A-PDF MERGER DEMO

Report n. 143

Monetary policy and Paretó improvability in a financial economy with restricted participation

Laura Carosi

Pisa, June 1999

Monetary policy and Pareto improvability in a financial economy with restricted participation

Laura Carosi
Department of Statistics and Applied Mathematics,
University of Pisa

Abstract

We analyze an economy with inside financial assets and outside money. Households have different restricted access on both types of assets. Using a well known approach in terms of needs of money to pay taxes, we first get existence of equilibria with positive price of money. We then prove generic regularity and inefficiency of equilibria. The presence of money suggests to model a monetary intervention in order to study Pareto Improvability. Since monetary policy can Pareto Improve upon the market equilibrium, money is not neutral.

KEY WORDS: General Equilibrium, Incomplete Markets, Monetary Policy.
Mathematical Subject Classification: 90A14,
Journal of Economic Literature Classification: D52, E52.

1 Introduction

In a standard general equilibrium framework with incomplete markets, consumers face the same opportunities to transfer wealth across spot markets. In real life, we can find many cases where the participation constraints on financial markets varies from a class of consumers to another. For example, we can think of collateral securities in American real estate market or of a credit line which is secured by financial assets.

In the recent literature we can find several models (see Balasko, Cass and Siconolfi (1990), Cass, Siconolfi and Villanacci (1992), Polemarchakis and Siconolfi (1997), Siconolfi (1988)) which present a wide range of restrictions on financial market participation. These kind of general equilibrium models are called "restricted participation model" and they can be seen as generalization of the incomplete market case. While Cass, Siconolfi and Villanacci (1992) propose a model where the individual participation constraint is described by a differentiable strictly quasi-concave function of consumer's assets a_h, our model in enriched by the presence of the outside money[1] whose exchange is restricted too. We assume a_h is a function of both consumer's assets and outside money demands ,i.e. $a_h(b_h, m_h^0) \geq 0$. From now on, unless it otherwise specified, money means outside money.

We can find many contributions in order to understand why money exists in a General Equilibrium Model. (See e.g. Lerner (1947), Starr (1974 and 1989), Hahn (1965)). Even when money exists in a general equilibrium context that does not imply the existence of a positive price for money (Hahn (1965)). For that reason, there are several additional assumptions in order to overcome the well-known hot potato problem (see Cass and Shell (1980), Dubey and Geanakoplos (1992), Grandmont and Younes (1975), Magill and Quinzii (1992), Starr (1974 and 1989)).

We do not try to explain why money exists, but, using a well known approach in terms of needs of money to pay taxes (e.g. Lerner (1947) , Starr (1974), Villanacci (1991 and 1993)), we study the basic properties of the model. We assume that households have to use money to pay taxes at the end of the second period; taxes are linear function of households' wealths.

The set up of the model is presented in section two while in section three we give the definition of Equilibrium and we state the Existence result, proved in Carosi (1999a).

Due to financial assets participations constraints and money constraints, the function F_ω which describes equilibria is not C^1 everywhere. The lack of differentiability does not allow to obtain regularity of equilibria as a trivial consequence of well-known results. Therefore, in section four, we first prove that differentiability is a generic property of function F_ω and then we get the generic regularity of equilibria. Consistently with the result in the incomplete market case, we verify that there exists an open and full measure subset of the set of economies, whose associated equilibrium allocations are not Pareto Optimal (section five).

The presence of money suggests to model a government monetary intervention in order to study Pareto Improvability of equilibria. More precisely, in section six we investigate what happens to consumers' utility levels if the government can modify only the amount of money endowments of one indi-

[1]By the term *Outside money* we refer to money which is a direct debt of the public sector, e.g. circulating currency, or is based on such debt, e.g. commercial bank deposits matched by bank holdings of public sector debt. Examples are fiat money, gold and foreign exchange reserves. On the other hand, *Inside money* is a form of money which is based on private sector debt.

vidual in period 1. The conclusion of the analysis is the following: monetary intervention can Pareto Improve upon the market equilibrium. That result is related to a crucial topics in macroeconomic theory: monetary policy can modify the real variables of the model that is, it is not neutral.

2 Set up of the model

We consider an exchange economy with two periods; today, which is called state 0, and tomorrow which is called period 1 and S states of the world are possible. The set of possible states of the world is $\{0, 1, ..., S\}$ with generic element s. In state 0, households receive endowments of goods and money, they exchange goods and assets and consume the goods they acquired. Households are not allowed to buy and sell assets freely, but they must take into account their own participation constraints. Tomorrow uncertainty is resolved, one of the S states occurs and households receive their endowments of goods and money. They exchange goods and fulfill the obligations underwritten in state 0. Finally households consume the goods they acquired and they use money to pay taxes. We will use the following notations:

- e_h^{sc} and x_h^{sc} are respectively, the endowment and demand of good c in state s, of household h.
- $e_h^s \equiv (e_h^{sc})_{c=1}^C$, $e_h = (e_h^s)_{s=0}^S$, $e = (e_h)_{h=1}^H$.
- $x_h^s \equiv (x_h^{sc})_{c=1}^C$, $x_h = (x_h^s)_{s=0}^S$, $x = (x_h)_{h=1}^H$.
- e_h^{sm} is the endowment of money in state s, owned by household h.
- $e_h^m = (e_h^{sm})_{s=0}^S$, $e^m = (e_h^m)_{h=1}^H$.
- b_h^i is the demand of asset i, of household h. $b_h \equiv (b_h^i)_{i=1}^I$.
- q^{sm} is the price of money in state s. $q^m = (q^{sm})_{s=0}^S$
- m_h^s is the demand of money in state s of household h. $m_h = (m_h^s)_{s=0}^S$, $m = (m_h)_{h=1}^H$.

Households' utility functions have the following properties.

Assumption 1 *i)* u_h *is a smooth function, i.e., a* C^∞ *function.*
ii) u_h *is differentiably strictly increasing, i.e.,* $Du_h(x_h) \gg 0$.
iii) the Hessian matrix $D^2 u_h$ *is negative definite*
iv) For any $\underline{u} \in \mathbb{R}$, $Cl\left\{x \in \mathbb{R}_{++}^C : u_h(x) \geq \underline{u}\right\} \subseteq \mathbb{R}_{++}^C$.

We assume consumers cannot issue outside money.

Assumption 2 $m_h^s \geq 0$ *for all* s *and all* h.

Prices of goods, money and assets are expressed in units of account. We assume that prices of goods are strictly positive.

