h^0 & \geq 0 \\
(s = 1, ..., S) & \widehat{p}^s x_h^s + \widehat{q}^{sm} m_h^s & \leq \sum\limits_{i=1}^{I} \widehat{q}^{sm} y^{si} b_h^i \widehat{q}^{sm} + \widehat{q}^{sm} \left(e_h^{ms} + m_h^0\right) \\
(s = 1, ..., S) & \sum\limits_{c=1}^{C} \tau_h^{sc} \widehat{p}^{sc} e_h^{sc} & \leq \widehat{q}^{sm} m_h^s \\
(s = 1, ..., S) & m_h^s & \geq 0 \\
& a_h \left(b_h, m_h^0\right) & \geq 0
\end{array}
\tag{2}
$$

Remark 14 *For a given economy $(\omega, \widehat{p}, \widehat{q}, \widehat{q}^m)$ the set of solution of $(P1\hat{\ })$ does not change if we divide the budget constraint of each state s by a positive number, for example if we normalize prices state by state. In order to eliminate technical complication, we normalize prices using the price of good C in state 0, while in the other states we normalize prices using the price of money. From now on we will always refer to the normalized prices. We have:*

$$
p^0 \equiv \frac{\widehat{p}^0}{p^{01}}, \quad q^{0m} \equiv \frac{\widehat{q}^{0m}}{p^{01}}, \quad q \equiv \frac{\widehat{q}}{p^{01}} \; \textit{and} \; p^s = \frac{\widehat{p}s}{q^{sm}} \; \textit{for } s > 0
$$

Let us denote

$Q = \left\{ \exists \left(\hat{q}, \hat{q}^m\right) \in \hat{Q} \text{ such that } q \equiv \frac{\widehat{q}}{p^{01}}, \; q^{0m} \equiv \frac{\widehat{q}^{0m}}{p^{01}}, \; q^{ms} = 1, \text{ for } s = 1..S \right\}$

Remark 15 *Since in every state of period 1 households use money only to pay taxes, no one wants to hold an amount of money greater than the one required to meet his tax obligations. This implies that there is no loss of generality if we substitutes $\widehat{q}^{sm} m_h^s$ with $\sum_{c=1}^{C} \tau_h^{sc} \widehat{p}^{sc} e_h^{sc}$ in the s following constraints*

$$
(s > 0) - \widehat{p}^s \left(x_h^s - e_h^s\right) + \widehat{q}^{sm} \left(m_h^0 - e_h^{ms} - m_h^s\right) + \sum_{i=1}^{I} \widehat{q}^{sm} y^{si} b_h^i \geq 0.
$$

Taking into account the prices normalizations and the previous Remark, we can rewrite household's maximization problem in the following way:

$$
\begin{array}{ll}
\max u_h(x_h) & s.t. \\
-\Phi \left(x_h - e_h\right) + q^m m_h^0 + \hat{U} e_h^m - \Psi \left(\tau_h, p\right) e_h + R b_h & \geq 0 \\
m_h^0 & \geq 0 \\
a_h \left(b_h, m_h^0\right) & \geq 0
\end{array}
\tag{3}
$$

where

- Φ is a $(S+1)\times G$ matrix,

$$\Phi \equiv \begin{bmatrix} p^0 & & \\ & \dots & \\ & & p^S \end{bmatrix}$$

with $p^0, p^1, ..., p^S \in \mathbb{R}^C_{++}$,

- q^m is an $(S+1)$ vector, $q^m \equiv \begin{bmatrix} -q^{om} \\ \mathbf{1} \end{bmatrix}$, and $\mathbf{1} = (1,...,1)^T$.

- $\hat{U}$ is an $(S+1)\times(S+1)$ diagonal matrix,

$$\hat{U} \equiv \begin{bmatrix} q^{0m} & \\ & -I_{S\times S} \end{bmatrix}$$

where $I_{S\times S}$ is the identy matrix whose dimension is S

- R is an $(S+1)\times I$ matrix, $R = \begin{bmatrix} -q \\ Y \end{bmatrix}$.

- $\Psi(\tau_h, p)$ is an $(S+1)\times G$ matrix,

$$\Psi(\tau_h, p) \equiv \begin{bmatrix} 0 & & & \\ & \tau_h^{11}p^{11}...\tau_h^{1C}p^{1C} & & \\ & & .. & \\ & & & \tau_h^{S1}p^{S1}...\tau_h^{SC}p^{SC} \end{bmatrix}$$

We define $\widehat{S} \equiv S+1$

Each consumer is price taker and demands goods, money and assets in the market, solving 3. Let us define the demand map of household. h. Given $\omega \in \Omega$, it associates with every vector prices, a vector of demand of goods, money and assets.

$$\left(x_h, b_h, m_h^0\right) : \mathbb{R}^G_{++} \times Q \to \mathbb{R}^G_{++} \times \mathbb{R}^I \times \mathbb{R}_+ \tag{4}$$

$$\left(x_h, b_h, m_h^0\right) : (p, q, q^m) \mapsto \arg\max(3). \tag{5}$$

Theorem 16 *i)For every consumer h the demand map is a function.*

Theorem 17 *ii)* (x_h, b_h, m_h^0) *is the solution to the problem 3 if and only if*

$(x_h, b_h, m_h^0, \lambda_h, \gamma_h, \mu_h)$ *is the solution of the following system of equations* :

$$\begin{array}{rll} D_{x_h} u_h(x_h) - \lambda_h \Phi = 0 & (F1) & (6) \\ -\Phi(x_h - e_h) + q^m m_h^0 + \hat{U} e_h^m - \Psi(\tau_h, p) e_h + R b_h = 0 & (F2) & \\ \lambda_h R + \mu_h D_{b_h} a_h(b_h, m_h^0) = 0 & (F3) & \\ \mu_h D_{m_h^0} a_h(b_h, m_h^0) + \lambda_h q^m + \gamma_h = 0 & (F4) & \\ (\forall j \in J_h) \quad \min\left[\mu_h^j, a_h^j(b_h, m_h^0)\right] = 0 & (F5) & \\ \min\left[\gamma_h, m_h^0\right] = 0 & (F6) & \end{array}$$

where $\left(\lambda_h = (\lambda_h^s)_{s=0}^S, \mu_h = \left(\mu_h^j\right)_{j=1}^{\sharp J_h}, \gamma_h\right) \in \mathbb{R}^{\hat{S}} \times \mathbb{R}^{\# J_h} \times \mathbb{R}$ *are the Lagrange multipliers.*

Proof. i) As a first step we prove the existence (Step a) and then the uniqueness of the solution for problem (3) (Step b).

(Step a) Existence.

We can easily show that for any $(p, q, q^m) \in \mathbb{R}_{++}^G \times Q$, the set
$B_h(p, q, q^m) \equiv \{(x_h, b_h, m_h^0) \in \left(\mathbb{R}_{++}^G \times \mathbb{R}^I \times \mathbb{R}\right) : a_h(b_h, m_h^0) \geq 0,$
$-\Phi(x_h - e_h) + q^m m_h^0 + \hat{U} e_h^{m0} - \Psi(\tau_h, p) e_h + R b_h = 0,$
$u_h(x_h) \geq u_h(e_h), m_h^0 \geq 0\}$ is compact.

Then, the maximization problem

$$\begin{array}{l} \max u_h(x_h) \; s.t. \\ (x_h, b_h, m_h^0) \in B_h \end{array} \qquad (7)$$

admits solution since B_h is compact and non empty and U_h is a continuous function.

Since the consumer problem is equivalent to (7) it admits a solution.

(Step b) Uniqueness.

Suppose to the contrary, that $\left(\hat{x}_h, \hat{b}_h, \hat{m}_h^0\right)$, $\left(\ddot{x}_h, \ddot{b}_h, \ddot{m}_h^0\right)$ are both solution to (7)

From Assumption 6 and the strict concavity of u_h, it follows $\hat{x}_h = \ddot{x}_h$.

Taking into account the Assumption 4 we can easily prove that $\left(\hat{b}_h, \hat{m}_h^0\right) = \left(\ddot{b}_h, \ddot{m}_h^0\right)$ and then we have the wanted result.

ii) From Assumption 7.ii) on $a_h(b_h, m_h^0)$ the indepedence constraint qualification is verified ([Bazaraa, M.S., H. D. Sheraly and C.M.Shetty 1993]). Hence the thesis follows directly from K.T. conditions ∎

Now, we want to find out if (x_h, b_h, m_h^0) is a C^1 function. In order to do this we consider the function

$$F_h : \Xi_h \times \mathbb{R}^G_{++} \times Q \times \Omega \to \mathbb{R}^G_{++} \times \mathbb{R}^{\widehat{S}}_{++} \times \mathbb{R}^I \times \mathbb{R} \times \mathbb{R}^{\sharp J} \times \mathbb{R}$$

where $\Xi_h = \mathbb{R}^G_{++} \times \mathbb{R}^{\widehat{S}}_{++} \times \mathbb{R}^I \times \mathbb{R} \times \mathbb{R}^{\sharp J} \times \mathbb{R}$,

$$F_h : (\xi_h, p, q, q^m, \omega) \mapsto \begin{pmatrix} D_{x_h} u_h(x_h) - \lambda_h \Phi \\ \Phi(e_h - x_h) + q^m m_h^0 + \hat{U} e_h^m - \Psi(\tau_h, p) e_h + R b_h \\ \lambda_h R + \mu_h D_{b_h} a_h(b_h, m_h^0) \\ \mu_h D_{m_h^0} a_h(b_h, m_h^0) + \lambda_h q^m + \gamma_h \\ \left(\min\left[\mu_h^j, a_h^j(b_h, m_h^0)\right]\right)_{j=1}^{J_h} \\ \min\left[\gamma_h, m_h^0\right] \end{pmatrix}$$

where $\xi_h = (x_h, \lambda_h, b_h, m_h^0, \mu_h, \gamma_h)$.

If F_h were differentiable in every points of its domain, then we would be able to get the differentiability of demand function as an easy application of Implicit Function Theorem to the function F_h. Unfortunately, it easy to verify that F_h is not differentiable in every points of its domain. When $\gamma_h = m_h^0 = 0$ or $\mu_h^j = a_h^j(b_h, m_h^0) = 0$ with $j \in J_h$, F_h is not differentiable. Therefore, we show that these cases are rare, i.e. the set of parameters such that the corresponding F_h is not differentiable is closed and it has measure zero. Then we claim that the demand function (x_h, b_h, m_h^0) is continuous and C^1 on an open and full measure set. In order to prove this result we need some Lemmas. The first one shows that if we eliminate the cases where either $\gamma_h = m_h^0 = 0$ or $\mu_h^j = a_h^j(b_h, m_h^0) = 0$, the Jacobian of F_h has full rank. Lemma 19 shows that the subset of $F_h^{-1}(0)$ where F_h is not differentiable has zero measure and it is closed. Then, there is an open and full measure set of $\mathbb{R}^G_{++} \times Q \times \Omega$ where the consumer's demand function is C^1 (see Lemma 20).

Let us consider the following partition of $J_h = [J_h^1, J_h^2, J_h^3]$ with

$$\begin{array}{l} J_h^1 = \{j \in J_h : \mu_h^j > 0, a_h^j(b_h, m_h^0) = 0\} \\ J_h^2 = \{j \in J_h : \mu_h^j = 0, a_h^j(b_h, m_h^0) = 0\} \\ J_h^3 = \{j \in J_h : \mu_h^j = 0, a_h^j(b_h, m_h^0) > 0\} \end{array}.$$

According to the partition we consider $\mu_h = [\mu_h^1, \mu_h^2, \mu_h^3]$ and $a_h(b_h, m_h^0) = [a_h^1(b_h, m_h^0), a_h^2(b_h, m_h^0), a_h^3(b_h, m_h^0)]$ where

$$\begin{array}{ll} \mu_h^1 \gg 0 & a_h^1(b_h, m_h^0) = 0 \\ \mu_h^2 = 0 & a_h^2(b_h, m_h^0) = 0 \\ \mu_h^3 = 0 & a_h^3(b_h, m_h^0) \gg 0 \end{array}.$$

Lemma 18 *If there is no $j \in J_h$ such that $\mu_h^j = a_h^j\left(b_h, m_h^0\right) = 0$ and if either $\gamma_h > 0$ or $m_h^0 > 0$ then the Jacobian matrix DF_h has full rank.*

Proof. According to the partition of J_h we have : $J_h^2 = \emptyset$. Moreover we have to distinguish the following cases:

Case A: $J_h^1 \neq \emptyset$ and $J_h^3 \neq \emptyset$.

Case B: $J_h^1 \neq \emptyset$ and $J_h^3 = \emptyset$.

Case C: $J_h^1 = \emptyset$ and $J_h^3 \neq \emptyset$.

Since the strategy of the proof is very similar, we limit our analysis to the case A.

CASE A.

Taking into account the partition, the Jacobian of F_h can be written in the following way:

	G	$\widehat{S}$	I	1	$\sharp J_h^1$	$\sharp J_h^3$	1
	x_h	λ_h	b_h	m_h^0	μ_h^1	μ_h^3	γ_h
G	D_h^2	$-\Phi^T$					
$\widehat{S}$	$-\Phi$		R	q^m			
I		R^T	β_{b_h}	β_{b_h,m_h^0}	$\left(D_{b_h} a_h^1\right)^T$	$\left(D_{b_h} a_h^3\right)^T$	
1		$(q^m)^T$	$\left(\beta_{b_h,m_h^0}\right)^T$	$\beta_{m_h^0}$	$\left(D_{m_h^0} a_h^1\right)^T$	$\left(D_{m_h^0} a_h^3\right)^T$	1
$\sharp J_h^1$			$D_{b_h} a_h^1$	$D_{m_h^0} a_h^1$			
$\sharp J_h^3$						$I_{\sharp J_3}$	
1				$\chi_{h[m_h^0=0]}$			$\chi_{h[\gamma_h=0]}$

where: D_h^2 is the Hessian matrix of the utility function u_h,

β_{b_h} is the following $I \times I$ symmetric matrix

	b_h^1	$\dots$	b_h^I
b_h^1	$\sum_{j\in J_h^1} \mu_h^{1j} D_{b_h^1}^2 a_h^{1j}$	$\dots$	$\sum_{j\in J_h^1} \mu_h^{1j} D_{b_h^1, b_h^I}^2 a_h^{1j}$
$\vdots$		$\ddots$	
b_h^I	$\sum_{j\in J_h^1} \mu_h^{1j} D_{b_h^I b_h^1}^2 a_h^{1j}$	$\dots$	$\sum_{j\in J_h^1} \mu_h^{1j} D_{b_h^I}^2 a_h^{1j}$

$\beta_{m_h^0} \in \mathbb{R}$, $\beta_{m_h^0} = \sum_{j\in J_h^1} \mu_h^{1j} D_{m_h^0}^2 a_h^{1j}$.

$$\beta_{b_h,m_h^0} \in \mathbb{R}^{I\times 1},\ \beta_{b_h,m_h^0} = \begin{bmatrix} \sum_{j\in J_h^1} \mu_h^{1j} D_{b_h^1 m_h^0}^2 a_h^{1j} \\ \dots \\ \sum_{j\in J_h^1} \mu_h^{1j} D_{b_h^I m_h^0}^2 a_h^{1j} \end{bmatrix}.$$

Moreover $\chi_{h[m_h^0=0]} = \begin{cases} 1 & \text{if } m_h^0 = 0 \\ 0 & \text{if } m_h^0 \neq 0 \end{cases}$ and $\chi_{h[\gamma_h=0]} = \begin{cases} 1 & \text{if } \gamma_h^0 = 0 \\ 0 & \text{if } \gamma_h^0 \neq 0 \end{cases}$

In this case we have $\chi_{h[m_h^0=0]} = 0 \Leftrightarrow \chi_{h[\gamma_h^0=0]} \neq 0$ and $\chi_{h[m_h^0=0]} \neq 0 \Leftrightarrow \chi_{h[\gamma_h^0=0]} = 0$.

CASE A1 $\gamma_h = 0, m_h^0 > 0$) We first consider the submatrix $\overline{JF}_h$, obtained by erasing the last row and the last column. We claim that $\overline{JF}_h$ has full rank, then the desired conclusion follows.

Suppose to the contrary $\overline{JF}_h$ does not have full rank. Then there exists a vector $\Delta = (\Delta x_h, \Delta\lambda_h, \Delta b_h, \Delta m_h^0, \Delta\mu_h^1, \Delta\mu_h^3)$ such that $\Delta \neq 0$ and $\left[\overline{JF}_h\right]$ $\Delta = 0$, i.e.,

$$\begin{array}{llr}
D_h^2\Delta x_h - \Phi^T\Delta\lambda_h & = 0 & (1)\\
-\Phi\Delta x_h + R\Delta b_h + q^m\Delta m_h^0 & = 0 & (2)\\
R^T\Delta\lambda_h + \beta_{b_h}\Delta b_h + \beta_{b_h,m_h^0}\Delta m_h^0 + (D_{b_h}a_h^1)^T\Delta\mu_h^1 + (D_{b_h}a_h^3)^T\Delta\mu_h^3 & = 0 & (3)\\
q^{mT}\Delta\lambda_h + \left(\beta_{b_h,m_h^0}\right)^T\Delta b_h + \beta_{m_h^0}\Delta m_h^0 + \left(D_{m_h^0}a_h^1\right)^T\Delta\mu_h^1 + \left(D_{m_h^0}a_h^3\right)^T\Delta\mu_h^3 & = 0 & (4)\\
D_{b_h}a_h^1\Delta b_h + D_{m_h^0}a_h\Delta m_h^0 & = 0 & (5)\\
\Delta\mu_h^3 I_{\sharp J_2} & = 0 & (6)
\end{array} \qquad (8)$$

From (6) of system (8), we get $\Delta\mu_h^3 = 0$. We claim that $\Delta x_h \neq 0$. If $\Delta x_h = 0$, then from (1) of system (8) we would have $\Delta\lambda_h = 0$ and from (2) $R\Delta b_h + q^m\Delta m_h^0 = 0$. By the full rankness of $[R, q^m]$ we would get $(\Delta b_h, \Delta m_h^0) = 0$. Hence $\Delta\mu_h^1 = 0$ and that cannot be true, because $\Delta \neq 0$.