Assumption 3 $p^{sc} \in \mathbb{R}_{++}$ *for all* s *and all* c,*where* p^{sc} *is the price of good* c *in state* s.

We denote the matrix of assets yields by $Y = \begin{bmatrix} y^{11} & .. & y^{1I} \\ & & \\ y^{S1} & & y^{SI} \end{bmatrix}$; Y is a $S \times I$ matrix . Moreover $Y^M = \begin{bmatrix} Y & \mathbf{1} \end{bmatrix}$ is a $S \times (I+1)$ matrix.

It greatly simplifies our analysis to assume that

Assumption 4 $S > I + 1$, $RankY = I$ *and* $rankY^M = I + 1$.

Remark 5 *he previous Assumption means there are no redundant assets in the economy. As Cass, Siconolfi and Villanacci (1992) say, "..In this context, Assumption 4 is not at all innocuous. When their portfolio holdings are constrained, households may very well benefit from the opportunity afforded by the availability of additional bonds whose yields are not linearly independent".*

Households deal with two different kinds of constraints in the assets market. On one hand they must take into account the incompleteness of the asset market (i.e. $rankY = I < S$) and on the other hand, they must consider their own participation constraint. The latter is expressed by the following function:

$$\begin{aligned} a_h^{\#J_h} &: \mathbb{R}^I \times \mathbb{R} \to \mathbb{R}^{\#J_h} \\ a_h^j &: \left(b_h, m_h^0\right) \mapsto a_h^j \left(b_h, m_h^0\right) \quad j = 1, ..., \#J_h \end{aligned} \tag{1}$$

where $a_h^{\#J_h} = \left[a_h^j \left(b_h, m_h^0\right)\right]_{j=1}^{\#J_h}$, J_h is a set of indexes such that $J_h \subseteq I$. a_h^j verifies the following Assumption.

Assumption 6 a_h^j *is a* C^2, *differentiably strictly quasi-concave function, i.e. for every* $(b_h, m_h^0) \in \mathbb{R}^{I+1}$ *and every* $\Delta \in \mathbb{R}^{I+1}$

$$Da_h^j \left(b_h, m_h^0\right) \Delta = 0 \Rightarrow \Delta^T D^2 a_h^j \left(b_h, m_h^0\right) \Delta < 0.$$

If the set J_h is clearly specified and there is no possibility of misunderstanding about it, we can drop J_h from a_h. Moreover this function a_h verifies the following further conditions:

Assumption 7 *i)* $a_h(0,0) > 0$.

ii) For every $(b_h, m_h^0) \in \mathbb{R}^{I+1}$ *such that* $a_h^{J_h'}(b_h, m_h^0) = 0$,
rank $\left(Da_h^{J_h'}(b_h, m_h^0)\right) = \#J_h'$, *for every index subset* $J_h' \subseteq J_h$.

iii) For every asset i*, there exists at least one consumer* h' *such that for every* $(b_{h'}, m_{h'}^0) \in \mathbb{R}^{I+1}$ *the following condition holds* $:D_{b_{h'}^i} a_{h'}(b_{h'}, m_{h'}^0) = 0$

iv) there exists at least one consumer h' *such that :* $D_{m_{h'}^0} a_{h'}(b_{h'}, m_{h'}^0) = 0$.

Remark 8 *Assumption 7 has important economic meanings.*

i) people are not obliged to operate in the assets and/or money markets. Moreover, there is a small neighborhood of $(0,0)$ *where every consumer can freely operate.*

iii) for every asset there exists at least one household who is unrestricted on that asset market.

iv) there exists at least one consumer who can arbitrary vary his money demand.

Assumption 7 i),...,iv) are used in order to prove the existence of the competitive equilibrium and their generic regularity.

In order to get regularity of equilibria, we impose the following Assumption on the relationship between the number of consumers and the number of assets.

Assumption 9 $S \geq H > I + 1$.

Remark 10 *Assumption 9 will be used in order to prove the non neutrality of policy intervention. In fact* $H > I + 1$ *allows to simplify the computations (see case 2 and 3 page. 32) and* $S \geq H$ *guarantees that the number of independent policy instruments is greater than the number of households.*

Households are not able to create wealth by acting on the assets and money markets. Hence we obtain the following no arbitrage condition that allows us to define the set of no arbitrage assets and money prices

Definition 11 *Let us define the no-arbitrage asset and money price set as:*
$\widehat{Q}_h = \{\hat{q} = (q, q^m) \in \mathbb{R}^I \times \mathbb{R}^{S+1} : \nexists (b_h, m_h^0),$ *such that* $a_h(b_h, m_h^0) \geq 0$ *and*
$\begin{bmatrix} -q & -q^{m0} \\ Y & q^{m1} \end{bmatrix} \begin{bmatrix} b_h \\ m_h^0 \end{bmatrix} > 0 \; \}$

where q^{m1} *is the vector* $q^{m1} = (q^{ms})_{s=1}^{S} = (q^{m1}, .., q^{ms})$ *of dimension* $S \times 1$.

$\widehat{Q} = \bigcap_{h \in H} \widehat{Q}_h$ *is the set of no arbitrage.*

In period 1, Mr. h pays taxes using money; taxes are proportional to the value of his endowments.

- $\tau_h^{sc} \in [0,1] \subseteq \mathbb{R}$ is the percentage of taxes that Mr. h has to pay for good c, in state s. $\tau_h^s = (\tau_h^{sc})_{c=1}^{C}$ $\tau_h = (\tau_h^s)_{s=1}^{S}$,$\tau = (\tau_h)_{h=1}^{H}$.

An economy is described by a vector $\omega = (e, e^m, \tau)$ of endowments of goods and money and tax parameters.