Now we show that $\Delta x_h^T D_h^2 \Delta x_h = 0$. Since D_h^2 is negative definite we have a contradiction and the result follows immediately. Premultiplying equation (1) of system (8) by Δx_h^T ; we obtain

$$\Delta x_h^T D_h^2\Delta x_h = \Delta x_h^T\Phi^T\Delta\lambda_h = \left(\Delta\lambda_h^T\Phi\Delta x_h\right)^T.$$

Premultiplying (2) by $\Delta\lambda_h^T$, we have

$$\left(\Delta\lambda_h^T\Phi\Delta x_h\right)^T = \left(\Delta\lambda_h^T R\Delta b_h + \Delta\lambda_h^T q^m\Delta m_h^0\right)^T = \Delta b_h^T R^T\Delta\lambda_h + \Delta m_h^{0T} q^{mT}\Delta\lambda_h.$$

Using equations $(F3)$ and $(F4)$ of system (6) we get

$$\lambda_h R\Delta b_h = -\mu_h^1 D_{b_h} a_h^1\Delta b_h$$

and

$$\lambda_h q^m\Delta m_h^0 = -\mu_h^1 D_{m_h^0} a_h^1\Delta m_h^0.$$

From the last two conditions the following equation holds:

$$\lambda_h\left(\Delta b_h^T R^T + \Delta m_h^{0T} q^{mT}\right)\Delta\lambda_h = -\left(\mu_h^1 D_{b_h} a_h^1\Delta b_h + \mu_h^1 D_{m_h^0} a_h^1\Delta m_h^0\right)\Delta\lambda_h.$$

From (5) of system (8), we get
$-\left(\mu_h^1 D_{b_h} a_h^1 \Delta b_h + \mu_h^1 D_{m_h^0} a_h^1 \Delta m_h^0\right) = 0.$
Hence $\lambda_h \left(\Delta x_h^T \Phi^T \Delta\lambda_h\right) = \lambda_h \left(\Delta b_h^T R^T + \Delta m_h^{0T} q^{mT}\right) \Delta\lambda_h = 0.$
Since $\lambda_h \gg 0$, $\Delta x_h^T \Phi \Delta\lambda_h = 0$, and so $\Delta x_h^T D^2 \Delta x_h = 0$.
CASE A2 $\gamma_h > 0, m_h^0 = 0)$
The proof is similar to the case A1. ■
Let us define the set
$B_h^* = \{(\omega, p, q, q^m) : (\omega, \xi_h, p, q, q^m) \in F_h^{-1}(0),\ \exists j \in \{1, ..., (\sharp J_h + 1)\}$ such that $\zeta_h^j = 0, l_h^j(b_h, m_h^0) = 0\}$, where

$$\begin{cases} l_h^j(b_h, m_h^0) = a_h^j(b_h, m_h^0) & \text{if } j = 1, ..., \sharp J_h \\ l_h^j(b_h, m_h^0) = m_h^0 & \text{if } j = \sharp J_h + 1 \end{cases}$$

and

$$\begin{cases} \zeta_h^j = \mu_h^j & \text{if } j = 1, ..., \sharp J_h \\ \zeta_h^j = \gamma_h & \text{if } j = \sharp J_h + 1 \end{cases}$$

The following Lemma shows B_h^* is a zero measure set of $\Omega \times \mathbb{R}_{++}^G \times Q$.

Lemma 19 *i) B_h^* is a zero measure set of $\Omega \times \mathbb{R}_{++}^G \times Q$.*
ii) B_h^* is a closed subset of $\Omega \times \mathbb{R}_{++}^G \times Q$.

Proof. i) Since the proof is long and not very easy to read we first present the sketch of the proof. We construct a particular set $B_h^{\widehat{J}_{h2}}$ (we will define it later at step a)), whose elements depend on the set of indexes $\widehat{J}_{h2} \subseteq \tilde{J}_h$ where $\tilde{J}_h = \{1, 2, ..., (\sharp J_h + 1)\}$. In step a) we prove that the set $B_h^{\widehat{J}_{h2}}$, corresponding to the set $\widehat{J}_{h2}$, has zero measure. Then by varying all the possible subsets of $\tilde{J}_h$ we can consider the set $\bigcup_{\widehat{J}_{h2} \in \mathfrak{P}_h} B_h^{\widehat{J}_{h2}}$ where $\mathfrak{P}_h$ is the set of all possible subsets of $\tilde{J}_h$. Of course $\bigcup_{\widehat{J}_{h2} \in \mathfrak{P}_h} B_h^{\widehat{J}_{h2}}$ has zero measure since is the finite union of zero measure sets. Hence it has measure zero and we get the wanted result since $B_h^* \subseteq \bigcup_{\widehat{J}_{h2} \in \mathfrak{P}_h} B_h^{\widehat{J}_{h2}}$.

Step a) $B_h^{\widehat{J}_{h2}}$ has measure zero.
Let us consider the correspondences $J_{h1}, J_{h2}, J_{h3} : \Omega \times \mathbb{R}^G \times \mathbb{R}^I \times \mathbb{R} \rightarrow \tilde{J}_h$, where $\tilde{J}_h = \{1, 2, ..., (\sharp J_h + 1)\}$ and

$$J_{h1} : (\omega, p, q, q^m) \mapsto \{j \in \tilde{J}_h : \zeta_h^j > 0, l_h^j\left(b_h, m_h^0\right) = 0, (\xi_h, p, q, q^m, \omega) \in F_h^{-1}(0)\}$$
$$J_{h2} : (\omega, p, q, q^m) \mapsto \{j \in \tilde{J}_h : \zeta_h^j = 0, l_h^j\left(b_h, m_h^0\right) = 0, (\xi_h, p, q, q^m, \omega) \in F_h^{-1}(0)\}$$
$$J_{h3} : (\omega, p, q, q^m) \mapsto \{j \in \tilde{J}_h : \zeta_h^j = 0, l_h^j\left(b_h, m_h^0\right) > 0, (\xi_h, p, q, q^m, \omega) \in F_h^{-1}(0)\}$$

Observe that from Theorem 16, given (ω, p, q, q^m) there always exists a unique $(\xi_h, p, q, q^m, \omega,)$ belonging to $F_h^{-1}(0)$. Then for every (ω, p, q, q^m) ,the images of the functions J_{h1}, J_{h2} and J_{h3} form a partition over the set $\tilde{J}_h = \{1, 2, ..., (\sharp J_h + 1)\}$.

Moreover we consider an arbitrary subset $\widehat{J}_{h2}$ of $\tilde{J}_h$ and we define the following set:

$$B_h^{\hat{J}_{h2}} = \{(\xi_h, p, q, q^m, \omega) \in F^{-1}(0) : \text{if } j \in \widehat{J}_{h2} \subseteq \tilde{J}_h, \text{ then } \zeta_h^j = 0, l_h^j(b_h, m_h^0) = 0\}$$

Let us consider the following auxiliary maximization problem:

$$\begin{array}{llr}
 & \max\limits_{(x_h, b_h, m_h^0)} u_h(x_h) & s.t. \\
 & \Phi(e_h - x_h) + q^m m_h^0 + \hat{U} e_h^{m0} - \Psi(\tau_h, p) e_h + R b_h = 0 & (\lambda_h) \\
(j \in \widehat{J}_h) & l_h^j(b_h, m_h^0) = 0 & (\zeta_h^j)_{j \in \widehat{J}_h}
\end{array} \quad (9)$$

where $\widehat{J}_h$ is a subset of $\tilde{J}_h$ such that $\widehat{J}_{h2} \subseteq \widehat{J}_h \subsetneq \tilde{J}_h$, and $\lambda_h, (\zeta_h^j)_{j \in \widehat{J}_h}$ are the Lagrange multipliers associated with the maximization problem. With respect to Lagrange conditions associated with problem (9) we have to distinguish two different cases: we consider the Lagrange conditions when $\sharp J + 1 \notin \widehat{J}_h$ and we left to the reader the case $\sharp J + 1 \in \widehat{J}_h$, since the strategy of the proof is similar.

CASE A: $\sharp J + 1 \notin \widehat{J}_h$

The Lagrange conditions associated with the maximization problem (9) are the following :

$$\begin{array}{llrr}
 & D_{x_h} u_h(x) - \lambda_h \Phi & = 0 & (1) \\
 & -\Phi(x_h - e_h) + q^m m_h^0 + \hat{U} e_h^m - \Psi(\tau_h, p) e_h + R b_h & = 0 & (2) \\
 & \lambda_h R + \sum\limits_{j_2 \in \hat{J}_2} \zeta_h^{j_2} D_{b_h} a_h^{j_2}(b_h, m_h^0) + \sum\limits_{j^* \in \widehat{J}_h \setminus \hat{J}_2} \zeta_h^{j^*} D_{b_h} a_h^{j^*}(b_h, m_h^0) & = 0 & (3) \\
 & \sum\limits_{j_2 \in \hat{J}_2} \zeta_h^{j_2} D_{m_h^0} a_h^{j_2}(b_h, m_h^0) + \sum\limits_{j^* \in \widehat{J}_h \setminus \hat{J}_2} \zeta_h^{j^*} D_{m_h^0} a_h^{j^*}(b_h, m_h^0) + \lambda_h q^m & = 0 & (4) \\
j_2 \in \widehat{J}_{h2} & l_h^{j_2}(b_h, m_h^0) & = 0 & (5) \\
j_h^* \in \widehat{J}_h \setminus \widehat{J}_{h2} & l_h^{j^*}(b_h, m_h^0) & = 0 & (6)
\end{array} \quad (10)$$

where $j^* \in J_h^* = \widehat{J}_h \setminus \widehat{J}_{h2}$. Let us define $\widehat{J}_h' = \{J_h \setminus \widehat{J}_h\} \cup \{\sharp J_h + 1\}$; hence we have a new partition on J_h; that is

$$\tilde{J}_h = \{\widehat{J}_h', J_h^*, \widehat{J}_{h2},\} \quad (11)$$

Let us define $\zeta_h^2 = \left(\zeta_h^{j_2}\right)_{j_2 \in \widehat{J}_2}$ and $\zeta_h^* = \left(\zeta_h^{j^*}\right)_{j^* \in J_h^*}$; in the same way we introduce the following notations $a_h^2 = \left(a_h^{j_2}\right)_{j_2 \in \widehat{J}_2}$ and $a_h^* = \left(a_h^{j^*}\right)_{j^* \in J_h^*}$. Now let us consider again the real consumer's maximization problem (3) with the associate First Order Conditions (6). According to the partition on $\tilde{J}_h$ (11), we write $\mu_h^1 = \left(\mu_h^j\right)_{j \in J_h \setminus \widehat{J}_h}$, $\mu_h^2 = \left(\mu_h^{j_2}\right)_{j_2 \in \widehat{J}_2}$ and $\mu_h^* = \left(\mu_h^{j^*}\right)_{j^* \in J_h^*}$. Given (p, q, q^{om}, ω), if $(x_h, \lambda_h, b_h, m_h^0, \mu_h^1, \mu_h^2, \mu_h^*, \gamma_h)$ is the solution of the problem (3),(i.e. it solves system (6)), then $(x_h, \lambda_h, b_h, m_h^0, \zeta_h^2, \zeta_h^*)$ is the solution of (10) with $\widehat{J}_h' = J_{h3}$, $\widehat{J}_{h2} = J_{h2}$, $J_h^* = J_{h1}$, and $\mu_h^2 = \zeta_h^2$, $\mu_h^* = \zeta_h^*$. It is worthy to note that the opposite implication does not hold true, that is, if $(x_h, \lambda_h, b_h, m_h^0, \zeta_h^2, \zeta_h^*)$ is the solution of the auxiliary maximization problem, we cannot say that it is also the solution of the real consumer's problem.

Now consider the following system of equations

$$
\begin{array}{lll}
 & D_{x_h} u_h(x) - \lambda_h \Phi & = 0 \quad (1) \\
 & -\Phi(x_h - e_h) + q^m m_h^0 + \hat{U} e_h^m - \Psi(\tau_h, p)\, e_h + R b_h & = 0 \quad (2) \\
 & \lambda_h R + \sum_{j_2 \in \hat{J}_2} \zeta_h^{j_2} D_{b_h} a_h^{j_2}(b_h, m_h^0) + \sum_{j^* \in \hat{J}_h \setminus \hat{J}_2} \zeta_h^{j^*} D_{b_h} a_h^{j^*}(b_h, m_h^0) & = 0 \quad (3) \\
 & \sum_{j_2 \in \hat{J}_2} \zeta_h^{j_2} D_{m_h^0} a_h^{j_2}(b_h, m_h^0) + \sum_{j^* \in \hat{J}_h \setminus \hat{J}_2} \zeta_h^{j^*} D_{m_h^0} a_h^{j^*}(b_h, m_h^0) + \lambda_h q^m & = 0 \quad (4) \\
j_2 \in \widehat{J}_{h2} & l_h^{j_2}(b_h, m_h^0) & = 0 \quad (5) \\
j_h^* \in \widehat{J}_h \setminus \widehat{J}_{h2} & l_h^{j^*}(b_h, m_h^0) & = 0 \quad (6) \\
 & \zeta_h^{j'} & = 0 \quad (7)
\end{array}
\qquad (12)
$$

with j' is an arbitrary element of $\widehat{J}_{h2}$. By a Transversality argument we can prove that the set of vectors $(\omega, \xi_h, p, q, q^{om},)$ which solves system (12) has measure zero (The interested reader can see also [Villanacci, A. (1991)] and [Villanacci, A. (1993)]).Then we can conclude that $B_h^{\hat{J}_{h2}}$ has zero measure, because $B_h^{\hat{J}_{h2}}$ is contained in this set.

ii)

The set B_h^* coincides with the following set : $B_h = \{(\omega, p, q, q^m) \in \Omega \times \mathbb{R}_{++}^G \times Q : \exists j \in \{1, ..., (\sharp J_h + 1)\}$ such that $\zeta_h^j(b_h(\omega, p, q, q^m), m_h^0(\omega, p, q, q^m)) = 0, l_h^j(b_h(\omega, p, q, q^m), m_h^0(\omega, p, q, q^m)) = 0\}$,where

$$
\begin{cases}
l_h^j(b_h(\omega, p, q, q^m), m_h^0(\omega, p, q, q^m)) = a_h^j(b_h(\omega, p, q, q^m), m_h^0(\omega, p, q, q^m)) & \text{if } j = 1, ..., \sharp J_h \\
l_h^j(b_h(\omega, p, q, q^m), m_h^0(\omega, p, q, q^m)) = m_h^0(\omega, p, q, q^m) & \text{if } j = \sharp J_h + 1
\end{cases}
$$

and

$$
\begin{cases}
\zeta_h^j = \mu_h^j(\omega, p, q, q^m) & \text{if } j = 1, ..., \sharp J_h \\
\zeta_h^j = \gamma_h(\omega, p, q, q^m) & \text{if } j = \sharp J_h + 1
\end{cases}
$$

We recall that the First order Conditions of the Consumer's Maximization problems define the following function $(x_h(\omega,p,q,q^m),\ b_h(\omega,p,q,q^m),\ m_h^0(\omega,p,q,q^m),\ \lambda_h(\omega,p,q,q^m), \mu_h^j(\omega,p,q,q^m), \gamma_h(\omega,p,q,q^m))$ of $(\omega,p,q,q^m) \in \Omega \times \mathbb{R}_{++}^G \times Q$. We know that this function is continuous and so for every $j = 1,...,\sharp J_h$,the function $a_h^j(b_h(\omega,p,q,q^m), m_h^0(\omega,p,q,q^m))$ is also a continuous function. Then $\zeta_h^j(b_h(\omega,p,q,q^m), m_h^0(\omega,p,q,q^m)), l_h^j(b_h(\omega,p,q,q^m), m_h^0(\omega,p,q,q^m))$ are continuous and so we get the desired result. ■

We are now able to state that the function F_h is differentiable on the open and full measure set. $D_h = \mathbb{R}_{++}^G \times Q \times \Omega \backslash B_h^*$.

Theorem 20 *The demand function of Mr. h is*
i) continuous on $\mathbb{R}_{++}^G \times Q \times \Omega$,
ii) C^1 *on* D_h.

Proof. i) It follows from the Maximum Theorem(see for example Beavis and Dobbs [Beavis B. and I. Dobbs (1990)])

ii) From Lemma 18, given $(p,q,q^{om},\omega) \in D_h$, the Jacobian matrix of F_h has full row rank and by applying Implicit function Theorem on F_h , we get the desired result ■

4 Definition of equilibrium

Definition 21 (Equilibrium) *Given an economy* $\omega = (e, e^m, \tau) \in \Omega$, *the vector* (p,q,q^m) *is an equilibrium prices system if and only if there exists* $(x_h, b_h, m_h^0)_{h=1}^H$ *such that*

1) (x_h, b_h, m_h^0) *solves consumer's maximization problem (for every h);*

2) $\sum_{h=1}^{H}(x_h^{sc} - e_h^{sc}) = 0$ *for every s, c i.e. markets for goods clear;*

3) $\sum_{h=1}^{H} b_h^i = 0$ *for every i i.e. markets for assets clear;*

4)

$$\begin{cases} \sum_{h=1}^{H}(m_h^0 - e_h^{m0}) = 0 \\ \sum_{h=1}^{H}\left(-m_h^0 - e_h^{ms} + \sum_{c=1}^{C}\tau_h^{sc}p^{sc}e_h^{sc}\right) = 0 \quad (s = 1..S) \end{cases} \tag{13}$$

i.e. money markets clear.

Remark 22 *Summing up with respect to* $h = 1,...,H$ *the constraints* $-\Phi(x_h - e_h) + q^m m_h^0 + \hat{U}e_h^m - \Psi(\tau_h, p)e_h + Rb_h = 0$, *we get* $\widehat{S}$ *Walras laws.*

If (13) holds and $\sum_{h=1}^{H}(x_h^{sc}-e_h^{sc})=0$ *for every* $(s,c)\neq(s,1)$, *then even the market of good* 1 *is in equilibrium in every state* s.