Assumption 12 $\omega \in \widehat{\Omega} = \mathbb{R}_{++}^{GH} \times X^m \times T$ *where*

i) $X^m \equiv \left\{ e^m \in \mathbb{R}^{(S+1)H} : \sum_{h=1}^{H} e_h^{m0} > 0 \text{ and for } s \geq 1, \sum_{h=1}^{H} (e_h^{m0} + e_h^{ms}) > 0 \right\}$

ii) $T \equiv \left\{ \begin{array}{l} \tau \in [0,1]^{SCH} : a) \forall s \geq 1, \\ \exists h \text{ and } \exists c : \tau_h^{sc} > 0 \\ b) \exists h^* \text{ such that } \forall s \geq 1, \exists c \ : \tau_{h^*}^{sc} \neq 1 \end{array} \right\}$

iii) $\omega \in \Omega \equiv \{\omega \in \widehat{\Omega} : \exists p \in \mathbb{R}_{++}^{G}$ *such that* $\forall h,\ p^0 e_h^0 + q^{0m} e_h^0 > 0$ *and for* $s = 1, ..., S\ \ p^s e_h^s + e_h^{ms} - \sum_{c=1}^{C} \tau_h^{sc} p^{sc} e_h^{sc} > 0\}$

Remark 13 *As Villanacci (1993) observes, condition i) implies that in each states of the world there exists a positive amount of money; moreover part a) of ii) means that taxes are a nontrivial function of wealth, while part b) says there exists at least one consumer, who, in every states, does not use all his wealth to pay taxes. Condition iii) says in every state of the world, households are able to pay their taxes using their initial endowments.*

3 Competitive Equilibria

Each household maximizes his utility function subject to his budget constraints which depends on his endowments and taxes and on participation constraints in both assets and money market. Note that in every state of period 1 households use money only to pay taxes, no one wants to hold an amount of money greater than the one required to meet his tax obligations.

For $(\omega, \widehat{p}, \widehat{q}, \widehat{q}^m) \in \Omega \times \mathbb{R}^G_{++} \times \hat{Q}$,we have :

$$
\textbf{(P1)} \qquad
\begin{array}{lll}
 & \max\limits_{(x_h,b_h,m_h)} u(x_h) & s.t. \\
 & \widehat{p}^0 x_h^0 + \widehat{q}^{0m} m_h^0 + \widehat{q} b_h & \leq \widehat{p}^0 e_h^0 + \widehat{q}^{0m} e_h^{m0} \\
 & m_h^0 & \geq 0 \\
(s = 1, ..., S) & \widehat{p}^s x_h^s + \sum_{c=1}^{C} \tau_h^{sc} \widehat{p}^{sc} e_h^{sc} & \leq \sum_{i=1}^{I} \widehat{q}^{sm} y^{si} b_h^i \widehat{q}^{sm} + \widehat{q}^{sm} \left(e_h^{ms} + m_h^0\right) \\
 & a_h \left(b_h, m_h^0\right) & \geq 0
\end{array}
\tag{2}
$$

Remark 14 *In order to eliminate technical complication, we normalize prices using the price of good C in state 0, while in the other states we normalize prices using the price of money. From now on we will always refer to the normalized prices. We have:*

$$p^0 \equiv \frac{\widehat{p}^0}{p^{01}}, \ q^{0m} \equiv \frac{\widehat{q}^{0m}}{p^{01}}, \quad q \equiv \frac{\widehat{q}}{p^{01}} \ \textit{and} \ p^s = \frac{\widehat{p}s}{q^{sm}} \ \textit{for} \ s > 0$$

Denote

$Q = \left\{ \exists \left(\hat{q}, \hat{q}^m\right) \in \hat{Q} \text{ such that } q \equiv \frac{\widehat{q}}{p^{01}}, \ q^{0m} \equiv \frac{\widehat{q}^{0m}}{p^{01}}, \ q^{ms} = 1, \ \text{for } s = 1..S \right\}$

Define the demand map of household h. Given $\omega \in \Omega$, it associates with every vector prices, a vector of demand of goods, money and assets.

$$\left(x_h, b_h, m_h^0\right) : \mathbb{R}^G_{++} \times Q \to \mathbb{R}^G_{++} \times \mathbb{R}^I \times \mathbb{R}_+ \tag{3}$$

$$\left(x_h, b_h, m_h^0\right) : (p, q, q^m) \mapsto \arg\max (P1). \tag{4}$$

It can be proved Carosi (1999a) that the demand function of Mr. h is continuous and it is C^1 in open and full measure set of $\mathbb{R}^G_{++} \times Q \times \Omega$.

From now on the maximizing behavior of households will be described by the following first order conditions that can be easily derived from Kuhn-Tucker necessary and sufficient conditions of the maximization problem.

$$
(Foc_h) \begin{pmatrix}
 & D_{x_h} u_h(x_h) - \lambda_h \Phi & = 0 \\
 & -\Phi(x_h - e_h) + q^m m_h^0 + \hat{U} e_h^m - \Psi(\tau_h, p) e_h + R b_h & = 0 \\
 & \lambda_h R + \mu_h D_{b_h} a_h \left(b_h, m_h^0\right) & = 0 \\
(\forall j \in J_h) & \min \left[\mu_h^j, a_h^j \left(b_h, m_h^0\right)\right] & = 0 \\
 & \lambda_h q^m + \mu_h D_m a_h \left(b_h, m_h^0\right) + \gamma_h & = 0 \\
 & \min \left[\gamma_h, m_h^0\right] & = 0
\end{pmatrix}
\tag{5}
$$

where $(\lambda_h, \mu_h, \gamma_h)$ are the Lagrange multipliers

- Φ is a $(S+1)\times G$ matrix,

$$\Phi \equiv \begin{bmatrix} p^0 & & \\ & \dots & \\ & & p^S \end{bmatrix}$$

with $p^0, p^1, ..., p^S \in \mathbb{R}^C_{++}$,

- q^m is an $(S+1)$ vector, $q^m \equiv \begin{bmatrix} -q^{om} \\ \mathbf{1} \end{bmatrix}$, and $\mathbf{1} = (1, ..., 1)^T$.

- $\hat{U}$ is an $(S+1)\times(S+1)$ diagonal matrix,

$$\hat{U} \equiv \begin{bmatrix} q^{0m} & \\ & -I_{S\times S} \end{bmatrix}$$

where $I_{S\times S}$ is the identity matrix whose dimension is S

- R is an $(S+1)\times I$ matrix, $R = \begin{bmatrix} -q \\ Y \end{bmatrix}$.

- $\Psi(\tau_h, p)$ is an $(S+1)\times G$ matrix,

$$\Psi(\tau_h, p) \equiv \begin{bmatrix} 0 & & & \\ & \tau_h^{11}p^{11}...\tau_h^{1C}p^{1C} & & \\ & & .. & \\ & & & \tau_h^{S1}p^{S1}...\tau_h^{SC}p^{SC} \end{bmatrix}$$

We define $\widehat{S} \equiv S+1$

We are ready to define the equilibrium for any given economy $\omega \in \Omega$.