From now on we use the following notations :

$$x^{\backslash}\equiv\left(x_h^{\backslash}\right)_{h=1}^{H}\equiv\left(\left(x_h^{s\backslash}\right)_{s=0}^{S}\right)_{h=1}^{H}\equiv\left(\left((x_h^{sc})_{c=2}^{C}\right)_{s=0}^{S}\right)_{h=1}^{H}$$

and

$$e^{\backslash}\equiv\left(e_h^{\backslash}\right)_{h=1}^{H}\equiv\left(\left(e_h^{s\backslash}\right)_{s=0}^{S}\right)_{h=1}^{H}\equiv\left(\left((e_h^{sc})_{c=2}^{C}\right)_{s=0}^{S}\right)_{h=1}^{H}$$

We define the function $F:\Xi\times\Omega\rightarrow\mathbb{R}^{\dim\Xi}$ with
$\Xi=\mathbb{R}_{++}^{GH}\times\mathbb{R}^{H\hat{S}}\times\mathbb{R}^{HI}\times\mathbb{R}^{\sum\#J_h}\times\mathbb{R}^{H}\times\mathbb{R}^{H}\times\mathbb{R}_{++}^{G-1}\times Q$

$$F:(\xi,\omega)\mapsto\left(\begin{array}{ll}(Foc_h) & \left(\begin{array}{r}D_{x_h}u_h(x_h)-\lambda_h\Phi\\ -\Phi(x_h-e_h)+q^m m_h^0+\hat{U}e_h^m-\Psi(\tau_h,p)\,e_h+Rb_h\\ \lambda_h R+\mu_h D_{b_h}a_h(b_h,m_h^0)\\ (\forall j\in J_h)\qquad \min\left[\mu_h^j,a_h^j(b_h,m_h^0)\right]\\ \lambda_h q^m+\mu_h D_m a_h(b_h,m_h^0)+\gamma_h\\ \min\left[\gamma_h,m_h^0\right]\end{array}\right)_{h=1}^{H}\\ (M1) & \sum_{h=1}^{H}\left(x_h^{\backslash}-e_h^{\backslash}\right)\\ (M2) & \sum_{h=1}^{H}b_h\\ (M3) & \sum_{h=1}^{H}(m_h^0-e_h^{m0})\\ (M4) & \sum_{h=1}^{H}\left(-m_h^0-e_h^{ms}+\sum_{c=1}^{C}\tau_h^{sc}p^{sc}e_h^{sc}\right)(s>0)\end{array}\right)\qquad(14)$$

with $\xi\equiv\left((x_h,\lambda_h,b_h,\mu_h,m_h^0,\gamma_h)_{h=1}^{H},p^{\backslash 01},q,q^{0m}\right)$.

Definition 23 *The set of equilibria associated with the economy* $\overline{\omega}=(\overline{e},\overline{e^m},\overline{\tau})\in\Omega$, *is given by* $EQ_{\overline{\omega}}\equiv F_{\overline{\omega}}^{-1}(0)$
where $F_{\overline{\omega}}$ *is the restriction of the function* F *to* $\overline{\omega}$, *i.e.* $F_{\overline{\omega}}:\xi\mapsto F(\xi,\overline{\omega})$

5 Existence of equilibria

We want to prove that $EQ_\omega \neq \emptyset$ for any given economy ω and so we want to verify that the Degree Theorem Assumptions hold. First, we define the so called "test economy".

We now consider an economy $\omega^* = (e^*, e^{*m}, \tau^*)$ with $e^* \in P.O.$ where $P.O.$ is the set of Pareto Optimal allocations. .

It is known (see for example [Balasko, Y. (1988)]) that when markets are complete, given a Pareto optimum $e^* \in P.O.$, there exists a unique prices vector that supports the equilibrium. We denote by p the prices vector associated with e^* while θ_h^* is the Lagrange multiplier associated with the budget constraints of the household's maximization problem when market are complete.

Definition 24 *A test economy is an economy* $\omega^* = (e^*, e^{*m}, \tau^*)$ *such that:*

$$e^* \in P.O$$

$$e_h^{*0m} = \frac{1}{n}\min_s p^{s1} e_h^{*s1} \text{ where } n \text{ is such that}$$

$$a_h\left(0, \frac{1}{n}\min_s p^{s1} e_h^{*s1}\right) > 0$$

$$e_h^{*sm} = \frac{1}{n}\min_s p^{s1} e_h^{*s1}$$

$$\tau^{*sc} = \begin{cases} 1 & if \quad (s,c) = (1,1)..(S,1) \\ 0 & \qquad otherwise \end{cases}$$

The following Lemma states that when market are incomplete, given every test economy, there exists a unique equilibrium allocation of goods, money and assets.

Lemma 25 *Given* $\omega^* = (e^*, e^{*m}, \tau^*)$ *:*
i) $\xi^* \equiv \left((x_h^*, \lambda_h^*, b_h^*, \mu_h^*, m_h^{*0}, \gamma_h^*)_{h=1}^H, p^{\backslash 01*}, q^*, q^{*0m} \right) \in F_{\omega^*}^{-1}(0)$ *with*

$$\begin{array}{lll} p^{\backslash 01*} = p^{\backslash 01} & q^* = \sum_{s=1}^S y^s & q^{*0m} = S \\ x_h^* = e_h^* & b_h^* = 0 & m_h^{*0} = e_h^{*0m} \\ \lambda_h^* = (\theta_h^*, \theta_h^* .., \theta_h^*) & \mu_h^* = 0 & \gamma_h^* = 0 \end{array}$$

ii) $F_{\omega^*}^{-1}(0) = \{\xi^*\}$

Proof. i) By computing the value of $F_{\omega*}$ in ξ^*, it is easy to verify that $F_{\omega*}(\xi^*) = 0$ and therefore i) holds.

ii) Suppose to the contrary there exists $\hat{\xi} \neq \xi^*$ such that $F_{\omega*}\left(\hat{\xi}\right) = 0$ with $\hat{\xi} = \left(\left(\hat{x}_h, \hat{\lambda}_h, \hat{b}_h, \hat{\mu}_h, \hat{m}_h^0, \hat{\gamma}_h\right)_{h=1}^H, \hat{p}^{\backslash 01}, \hat{q}, \hat{q}^{0m}\right)$.

From the strict concavity of the utility function , $x_h^* = \hat{x}_h$ for every h.

Since $p^{01} = 1$, $\hat{\lambda}_h^0 = D_{x_h} u_h^*(\hat{x}_h) = D_{x_h} u_h^*(x_h^*)$; hence $\hat{\lambda}_h^0 = \lambda_h^{*0} = \theta_h^*$.

From the First Order Conditions and the money market clearing condition we get:

$$\sum_h^H m_h^{*0} = \sum_h^H e_h^{*m0} =$$
$$= \sum_h^H \left(\sum_{c=1}^C \tau_h^{*sc} \widehat{p}^{sc} e_h^{*sc} - e_h^{*ms} \right) =$$
$$= \sum_h^H \left(\sum_{c=1}^C \tau_h^{*sc} \left(D_{x_{h'}^{sc}} u_{h'}^* (e_{h'}^*) \backslash \hat{\lambda}_{h'}^s \right) e_h^{*sc} - e_h^{*ms} \right)$$

and therefore

$$\hat{\lambda}_{h'}^s = \frac{\sum_h^H \left(\sum_{c=1}^C \tau_h^{*sc} \left(D_{x_{h'}^{sc}} u_{h'}^* (e_{h'}^*) \right) e_h^{*sc} \right)}{\sum_h^H \left(e_h^{*sc} + e_h^{*ms} \right)} = \lambda_{h'}^{*s} = \theta_{h'}^* \tag{15}$$

From equation (15) and again from the First Order Conditions it follows $\hat{p}^{\backslash 01} = p^{\backslash 01*}$.

Adding up the budget constraints of states $s = 1, ..., S$ we have that

$$\left(m_h^{*0} - \widehat{m}_h^0 \right) + Y \left(b_h^* - \widehat{b}_h \right) = 0.$$

From Rank Condition on matrix $\begin{bmatrix} 1 & Y \end{bmatrix}$ we can conclude that $m_h^{*0} = \widehat{m}_h^0$ and $b_h^* = \widehat{b}_h$.

Hence $m_h^0 > 0$ for every h and from Assumption 7 iv) there exists a h° such that $a_{h^\circ}(0, m_{h^\circ}^0) > 0$. From the First Order Conditions of Mr. h° we obtain $\mu_{h^\circ} = 0$, $\lambda_{h^\circ} R = 0$, and so $\hat{q} = \sum_{s=1}^S y^s = q^*$. Since $\hat{m}_{h^\circ}^0 > 0$, $\hat{\gamma}_{h^\circ} = 0$. Moreover $\lambda_{h^\circ} q^m = 0$, and so $\widehat{q}^{0m} = S = q^{*0m}$. ■

We now prove that $\omega^* = (e^*, e^{*m}, \tau^*)$ is a regular economies i.e., 0 is a regular value of F_{ω^*}. Let us denote the set of all regular economy with $\mathcal{R}^{in}$, where "*in*" stands for incomplete markets.

Lemma 26 *$\omega^* = (e^*, e^{*m}, \tau^*)$ is a regular economy, i.e., $\omega^* \in \mathcal{R}^{in}$.*

Proof. Let us consider $\xi^* = \left(\left(x_h^*, \lambda_h^*, b_h^*, \mu_h^*, m_h^{*0}, \gamma_h^*\right)_{h=1}^H, p^{\backslash 01*}, q^*, q^{*0m}\right)$. By previous Lemma, F is differentiable at ξ^* because there is no consumer h, such that $\gamma_h^* = m_h^{0*} = 0$ or $\mu_h^{j*} = a_h^{j*}(b_h, m_h^0) = 0$ (for some j). Moreover $\sharp F_{\omega^*}^{-1}(0) = 1$. We are left to show the Jacobian matrix of F $J^*F = [D_{\xi^*}F_{\omega^*}]$ has full row rank. In order to make the proof more readable, we drop the superscript $(*)$ from all variables

Let us assume that JF has not full row rank. Then there exists a vector $\Delta \neq 0$

$$\Delta = \left(\left(\Delta x_h, \Delta\lambda_h, \Delta b_h, \Delta m_h^0, \Delta\mu_h, \Delta\gamma_h\right)_{h=1}^H, \Delta p^{\backslash}, \Delta q, \Delta q^{om}, \Delta p^{\cdot 1}\right)$$

such that $JF\Delta = 0$ i.e

$$\begin{array}{llll}
 & D_h^2\Delta x_h - \Phi^T\Delta\lambda_h - \Lambda_h\Delta p^{\backslash} - \Lambda_h^1\Delta p^{\cdot 1} & = 0 & (h1)\\
 & -\Phi\Delta x_h + R\Delta b_h + q^m\Delta m_h^0 + \Theta_h^1\Delta p^{\cdot 1} & = 0 & (h2)\\
h = 1..H & R\Delta\lambda_h + D_b a_h(b_h, m_h^0)\Delta\mu_h + -\lambda_1^0 I_I\Delta q & = 0 & (h3)\\
 & q^{Tm}\Delta\lambda_h + \Delta\gamma_h + \lambda_h^0\Delta q^{om} & = 0 & (h4)\\
 & \Delta\mu_h & = 0 & (h5)\\
 & \Delta\gamma_h & = 0 & (h6)\\
 & \sum_h \Delta x_h^{\backslash} & = 0 & (M1)\\
 & \sum_h \Delta b_h & = 0 & (M2)\\
 & \sum_h \Delta m_h^0 & = 0 & (M3)\\
s = 1..S & \sum_h \left(-\Delta m_h^0 + e_h^{s1}\tau_h^{s1}\Delta p^{\cdot 1}\right) & = 0 & (M4)
\end{array} \tag{16}$$

Where $G^{\backslash} = G - \widehat{S}$,
$\Lambda_h^{\backslash}$ is a block diagonal matrix $G \times G^{\backslash}$,

		C $p^{0\backslash}$	... ...	C $p^{S\backslash}$
1	$D_{x_h^{01}}u_h(x_h) - \lambda_h^0 p^{01}$	0		
$C-1$	$D_{x_h^{0\backslash}}u_h(x_h) - \lambda_h^0 p^{0\backslash}$	$\lambda_h^0 I_{C-1}$		
...	...		$\ddots$	
1	$D_{x_h^{S1}}u_1(x_1) - \lambda_1 p^{S1}$			0
$C-1$	$D_{x_h^{S\backslash}}u_1(x_1) - \lambda_1 p^{S\backslash}$			$\lambda_h^S I_{C-1}$

Λ_h^1 is a $G \times S$ matrix such that

		1 p^{11}	... ...	1 p^{S1}
C	$D_{x_h^0} u_h(x_h) - \lambda_h^0 p^0$			
1	$D_{x_h^{11}} u_h(x_h) - \lambda_h^1 p^{11}$	λ_h^1		
$C-1$	$D_{x_h^{1\backslash}} u_h(x_h) - \lambda_h^1 p^{1\backslash}$	0		
	...		$\ddots$	
1	$D_{x_h^{S1}} u_h(x_h) - \lambda_h^S p^{S1}$			λ_h^S
$C-1$	$D_{x_h^{S\backslash}} u_h(x_h) - \lambda_h^S p^{S\backslash}$			0

Θ_h^1 is a $\widehat{S} \times S$ matrix such that

	p^{11}	...	p^{S1}
$\sum_{c \in C} - p^{0c}\left(x_h^{0c} - e_1^{0c}\right) + ...$			
$\sum_{c \in C} - p^{1c}\left(x_h^{1c} - e_1^{1c}\right) + ...$	$-x_h^{11} + e_h^{11} - \tau_h^{11} e_h^{11}$		
.....		$\ddots$	
$\sum_{c \in C} - p^{Sc}\left(x_h^{Sc} - e_1^{Sc}\right) + ...$			$-x_h^{S1} + e_h^{S1} - \tau_h^{S1} e_h^{S1}$

$0I$ is a $G^{\backslash} \times G$ matrix such that

		1 x^{01}	$C-1$ $x^{0\backslash}$	... ...	1 x^{S1}	$C-1$ $x^{S\backslash}$
$C-1$	$\sum_{h \in H}\left(x_h^{0\backslash} - e_h^{0\backslash}\right)$		I_{C-1}			
.....				$\ddots$		
$C-1$	$\sum_{h \in H}\left(x_h^{S\backslash} - e_h^{S\backslash}\right)$					I_{C-1}

and finally $\widetilde{\Theta}^1$ is a $S \times S$ diagonal matrix

	p^{11}	...	p^{S1}
$\sum_{h \in H} - m_h^0 - e_h^{m1}...$	$\sum_h \tau_h^{11} e_h^{11}$		
.....		$\ddots$	
$\sum_{h \in H} - m_h^0 - e_h^{mS}$			$\sum_h \tau_h^{S1} e_h^{S1}$

Since $\widetilde{\Theta}^1$ is a strictly positive diagonal matrix and $\sum_h \Delta m_h^0 = 0$, $\Delta p^{\cdot 1} = 0$.

Take into account the first order conditions and the rank condition on $[R, q^m]$, we have $\Delta x_h \neq 0$.

Performing some appropriate computations on equations $(h1)$ and $(h2)$ of system (16) we can claim $\sum_{h=1}^{H} \Delta x_h^T \frac{D^2}{\lambda_h} \Delta x_h = 0$. Since $\Delta x_h^T D^2 \Delta x_h < 0$ and $\theta_h > 0$ for every h , $\sum_{h=1}^{H} \Delta x_h^T \frac{D^2}{\theta_h} \Delta x_h < 0$ and that contradicts the previous claim. Hence the thesis follows. ■

The following Lemmas show that Assumptions of the Degree Theorem are satisfied in our model.

Lemma 27 *i)* $\Xi = \mathbb{R}_{++}^{GH} \times \mathbb{R}^{H(\widehat{S})} \times \mathbb{R}^{HI} \times \mathbb{R}^{\sum \# J_h} \times \mathbb{R}^{H} \times \mathbb{R}^{H} \times \mathbb{R}_{++}^{G-1} \times Q$ *is an open set.*

ii) $\mathbb{R}^n$ *with* $n = \dim \Xi$ *is an open set.*

iii) the set Ω *is path connected.*

Proof. i) We first prove Q is open and then the thesis follows directly from the properties of the n-dimensional real numbers spaces.

We prove that the complement set of Q_h, is closed and then the thesis follows from $Q = \bigcap_{h \in H} Q_h$.

Consider an arbitrary sequence $\left(q^\nu, (q^{0m})^\nu\right) \notin Q_h$ such that $\left\{\left(q^\nu, (q^{0m})^\nu\right)\right\} \to (q, q^{0m})$.

Since $\left\{\left(q^\nu, (q^{0m})^\nu\right)\right\} \notin Q$ then for every ν there exists a vector $\left((b_h)^\nu, (m_h^0)^\nu\right)$ such that

$$\begin{array}{l} \begin{bmatrix} -q^\upsilon & (-q^{0m})^\nu \\ Y & \mathbf{1} \end{bmatrix} \left((b_h)^\nu, (m_h^0)^\nu\right)^T > 0 \\ a_h((b_h)^\nu, (m_h^0)^\nu) \geq 0 \end{array} \tag{17}$$

Then we can have a sequence $\left\{\left((b_h)^\nu, (m_h^0)^\nu\right)\right\}$ such that (17) holds for every υ and it is easy to show (17) implies $Y^M \left((b_h)^\nu, (m_h^0)^\nu\right) > 0$.

If $\left\{\left((b_h)^\nu, (m_h^0)^\nu\right)\right\}$ is an unbounded sequence, i.e. $\left\|\left((b_h)^\nu, (m_h^0)^\nu\right)\right\| \to \infty$ we can take the sequence $\left\{\frac{\left((b_h)^\nu, (m_h^0)^\nu\right)}{\left\|\left((b_h)^\nu, (m_h^0)^\nu\right)\right\|}\right\}$. It admits a converging subsequence (w.l.o.g. the sequence itself), such that $a_h\left(\frac{\left((b_h)^\nu, (m_h^0)^\nu\right)}{\left\|\left((b_h)^\nu, (m_h^0)^\nu\right)\right\|}\right) \geq 0$ and $\left\{\frac{\left((b_h)^\nu, (m_h^0)^\nu\right)}{\left\|\left((b_h)^\nu, (m_h^0)^\nu\right)\right\|}\right\} \to \left(\bar{b}_h, \bar{m}_h^0\right)$. From continuity of $a_h(.,.)$ we have $a_h\left(\bar{b}_h, \bar{m}_h^0\right) \geq 0$ and therefore $\begin{bmatrix} -q & (-q^{0m}) \\ Y & \mathbf{1} \end{bmatrix} \left(\bar{b}_h, \bar{m}_h^0\right)^T \geq 0$.

Since $\left(\bar{b}_h, \bar{m}_h^0\right) \neq (0,0)$ and Y^M has full rank, we obtain $\begin{bmatrix} -q & (-q^{0m}) \\ Y & \mathbf{1} \end{bmatrix} \left(\bar{b}_h, \bar{m}_h^0\right)^T > 0$ and that implies $(q, q^{0m}) \notin Q_h$, i.e. the complement set of Q_h is closed.

CaseB. $\left\{\left((b_h)^\nu, (m_h^0)^\nu\right)\right\}$ is a bounded sequence. Observe that $\forall \nu \left((b_h)^\nu, (m_h^0)^\nu\right) \neq 0$

If $\left\{\left((b_h)^\nu, (m_h^0)^\nu\right)\right\}$ is a bounded sequence then it admits a converging subsequence in itself. Without any loss of generality we can consider the same sequence. Hence $\left\{\left((b_h)^\nu, (m_h^0)^\nu\right)\right\} \to \left(\bar{b}_h, \bar{m}_h^0\right)$. From the continuity of function a_h we have that $a_h\left(\bar{b}_h, \bar{m}_h^0\right) \geq 0$ and moreover $\begin{bmatrix} -q & (-q^{0m}) \\ Y & \mathbf{1} \end{bmatrix} \left(\bar{b}_h, \bar{m}_h^0\right)^T \geq 0$.