Definition 15 (Equilibrium) *Given an economy $\omega = (e, e^m, \tau) \in \Omega$, the vector (p, q, q^m) is an equilibrium prices system if and only if there exists $(x_h, b_h, m_h^0)_{h=1}^H$ such that*

1) (x_h, b_h, m_h^0) solves consumer's maximization problem (for every h);

2) $\sum_{h=1}^H (x_h^{sc} - e_h^{sc}) = 0$ for every s, c i.e. markets for goods clear;

3) $\sum_{h=1}^H b_h^i = 0$ for every i i.e. markets for assets clear;

4) money markets clear i.e.

$$\begin{cases} \sum_{h=1}^H (m_h^0 - e_h^{m0}) = 0 \\ \sum_{h=1}^H \left(-m_h^0 - e_h^{ms} + \sum_{c=1}^C \tau_h^{sc} p^{sc} e_h^{sc}\right) = 0 \quad (s = 1..S). \end{cases} \tag{6}$$

From now on we use the following notations :

$$x^{\backslash} \equiv \left(x_h^{\backslash}\right)_{h=1}^{H} \equiv \left(\left(x_h^{s\backslash}\right)_{s=0}^{S}\right)_{h=1}^{H} \equiv \left(\left((x_h^{sc})_{c=2}^{C}\right)_{s=0}^{S}\right)_{h=1}^{H}$$
$$e^{\backslash} \equiv \left(e_h^{\backslash}\right)_{h=1}^{H} \equiv \left(\left(e_h^{s\backslash}\right)_{s=0}^{S}\right)_{h=1}^{H} \equiv \left(\left((e_h^{sc})_{c=2}^{C}\right)_{s=0}^{S}\right)_{h=1}^{H}$$

We define the function $F : \Xi \times \Omega \to \mathbb{R}^{\dim \Xi}$ with
$\Xi = \mathbb{R}_{++}^{GH} \times \mathbb{R}^{H\hat{S}} \times \mathbb{R}^{HI} \times \mathbb{R}^{\sum \# J_h} \times \mathbb{R}^{H} \times \mathbb{R}^{H} \times \mathbb{R}_{++}^{G-1} \times Q$

$$F : (\xi, \omega) \mapsto \begin{pmatrix} \text{Left Hand Side of equations 5} & \\ (M1) & \sum_{h=1}^{H} \left(x_h^{\backslash} - e_h^{\backslash}\right) \\ (M2) & \sum_{h=1}^{H} b_h \\ (M3) & \sum_{h=1}^{H} (m_h^0 - e_h^{m0}) \\ (M4) & \sum_{h=1}^{H} \left(-m_h^0 - e_h^{ms} + \sum_{c=1}^{C} \tau_h^{sc} p^{sc} e_h^{sc}\right) (s > 0) \end{pmatrix} \tag{7}$$

with $\xi \equiv \left((x_h, \lambda_h, b_h, \mu_h, m_h^0, \gamma_h)_{h=1}^{H}, p^{\backslash 01}, q, q^{0m}\right)$.

Definition 16 *The set of equilibria associated with the economy $\overline{\omega} = (\overline{e}, \overline{e^m}, \overline{\tau}) \in \Omega$, is given by $EQ_{\overline{\omega}} \equiv F_{\overline{\omega}}^{-1}(0)$ where $F_{\overline{\omega}}$ is the restriction of the function F to $\overline{\omega}$, i.e. $F_{\overline{\omega}} : \xi \mapsto F(\xi, \overline{\omega})$*

The following existence result can be prove by means of a Degree Argument (see Carosi 1999a)

Theorem 17 (Existence) *For every economy $\omega \in \Omega$, $EQ_{(\omega)} \neq \emptyset$.*

We are interested in investigating the generic uniqueness and regularity of equilibria.

4 Regularity of Equilibria

The function F_ω is not differentiable on all of its domain; in fact, F_ω is not differentiable when for some h, either $a_h^{j_h}(b_h, m_h^0) = \mu_h^{j_h} = 0$ or $m_h^0 = \gamma_h = 0$,.

The lack of differentiability on the whole domain of F_ω does not allow us to obtain the Regularity result as a trivial extension of a well known analysis.

In order to prove that the "border line cases" (i.e., cases where F_ω is not differentiable) are "rare", we classify equilibria with respect to different border line cases and we then define a related concept of "auxiliary equilibria". Theorem 20 states that there exists a projection function pr that associates an auxiliary equilibrium with every "real equilibrium". Then Theorem 21 establishes that the set of economies whose corresponding auxiliary equilibria have border line cases has measure zero and it is closed. Taking into account the relationship between auxiliary equilibria and "real equilibria" we can conclude that there exists an open and full measure set of economies such that the corresponding F_ω is differentiable. Then the regularity of equilibria is proved with the usual argument.

First of all we show that the projection function $\pi : F^{-1}(0) \to \Omega$, $\pi : (F^{-1}(0)) \mapsto \omega$ is proper.

Theorem 18 *The function* $\pi : F^{-1}(0) \to \Omega$, $\pi : (F^{-1}(0)) \mapsto \omega$ *is proper.*

Proof. Let $K \subseteq \Omega$ be a compact set and let us show that $\pi^{-1}(K)$ is sequentially compact, i.e. compact. Take a sequence $\{(\xi^n, \omega^n)\} \subseteq \pi^{-1}(K)$ and show $\{(\xi^n, \omega^n)\}$ admits a converging subsequence in $\pi^{-1}(K)$. Since $\{\omega^n\} \subseteq K$, $\{\omega^n\}$ is bounded sequence it has converging subsequence $\{\omega^n\} \to \overline{\omega}$ (Without any loss of generality we can consider the sequence $\{\omega^n\}$ itself).

By using a similar argument of those we have already presented in (Carosi 1999a), we get that $\{\xi^n\}$ admits a converging subsequence $\{\xi^n\} \to \overline{\xi} \in \Xi$. Then the sequence $(\xi^n, \omega^n) \subseteq \left(F_{\omega^n}^{-1}(0), \omega^n\right)$ has a converging subsequence $(\xi^n, \omega^n) \to \left(\overline{\xi}, \overline{\omega}\right)$ and from the continuity of the function π we have $\pi(\xi^n, \omega^n) \to \pi\left(\overline{\xi}, \overline{\omega}\right)$ from which it follows $\left(\overline{\xi}, \overline{\omega}\right) \in \pi^{-1}(K)$ i.e. the desired result. ∎

We define

$$l_h^j\left(b_h, m_h^0\right) = \begin{cases} a_h^j\left(b_h, m_h^0\right) & \text{if } j = 1, ..\sharp J_h \\ m_h^0 & \text{if } j = \sharp J_h + 1 \end{cases} \quad \text{and } \zeta_h^j = \begin{cases} \mu_h^j & \text{if } j = 1, ..\sharp J_h \\ \gamma_h & \text{if } j = \sharp J_h + 1 \end{cases}$$

Let $\xi^{\widehat{J}} = \left(\left(\xi_h^{\widehat{J}_h}\right)_{h \in H}, p, q, q^{om}, \right), \xi^{\widehat{J}} \in \widehat{\Xi}$, where $\xi_h^{\widehat{J}_h} = \left(x_h, \lambda_h, b_h, m_h^0, \left(\zeta_h^j\right)_{j \in \widehat{J}_h}\right)$ and $\widehat{\Xi} = \mathbb{R}_{++}^{GH} \times \mathbb{R}_{++}^{\widehat{S}H} \times \mathbb{R}^{IH} \times \mathbb{R}^{H} \times \mathbb{R}^{\sum_h \sharp \widehat{J}_h} \times \mathbb{R}_{++}^{G-1} \times Q$. It is worth noting that $\widehat{J}_h$ is an arbitrary subset of J_h and $\widehat{J} = \{\widehat{J}_1, ...\widehat{J}_H\}$. Let us consider the following auxiliary maximization problem:

$$\begin{array}{lll} & \max\limits_{(x_h,b_h,m_h^0)} u_h(x_h) & \\ & -\Phi(x_h-e_h)+q^m m_h^0+\hat{U}e_h^{m0}-\Psi(\tau_h,p)\,e_h+Rb_h & =0 \quad (\lambda_h) \\ \left(j\in\widehat{J}_h\right) & l_h^j(b_h,m_h^0) & =0 \quad \left(\zeta_h^j\right)_{j\in\widehat{J}_h} \end{array} \tag{8}$$

We define $\widehat{F}_h^{\widehat{J}_h}:\mathbb{R}_{++}^G\times\mathbb{R}_{++}^{\widehat{S}}\times\mathbb{R}^I\times\mathbb{R}\times\mathbb{R}^{\sharp\widehat{J}_h}\rightarrow\mathbb{R}^G\times\mathbb{R}^{\widehat{S}}\times\mathbb{R}^I\times\mathbb{R}\times\mathbb{R}^{\sharp\widehat{J}_h}$, $\widehat{F}_h^{\widehat{J}_h}:\left(\xi_h^{\widehat{J}},\omega\right)\mapsto$ (left side of Lagrange conditions of the problem (8)).

Since $l_h^j(b_h,m_h^0)$ varies as the set of indexes $\widehat{J}_h$ varies, $\widehat{F}_h^{\widehat{J}_h}$ has a different form if the index $\sharp J+1$ belongs or does not to $\widehat{J}_h$. With this regard we have to distinguish two different cases.

CASE A $\sharp J+1\notin\widehat{J}_h$ and **CASE B** $\sharp J+1\in\widehat{J}_h$. We are going to handle only CASE A since the treatment of other one is very similar. From the First Order Conditions we have the following function

$\widehat{F}_h^{\widehat{J}_h}:\left(\xi_h^{\widehat{J}},\omega\right)\mapsto$

$$\left(\begin{array}{ll} & D_{x_h}u_h(x)-\lambda_h\Phi \\ & -\Phi(x_h-e_h)+q^m m_h^0+\hat{U}e_h^m-\Psi(\tau_h,p)\,e_h+Rb_h \\ & \lambda_h R+\sum\limits_{j_2\in\hat{J}_2}\zeta_h^{j_2}D_{b_h}a_h^{j_2}(b_h,m_h^0)+\sum\limits_{j^*\in\widehat{J}_h\backslash\hat{J}_2}\zeta_h^{j^*}D_{b_h}a_h^{j^*}(b_h,m_h^0) \\ & \sum\limits_{j_2\in\hat{J}_2}\zeta_h^{j_2}D_{m_h^0}a_h^{j_2}(b_h,m_h^0)+\sum\limits_{j^*\in\widehat{J}_h\backslash\hat{J}_2}\zeta_h^{j^*}D_{m_h^0}a_h^{j^*}(b_h,m_h^0)+\lambda_h q^m \\ j_2\in\widehat{J}_{h2} & l_h^{j_2}(b_h,m_h^0) \\ j_h^*\in\widehat{J}_h\backslash\widehat{J}_{h2} & l_h^{j^*}(b_h,m_h^0) \end{array}\right) \tag{9}$$

Observe that given ω, $\xi_h^{\widehat{J}}$ is the solution of the maximization problem (8) for every h if and only if $\left(\xi_h^{\widehat{J}},\omega\right)\in\widehat{F}_h^{\widehat{J}_h-1}(0)$ for every h.

Once we have fixed a subset of indexes $\widehat{J}=\{\widehat{J}_1,...\widehat{J}_H\}$ for every h, we introduce the following definition of equilibrium.

Definition 19 (Auxiliary equilibrium) *Given an economy $\omega\in\Omega$, $\xi^{\widehat{J}}$ is an auxiliary equilibrium if and only if*

i) $\left(\xi_h^{\widehat{J}},\omega\right)\in\left(\widehat{F}_h^{\widehat{J}_h}\right)^{-1}(0)$ *for every h.*

ii) $D_{\xi_h^{\widehat{J}}}\widehat{F}_h^{\widehat{J}_h}\left(\xi_h^{\widehat{J}},\omega\right)$ *has full rank for every h.*

iii) $\left(\xi^{\widehat{J}}, \omega\right)$ *is such that*

$$\begin{array}{lll} & \sum_{h=1}^{H}\left(x_h^{\setminus} - e_h^{\setminus}\right) & = 0 \quad (M1) \\ & \sum_{h=1}^{H} b_h & = 0 \quad (M2) \\ & \sum_{h=1}^{H}\left(m_h^0 - e_h^{m0}\right) & = 0 \quad (M3) \\ (s > 0) & \sum_{h=1}^{H}\left(-m_h^0 - e_h^{ms} + \sum_{c=1}^{C}\tau_h^{sc} p^{sc} e_h^{sc}\right) & = 0 \quad (M4) \end{array}$$

We define $E^{\widehat{J}} = \{\left(\xi^{\widehat{J}}, \omega\right) \in \widehat{\Xi} \times \Omega : i), ii)$ *and* $iii)$ *hold*$\}$

Note that markets clearing conditions in auxiliary equilibrium are the same as in the real one while households' maximization problem are different. In the auxiliary equilibrium, participation constraints are equality constraints and they are related to $\widehat{J}$ instead of J. Since the Jacobian of the function defined by the left hand side of equations (5) has full rank "almost" everywhere (Carosi (1999a), condition ii) is given in order to establish a relation between real and auxiliary equilibria. Observe that the set $E^{\widehat{J}}$ depends on the choice of $\widehat{J}$.