If $\begin{bmatrix} -q & (-q^{0m}) \\ Y & \mathbf{1} \end{bmatrix} \left(\bar{b}_h, \bar{m}_h^0\right)^T > 0$ we have the desired result, otherwise we can easily prove that there always exists a vector $\left(\tilde{b}_h, \tilde{m}_h^0\right)$ such that such that $\begin{bmatrix} -q & (-q^{0m}) \\ Y & \mathbf{1} \end{bmatrix} \begin{pmatrix} \tilde{b}_h \\ \tilde{m}_h^0 \end{pmatrix} > 0$ and $a_h\left(\tilde{b}_h, \tilde{m}_h^0\right) \geq 0$. In this way we obtain $(q, q^{0m}) \notin Q_h$.

ii) Obvious

iii) Let us consider the set $L = \{\omega = (e, e^m, \tau) \in \mathbb{R}_{++}^{GH} \times \mathbb{R}^{(\hat{S})H} \times \mathbb{R}^{SCH}$:

$$e_h^{ms} = \begin{cases} 1 & \text{if } h = 1,\ s \geq 0 \\ 0 & \text{if } h \neq 1,\ s \geq 0 \end{cases}$$

$$\tau_h^{sc} = \begin{cases} 1 & \text{if } h = 1,\ c = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \}$$

The set $L \subseteq \Omega$ is convex and any point of Ω can be connected trough a segment to a point in L. Hence Ω is path connected. ■

Lemma 28 *The function F_ω verifies the following properties:*

i) F_ω is continuous for every $\omega \in \Omega$.

ii) $F_{\omega*}$ is a C^1 function in a neighborhood of $F_{\omega*}^{-1}(0)$, 0 is a regular value of $F_{\omega*}$ and $\deg F_{\omega*} = 1$.

Proof. i)-ii) The result follows from the previous Lemmas ■

Lemma 29 *Let us consider $\alpha : [0,1] \to \Omega$, $\alpha(t) \mapsto (1-t)\omega^* + t\omega$ and the homotopy $F_{\text{hom}} : \Xi \times [0,1] \to \mathbb{R}^n$ $F_{\text{hom}} : (\xi, t) \mapsto F(\xi, \alpha(t))$. Let F_t be a function such that $F_t : \Xi \to \mathbb{R}^n$, $F_t : \xi \mapsto F(\xi, \alpha(t))$.*

The following conditions hold:

i) α is continuous.

ii) F_{hom} is continuous.

iii)$F_{\text{hom}}^{-1}(0)$ is compact.

iv) $F_\omega^{-1}(0)$ is compact for every $\omega \in \Omega$.

Proof. i) Obvious.

ii) Since F_{hom} is the composition of continuous functions, F_{hom} is continuous.

iii) We claim $F_{\text{hom}}^{-1}(0)$ is sequentially compact. In order to show that result, we take a sequence $\{(t^n, \xi^n)\}$ such that $(\alpha(t^n), \xi^n) \in F_{\text{hom}}^{-1}(0)$, where $\xi^n = \left(\left(x_h^n, \lambda_h^n, b_h^n, (m_h^0)^n, \mu_h^n, \gamma_h^n\right)_{h\in H}, \left(p^{\setminus 01}\right)^n, q^n, \left(q^{0m}\right)^n\right)$ and $\alpha(t^n) = (e(t^n), e^m(t^n), \tau(t^n)) \in \Omega$. Clearly $e(t^n) = (e_h(t^n))_{h=1}^H = \left(\left(\left(e_h^{sc}(t^n)\right)_{c=1}^C\right)_{s=0}^S\right)_{h=1}^H$, $e^m(t^n) = (e_h^m(t^n))_{h=1}^H = \left(\left(e_h^{ms}(t^n)\right)_{s=0}^S\right)_{h=1}^H$ and $\tau(t^n) = (\tau_h(t^n))_{h=1}^H = \left(\left(\left(\tau_h^{sc}(t^n)\right)_{c=1}^C\right)_{s=0}^S\right)_{h=1}^H$.

Since $\{t^n\} \subseteq [0,1]$ it admits a converging subsequence $\{t^n\} \to \tilde{t}$.

Hence $\omega^n = \alpha(t^n) \to \alpha\left(\tilde{t}\right) = \widetilde{\omega} \in \Omega$.

Note that $F_{\text{hom}} : (t^n, \xi^n) \mapsto F(\xi^n, \alpha(t^n))$

We get the wanted result by proving that every component of the sequence $\{\xi^n\} = \{\left(x_h^n, \lambda_h^n, b_h^n, (m_h^0)^n, \mu_h^n, \gamma_h^n\right)_{h\in H}, p^{\setminus 01n}, q^n, (q^{0m})^n\}$ such that $(\xi^n, \alpha(t^n)) \in F_{\text{hom}}^{-1}(0)$ admits a converging subsequence in Ξ.

Then $\xi^n \to \widetilde{\xi} = \left(\left(\widetilde{x}_h, \widetilde{\lambda}_h, \widetilde{b}_h, \widetilde{\mu}_h, \widetilde{\gamma}_h\right)_{h\in H}, \widetilde{p}^{\setminus 01}, \widetilde{q}, \widetilde{q}^{0m}\right)$ and from the continuity of F we get $\widetilde{\xi} \in F_{\text{hom}}^{-1}(0)$.

Since F_{hom} is continuous, $F_{\text{hom}}(t^n, \xi^n) \to F_{\text{hom}}\left(\widetilde{t}, \widetilde{\xi}\right)$ and so $F_{\text{hom}}\left(\widetilde{t}, \widetilde{\xi}\right) = 0$.

The following steps 1-10 show that every component of the sequence $\{\xi^n\}$ admits a converging subsequence in Ξ.

Step 1. $\{x_h^n\}$ **has a converging subsequence in** $\mathbb{R}_{++}^{GH}$.

By definition we have $x_h^n \gg 0$, for every h and therefore $\{x_h^n\}$ is bounded below.

Since $\sum_h \left(x_h^{\setminus n} - e_h^{\setminus}(t^n)\right) = 0$ we get $\sum_h x_h^{\setminus n} \leq \sum_h e_h^{\setminus}(t^n)$ and so $(x_h^{sc})^n \leq \sum_h e_h^{sc}(t^n)$. By noting $e_h^{\setminus}(t^n)$ is continuous and $\{t^n\}$ is a converging sequence, we obtain $\{x_h^n\}$ is bounded above and hence it admits a converging subsequence $\{x_h^n\} \to \widetilde{x}_h$. We are left to show that $\widetilde{x}_h \in \mathbb{R}_{++}^{GH}$. Since $\{e_h^n\} \to \widetilde{e}_h$ there exists a compact set $I_{\widetilde{e}_h}$ such that $\widetilde{e}_h \in I_{\widetilde{e}_h}$ and for any n which is sufficiently big, we have $e_h^n \in I_{\widetilde{e}_h}$. Let be $\widetilde{u} = \min_{e_h \in I_{\widetilde{e}_h}} u_h(e_h)$. For a well chosen n', since $(\xi^n, \alpha(t^n)) \in F_{\text{hom}}^{-1}(0)$, $u_h(x_h^n) \geq \widetilde{u}$ for every $n > n'$. hence $x_h^n \in L_{\widetilde{u}} = \{x_h \in \mathbb{R}_{++}^G : u_h(x_h^n) \geq \widetilde{u}\}$. From Assumption on the utility function $clL_{\widetilde{u}} \subseteq \mathbb{R}_{++}^G$, hence $\widetilde{x}_h \gg 0$.

- **Step 2.**$\{m_h^{0n}\}$ **has a converging subsequence in** $\mathbb{R}_+$.

By using equation M3 of 14 and the non negativity constraint we obtain

$\{m_h^{0n}\}$ is a bounded sequence and then we get $m_h^{0n} \to \tilde{m}_h^0$ with $\tilde{m}_h^0 \geq 0$.

- **Step 3. $\{p_h^{sn}\}$ has a converging subsequence in $\mathbb{R}_{++}$** for every $s > 0$

From the First Order Conditions we have $p^{sn} = \frac{D_{x_h^s C} u_h(x_h^n)}{\lambda_h^{sn}}$ for every $h \in H$.