Define $\widehat{E} = \bigcup_{\widehat{J} \in \mathfrak{J}} E^{\widehat{J}}$ where $\mathfrak{J}$ represents the set of the all possible subsets of $J = \{J_1 \times .. \times J_H\}$ i.e. $\mathfrak{J} = \times_h \mathfrak{P}_h$. The following Theorem describe the relationship between the set $E^{\widehat{J}}$ and the set of equilibria $E = (EQ_\omega, \omega)$.

Theorem 20 *If* $(\xi, \omega) \in E$, *then there exists a* $\widehat{J}$ *such that* $pr(\xi, \omega) \in E^{\widehat{J}}$, *where* $pr : \Xi \times \Omega \to \widehat{\Xi} \times \Omega$, $pr : (\xi, \omega) \mapsto \left(\xi^{\widehat{J}}, \omega\right)$, *i.e. pr is the projection of* $\Xi \times \Omega$ *on* $\widehat{\Xi} \times \Omega$.

Proof. We consider $(\xi, \omega) \in E$ and we take $\widehat{J}_h = J_{2h} \cup J_{3h}$ where $\widehat{J}_{2h} = J_{2h}$ and $\widehat{J}_h^* = J_{3h}$. By the definition of equilibrium (ξ, ω) satisfies the First Order Conditions of the true maximization problem. That implies $pr(\xi, \omega)$ satisfies condition (i) of the definition of Auxiliary equilibrium. Taking into account the definition of equilibria, it is easy to verify the condition iii) of 19 is verified. Condition ii) follows from Lemma in Carosi (1999a) since $\widehat{F}_h^{\widehat{J}_h}$ coincides with $\widetilde{F}_h$. ■

Now let us consider the restriction of the function $\widehat{F}^{\widehat{J}}$ on the set of those economies such that $D_{\xi_h^{\widehat{J}}} \widehat{F}_h \left(\xi_h^{\widehat{J}}, \omega\right)$ has full rank for every h and then define the function $\bar{F} : \widehat{\Xi}^{\widehat{J}} \times \Omega^{\widehat{J}} \to \mathbb{R}^{\dim \widehat{\Xi}} \times \mathbb{R}$ where

$\{\widehat{\Xi}^{\widehat{J}} \times \Omega^{\widehat{J}} = \{\left(\xi^{\widehat{J}}, \omega\right) \in \widehat{\Xi} \times \Omega : D_{\xi_h^{\widehat{J}}}\ \widehat{F}_h^{\widehat{J}}\ \left(\xi_h^{\widehat{J}}, \omega\right)$ has full rank for every $h\}$ and

$$\bar{F} : \left(\widehat{\xi} \times \omega\right) \mapsto \begin{pmatrix} & \left(\widehat{F}_h^{\widehat{J}}\right)_{h \in H} \\ & \sum_{h=1}^{H} \left(x_h^{\backslash} - e_h^{\backslash}\right) \\ & \sum_{h=1}^{H} b_h \\ & \sum_{h=1}^{H} \left(m_h^0 - e_h^{m0}\right) \\ (s > 0) & \sum_{h=1}^{H} \left(-m_h^0 - e_h^{ms} + \sum_{c=1}^{C} \tau_h^{sc} p^{sc} e_h^{sc}\right) \\ & \zeta_h^j \end{pmatrix}$$

The following Theorem shows that the set of economies such that $\bar{F}_\omega^{-1}(0) \neq \emptyset$ has zero measure. If $(\xi, \omega) \in \bar{F}^{-1}(0)$ that means (ξ, ω) satisfies the auxiliary equilibrium conditions and there exists at least one consumer h such that $\zeta_h^j = l_h^j(b_h, m_h^0) = 0$ is verified at least for one asset j. From the true equilibrium side, if $(\xi, \omega) \in \bar{F}^{-1}(0) \cap F^{-1}(0)$ then ξ is such that the function F_ω is not differentiable at ξ.

In order to get the desired result we consider the following set :

$B^{\widehat{J}} = \{\left(\xi^{\widehat{J}}, \omega\right) \in E^{\widehat{J}}$: there exists h and $j_h \in \widehat{J}_h$ such that $\zeta_h^j = l_h^j(b_h, m_h^0) = 0\}$ and the following projection function $\widehat{\pi} : \widehat{\Xi} \times \Omega \to \Omega$, $\widehat{\pi} : \left(\xi^{\widehat{J}}, \omega\right) \mapsto \omega$

Theorem 21 *$\widehat{\pi}\left(B^{\widehat{J}}\right)$ is a zero measure subset of Ω.*

Proof. By means of perturbation methods we can showthat 0 is a regular value of $\bar{F}$ (The reader who is interested in all details can see step a)). Then from the Transversality Theorem there exists a full measure set $\Omega' \subseteq \Omega$ such that, for every $\omega \in \Omega'$, 0 is a regular value for $\bar{F}_\omega$ which is the restriction of $\bar{F}$ on ω. Hence $\bar{F}_\omega^{-1}(0) = \emptyset$ for every $\omega \in \Omega'$. From the definition of $B^{\widehat{J}}$, $\widehat{\pi}\left(B^{\widehat{J}}\right) \subseteq \Omega \backslash \Omega'$ has zero measure

Step a) We must distinguish different cases.

- Case A: there exists at least a $h : (\sharp J_h + 1) \in \widehat{J}_{h2}$
- Case B :for every h we have $\sharp J_h + 1 \notin \widehat{J}_{h2}$.

Case A)

Without any loss of generality we can assume that $(\sharp J_H + 1) \in \widehat{J}_{H2}$ and Mr. H verifies Assumption 12.ii) for good 1; that implies $\Sigma_H^{\backslash 1}$ has full rank. We can assume that $\zeta_h^{j'} = \zeta_H^{\sharp \widehat{J}_{H2}}$ and $\sharp \widehat{J}_{H2} \neq \sharp J_h + 1$.

From the Jacobian of $\bar{F}$ we have:

		ξ_1		G x_H	$\widehat{S}$ λ_H	I b_H	1 m_H^0	$\sharp J_H^{*\setminus}$ ζ_H^*	$\sharp \hat{J}_{H2}^{\setminus}$ ζ_H^2	1 $\zeta_H^{(J_H+1)}$	I q	1 q^{m0}	$G^\setminus$ $e_1^\setminus$	$\widehat{S}$ $e_1^{\bullet 1}$	...	$G^\setminus$ $e_H^\setminus$	$\widehat{S}$ $e_H^{\bullet 1}$	1 c_1^{0m}	..	1 c_H^{0m}	S $\tau_1^{\bullet 1}$	..	S $\tau_H^{\bullet 1}$	S c_1^{1m}	..	S c_H^{1m}
$(1_1)\ G$		$*$																								
$(2_1)\ \widehat{S}$		$*$									$*$	$*$	$\Sigma_1^\setminus$	$\Sigma_1^\mid$				$-q^{0m}$ $\mathbf{0}$			$\Pi_1^{\bullet 1}$			0 I		
	.....																									
$(1_H)\ G$	$D_{x_H} u_H ...$			D_H^2	$-\Phi^T$																					
$(2_H)\ \widehat{S}$	$-\Phi x_H - ..$			$-\Phi$		R	q^m				b_H $\mathbf{0}$	z_H^{m0} $\mathbf{0}$				$\Sigma_H^\setminus$	$\Sigma_H^\mid$			$-q^{0m}$ $\mathbf{0}$			$\Pi_H^{\bullet \mid}$			0 I
$(3_H)\ I$	$\lambda_H R + ..$				R^T	$\overline{\overline{\Upsilon}}_{b_H}$	$\overline{\overline{\Upsilon}}_{b_H m_H^0}$	$\left[D_{b_H} a_H^{*\setminus}\right]^T$	$\left[D_{b_H} a_H^{2\setminus}\right]^T$		$-\lambda_0 I_I$															
$(4_H)\ 1$	$\lambda_H q^m + ...$				$(q^m)^T$	$\left[\overline{\overline{\Upsilon}}_{b_H m_H^0}\right]^T$	$\overline{\overline{\Upsilon}}_{m_H^0}$	$\left[D_{m_H^0} a_H^{*\setminus}\right]^T$	$\left[D_{m_H^0} a_H^{2\setminus}\right]^T$	$\mathbf{1}$		$-\lambda_0$														
$(5_H)\ \sharp J_H^{*\setminus}$	$\iota_H^{*\setminus}$					$D_{b_H} a_H^{2\setminus}$	$D_{m_H^0} a_H^{2\setminus}$																			
$(6_H)\ \sharp \hat{J}_{H2}^{\setminus}$	$\iota_{H2}^{\setminus}$					$D_{b_H} a_H^{*\setminus}$	$D_{m_H^0} a_H^{*\setminus}$																			
$(7_H)\ 1$	$\iota_{H2}^{\sharp J_H+1}$						1																			
$(8)\ G^\setminus$	$\sum_{h=1}^{H} x_h^\setminus - ..$	$0I$		$0I$																						
$(9)\ I$	$\sum_{h=1}^{H} b_h$	I				I							$-I$			$-I$										
$(10)\ 1$	$\sum_{h=1}^{H} m_h^0 - .$	1					1											-1		-1						
$(11)\ S$	$\sum_{h=1}^{H} - m_h^0 ..$	$-\mathbf{1}$					$-\mathbf{1}$						$*$	$*$		$\widetilde{\Sigma}_H^\setminus$	$\widetilde{\Sigma}_H^\mid$				$\widetilde{\Pi}_1^{\bullet 1}$		$\widetilde{\Pi}_H^{\bullet \mid}$	$-I$		$-I$
$(12)\ 1$	$\zeta^{\sharp J_H+1}$									$\mathbf{1}$																

where $\Sigma_h^{\backslash}$ is a matrix $\widehat{S} \times G^{\backslash}$ such that

		$C-1$			...	$C-1$		
		e_h^{02}	...	e_h^{0C}	...	e_h^{S2}	...	e_h^{SC}
1	0	p^{02}	...	p^{0C}				
1	1	$p^{12}\left(1-\tau_h^{12}\right)$	...	$p^{1C}\left(1-\tau_h^{1C}\right)$				
..	...							
1	S					$p^{S2}\left(1-\tau_h^{S2}\right)$	...	$p^{SC}\left(1-\tau_h^{SC}\right)$

while Σ_h^1 is a diagonal matrix $\widehat{S} \times \widehat{S}$ such that

		1	...	1
		e_h^{01}	...	e_h^{S1}
1	0	1	...	
1	1	$p^{11}\left(1-\tau_h^{11}\right)$	...	
..	...		$\ddots$	
1	S			$p^{S1}\left(1-\tau_h^{S1}\right)$

The matrix $\widetilde{\Sigma}_h^{\backslash}$ is a $S \times \left(G-\widehat{S}\right)$ matrix such that

		C	$-$	1	...	C	$-$	1	..	C	$-$	1
		e_h^{02}	..	e_h^{0C}	...	e_h^{12}	..	e_h^{0C}		e_h^{S2}	..	e_h^{SC}
1	1	0	..	0		$p^{12}\tau_h^{12}$		$p^{1C}\tau_h^{1C}$				
..	...								$\ddots$			
1	S	0		0						$p^{S2}\tau_h^{S2}$	..	$p^{SC}\tau_h^{SC}$

and the matrix $\widetilde{\Sigma}_h^1$ has dimension $S \times \widehat{S}$ such that

		1	1	...	1
		e_h^{01}	e_h^{11}	...	e_h^{S1}
1	1	0	$p^{11}\tau_h^{11}$	...	
..	...			$\ddots$	
1	S	0			$p^{S1}\tau_h^{S1}$

$\Pi_h^{\bullet 1}$ is a $\widehat{S} \times S$ matrix such that

		1	...	1
		τ_h^{11}	...	τ_h^{S1}
1	0	0		0
1	1	$-p^{11}e_h^{11}$	...	
..	...		$\ddots$	
1	S			$-p^{S1}e_h^{S1}$

$\widetilde{\Pi}_h^{\bullet 1}$ is a $S \times S$ diagonal matrix such that

		1	...	1
		τ_h^{11}	...	τ_h^{S1}
1	1	$p^{11}e_h^{11}$	...	
..	...		$\ddots$	
1	S			$p^{S1}e_h^{S1}$

Let us consider the submatrix obtained by erasing the superrows (8) .. (12) and by erasing the last two supercolumns. It is known (Carosi 1999a) that this submatrix has full row rank. Then we get the result by using the perturbations method. In order to perturb the (8) .. (10) we erase the last H supercolumns.