Taking into account that $p^{sn} = \frac{D_{x_{h'}^s C} u_{h'}(x_{h'}^n)}{\lambda_{h'}^{sn}}$ for $s > 0$ and $\sum_{h=1}^{H} m_h^{0n} = \sum_{h=1}^{H} e_h^{m0n}$ we get

$$\sum_{h=1}^{H} \left(-e_h^{0mn} - e_h^{smn} + \sum_{c=1}^{C} \tau_h^{scn} \frac{D_{x_{h'}^s C} u_{h'}(x_{h'}^n)}{\lambda_{h'}^{sn}} e_h^{scn} \right) = 0 \quad s > 0$$

and therefore

$$\lambda_{h'}^{sn} = \frac{\sum_{h=1}^{H} \sum_{c=1}^{C} \tau_h^{scn} D_{x_{h'}^s C} u_{h'}(x_{h'}^n) e_{h'}^{scn}}{\sum_{h=1}^{H} (e_h^{0mn} + e_h^{smn})}$$

hence the sequence $\{\lambda_{h'}^{sn}\}$ converges as follows :

$$\lambda_{h'}^{sn} \to \frac{\sum_{c=1}^{C} \widetilde{\tau}_h^{sc} D_{x_{h'}^s C} u_{h'}(\widetilde{x}_{h'}) \widetilde{e}_{h'}^{sc}}{\sum_{h=1}^{H} (\widetilde{e}_h^{0m} + \widetilde{e}_h^{sm})} > 0$$

where the strictly inequality comes from $D_{x^s C} u_{h'}(\widetilde{x}_{h'}) > 0$,and from i) and ii) of Assumption 11. Since $p^{sn} = \frac{D_{x_{h'}^s C} u_{h'}(x_{h'}^n)}{\lambda_{h'}^{sn}}$, $p^{sn} \to \frac{D_{x_{h'}^s C} u_{h'}(\widetilde{x}_{h'})}{\lambda_{h'}^{s}} > 0$.

- **Step 4. λ_h^{0n} has a converging subsequence in $\mathbb{R}_{++}$**

Since $D_{x^0 C} u_h(x_h^n) = \lambda_h^{0n}$. From step 1 we know $D_{x^0 C} u_h(x_h^n)$ has a converging subsequence, then $\lambda_h^{0n} \to \widetilde{\lambda}_h^0 = D_{x_h^0 C} u_h(\widetilde{x}_h) > 0$.

Step 5. p^{0c} has a converging subsequence in $\mathbb{R}_{++}$ for $c = 2, ..., 1$

It follows directly from the first order condition and step 4.

Step 6. λ_h^{sn} has a converging subsequence in $\mathbb{R}_{++}$ $(s > 0)$

For $h = h'$, we proved the result in step 3. For $h \neq h'$ since $\{x_h^{sn}\}, \{p^{sn}\}$ admits converging subsequence, and $\lambda_h^{sn} = \frac{D_{x_h^{s1}} u_h(x_h^n)}{p^{s1n}}$ we can easily note that $\{\lambda_h^{sn}\} \to \frac{D_{x_h^{s1}} u_h(\widetilde{x}_h)}{\widetilde{p}^{sc}} \gg 0$.

Step 7. $\{q^{in}\}$ **has a converging subsequence**

Due to Assumption iii) of 7, for every asset i, there exists a h' such that $\lambda_{h'}R^i + \mu_{h'}D_{b^i_{h'}}a_{h'}(b_{h'}, m^0_{h'}) = \lambda_{h'}R^i = 0$

Then the boundness of the sequence$\{q^{in}\}$ follows; in fact we have $\lambda^{0n}_{h'}q^{in} = \sum_s \lambda^{sin}_{h'}y^{si}$ for every n. That implies $q^{in} = \frac{\sum_s \lambda^{sn}_{h}y^{si}}{\lambda^{no}_h}$ for every i and every n. Hence $\{q^{in}\} \to \frac{\sum_s \tilde{\lambda}^s_h y^{si}}{\tilde{\lambda}^0_h}$.

Step 8. $\left\{\left(b^n, (m^0_h)^n\right)\right\}$ **has a converging subsequence in** $\mathbb{R}^I \times \mathbb{R}_+$

From $-\Phi(x_h - e_h) + q^m m^0_h + \hat{U}e^m_h - \Psi(\tau_h, p)e_h + Rb_h = 0$ we get $Rb_h = -q^m m^0_h - \Phi(x_h - e_h) - \hat{U}e^m_h + \Psi(\tau_h, p)e_h$ Consider the S equations referring to states $s = 1, ..., S$. Recall that $rank[Y] = I$, the vector (b_h) is a continuous function of $\left(x_h, (m^0_h)^n, e_h, e^m_h, p\right)$. Then since $\left\{\left(x^n_h, (m^0_h)^n e^n_h, e^{nm}_h, p^{\backslash 01n}\right)\right\}$ admits a converging subsequence we have $\{(b^n_h)\} \to \tilde{b}_h$.

Step 9. $\{(q^{0m}_h)\}$ **has a converging subsequence in** $\mathbb{R}_+$ **and** $\{(q^n, q^m_h)\} \to (\tilde{q}, \tilde{q}^{0m}) \in Q$

Since there exists a consumer $h\prime$ such that $m^0_{h'} > 0$, from budget constraint referring to state $s = 0$ we can derive q^{0m}_h as a continuous function of $\left\{\left(x^{0n}_{h'}, (m^0_{h'})^n, b^n_{h'}, e^{0n}_{h'}, e^{0mn}_{h'}, p^{0n}, q^n\right)\right\}$. Since the latter sequence admits a converging sequence we get $(q^{0m})^n \to \tilde{q}^{0m} > 0$.

Since $\tilde{\lambda}^s_h > 0$ for every s, and $\tilde{q}^{0m} > 0$, then $\tilde{q}^{0m}$ is a no arbitrage money price (see Definition 10) is verified.

Step 10. $\left\{\left(\mu^n, (\gamma^0_h)^n\right)\right\}$ **has a converging subsequence in** $\mathbb{R}^{\sharp J_h} \times \mathbb{R}_+$

Since $a_h(b_h, m^0_h)$is a continuous function, $a_h(b^n_h, m^{0n}_h) \geq 0$ implies

$a_h\left(\tilde{b}_h, \tilde{m}^0_h\right) \geq 0$. Let $J_h = \left\{J^A_h, J^B_h\right\}$ be a partition of the set of index J_h such that $J^A_h = \left\{j \in J_h : a^j_h\left(\tilde{b}_h, \tilde{m}^0_h\right) = 0\right\}$ and $J^B_h = \left\{j \in J_h : a^j_h\left(\tilde{b}_h, \tilde{m}^0_h\right) \gg 0\right\}$. If $j \in J^B_h$, by a well known limit theorem, there exists a n^* such that $a^j_h\left(b^n_h, (m^0_h)^n\right) \gg 0$ for every $n > n^*$. Hence for every $n > n^*$ we have $\mu^{jn}_h = 0$ i.e. $\left\{\mu^{jn}_h\right\} \to 0$ for every $j \in J^B_h$.

If $j \in J^A_h$, from iv) of (7), $rank\left(D_{b_h}a^{jA}_h\left(\tilde{b}_h, \tilde{m}^0_h\right)\right) = \sharp J^A_h$.

Let $D_{b_h}a^{*JA}_h\left(\left(\tilde{b}_h, \tilde{m}^0_h\right)\right)$ be the square submatrix of $D_{b_h}a_h$ (whose dimension is $\sharp J^A_h \times \sharp J^A_h$) such that $\left|detD_{b_h}a^*_h\left(\tilde{b}_h, \tilde{m}^0_h\right)\right| > 0$ and $D_{b_h}a^{\hat{}JA}_h\left(\tilde{b}_h, \tilde{m}^0_h\right)$ is the matrix of dimension $\left(\sharp J_h - \sharp J^A_h\right) \times \left(I - \sharp J^A_h\right)$ which is the complement of $D_{b_h}a^{*JA}_h$. By a well known limit theorem, there exists $n\prime$ such that $\left|detD_{b_h}a^{*jA}_h\left(b^n_h, (m^0_h)^n\right)\right| > 0$ for every $n > n\prime$. Let us take $n^{**} = max\{n^*, n'\}$.

Making the proper permutations, we get:

$$\begin{array}{c|ccc|c} & 1 & I & 1 & \\ \hline \begin{array}{c}\sharp J_h^B\\ \sharp J_h^A\end{array} & \begin{bmatrix} \mu_h^{j_h^B n} \\ \mu_h^{j^A n} \end{bmatrix}^T & \begin{bmatrix} D_{b_h} a_h^{j_h^B}\left(b_h^n,(m_h^0)^n\right) \\ D_{b_h} a_h^{j^A}\left(b_h^n,(m_h^0)^n\right) \end{bmatrix} & \equiv [\eta^n] & I \end{array}$$

for every $n > n^{**}$ i.e

$$\begin{bmatrix} 0 \\ \mu_h^{jAn} D_{b_h} a_h^{j^A}\left(b_h^n,(m_h^0)^n\right) \end{bmatrix} = \begin{bmatrix} \eta^{\cdot n} \\ \eta^{*n} \end{bmatrix}$$

Then $\mu_h^{JAn} = \eta^{n*}\left[D_{b_h}^{-1} a_h^{J^A}\left(b_h^n,(m_h^0)^n\right)\right]$ follows and so $\mu_h^{j^A n} \to \widetilde{\mu}_h^{j^A}$.

From $\lambda_h^n q^{mn} + \mu_h^n D_m a_h\left(b_h^n, m_h^{0n}\right) + \gamma_h^n = 0$ we have $\gamma_h^n \to \widetilde{\gamma}_h = \widetilde{\lambda}_h \widetilde{q}^m + \mu_h D_m a_h\left(\widetilde{b}_h, \widetilde{m}_h^0\right)$

iv) It follows directly from iii). ■

We are now ready to claim and prove the existence result.

Theorem 30 (Existence) *For every economy* $\omega \in \Omega$, $EQ_{(\omega)} \neq \emptyset$.

Proof. From previous Lemmas, Assumptions of Degree Theorem are satisfied. Since $\deg(g,0) = 1$,we have $\deg(f,0) = \deg(g,0) = 1$. Then the result follows. ■

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