Perturbation of (8):

$(8) \longleftarrow \Delta e_H^{\setminus} \rightsquigarrow (2_H) \leftarrow \Delta e_H^{\bullet 1}$

Perturbation of (9) :

We recall that from Assumption 7.iv) and 7.v), for every asset i, there exists there exists at least one consumer h' such that for every $(b_{h'}, m_{h'}^0) \in \mathbb{R}^{I+1}$ $D_{b_{h'}^i} a_{h'}(b_{h'}, m_{h'}^0) = 0$. Then if for every asset i, we perturb the corresponding row of (9) with $\Delta b_{h'}^{\setminus}$, this alter only the row $(2_{h'})$.

Hence we get:

$$(9^i)_{i=1}^I \longleftarrow \Delta b_{h'}^i \rightsquigarrow (2_{h'}^0) \leftarrow \Delta e_{h'}^{0m}$$
$$\rightsquigarrow (2_{h'}^s)_{s=1}^S \leftarrow \Delta \tau_{h'}^{\bullet 1}$$

Perturbation of (10) :

$(10) \longleftarrow \Delta e_H^{m0} \rightsquigarrow (2_H^0) \leftarrow \Delta e_H^{01}$

Perturbation of (11) and (12) :

In order to perturb the last two supperrows we now consider the Jacobian matrix. We consider a supercolumn corresponding to $e_{h^*}^{1m}$ such that h^* has not be used to perturb the previous columns. Without any loss of generality we assume $h^* = 1$. By using the columns e_1^{1m} we erase the columns of $\Sigma_1^{\bullet 1}$ corresponding to $(e_1^{s1})_{s=1}^S$, at superrow (11).

Hence we get the following matrix.

			G	$\widehat{S}$	I	1	$\sharp J^{*}_{H}$	$\sharp \widehat{J}^{\backslash}_{H2}$	$\sharp J_{H+1}$	I	1	$G^{\backslash}$	$\widehat{S}$	..	$G^{\backslash}$	$\widehat{S}$	1	..	1	S	..	S	S	..	S
		ξ_1	x_H	λ_H	b_H	m^0_H	ζ^{*}_{H}	ζ^{2}_{H}	ζ^{2}_{H}	q	q^{m0}	$e^{\backslash}_{1}$	$e^{\bullet 1}_{1}$	..	$e^{\backslash}_{H}$	$e^{\bullet 1}_{H}$	e^{0m}_{1}	..	e^{0m}_{H}	$\tau^{\bullet 1}_{1}$	..	$\tau^{\bullet 1}_{H}$	c^{1m}_{1}	..	c^{1m}_{H}
$(1_1)\ G$		*																							
$(2_1)\ \widehat{S}$		*								*	*	$\Sigma^{\backslash}_{1}$	p^{01} 0 0				$-q^{0m}$ 0			$\Pi^{\bullet 1}_{1}$			0 I		
	.....																								
$(1_H)\ G$	D_{x_H}		D^2_H	$-\Phi^T$																					
$(2_H)\ \widehat{S}$	$-\Phi x_H-$		$-\Phi$		R	q^m				b_H 0	z^{m0}_{H} 0				$\Sigma^{\backslash}_{H}$	Σ^{1}_{H}			$-q^{0m}$ 0			$\Pi^{\bullet 1}_{H}$			0 I
$(3_H)\ I$	$\lambda_H R+..$			R^T	$\overline{\overline{\Upsilon}}_{b_H}$	$\overline{\overline{\Upsilon}}_{b_H m^0_H}$	$[D_{b_H} a^{*}_{H}]^T$	$[D_{b_H} a^{2}_{H}]^T$		$-\lambda_0 I_I$															
$(4_H)\ 1$	$\lambda_H q^m+.$			$(q^m)^T$	$\left[\overline{\overline{\Upsilon}}_{b_H m^0_H}\right]^T$	$\overline{\overline{\Upsilon}}_{m^0_H}$	$\left[D_{m^0_H} a^{*}_{H}\right]^T$	$\left[D_{m^0_H} a^{2}_{H}\right]^T$	1		$-\lambda_0$														
$(5_H)\ \sharp J^{*}_{H}$	l^{*}_{H}				$D_{b_H} a^{2}_{H}$	$D_{m^0_H} a^{2}_{H}$																			
$(6_H)\ \sharp \widehat{J}^{\backslash}_{H2}$	l_{H2}				$D_{b_H} a^{*}_{H}$	$D_{m^0_H} a^{*}_{H}$																			
$(_7H)$	$l^{\sharp J_{H+1}}_{H}$					1																			
$(8)\ G$	$\sum_{h=1}^{H} x^{\backslash}_{h}-.$	$0I$	$0I$																						
$(9)\ I$	$\sum_{h=1}^{H} b_h$	I			I							$-I$			$-I$										
$(10)\ 1$	$\sum_{h=1}^{H} m^0_h-$	1				1											-1		-1						
$(11)\ S$	$\sum_{h=1}^{H} m^0_h..$	-1				-1						$\widetilde{\Sigma}^{\backslash}_{1}$	$\Phi^{\bullet 1}$		$\widetilde{\Sigma}^{\backslash}_{H}$	$\widetilde{\Sigma}^{1}_{H}$				$\widetilde{\Pi}^{\bullet 1}_{1}$		$\widetilde{\Pi}^{\bullet 1}_{H}$	$-I$		$-I$
$(12)\ 1$	$\zeta^{\sharp J_H+1}$								1																

where $\Phi^{\bullet 1}$ is a $S\times\widehat{S}$ matrix such that

$$\Phi^{\bullet 1} = \begin{bmatrix} 0 & & & \\ & p^{11} & & \\ & & \ddots & \\ & & & p^{S1} \end{bmatrix}$$

Let us consider the submatrix obtained by erasing the last two superrows of the new matrix. We have previously stated this submatrix has full row rank. We are left to perturb the last superrows.

Perturbation of (11) :

$(11) \longleftarrow (\Delta e_1^{\bullet s1})_{s=1}^{S}$

Perturbation of (12) :

(12)	$\longleftarrow$	$\Delta\zeta_H^{\sharp J_H+1}$	$\rightsquigarrow$	(4_H)	$\leftarrow$	$\Delta\lambda_H^0$	$\rightsquigarrow$	(3_H^0) (4_H^0) (1_H)
$(3_H^0)^{(*)}$ (4_H^0)	$\longleftarrow$	$(\Delta\lambda_H^s)_{s\neq 0}$	$\rightsquigarrow$	(1_H)				
(1_H)	$\longleftarrow$	Δx_H^1	$\rightsquigarrow$	(2_H) (8)				
(8)	$\leftarrow$	$\Delta e_H^{\setminus}$	$\rightsquigarrow$	(2_H) (11)				
(2_H)	$\leftarrow$	$\Delta e_H^{\bullet 1}$	$\rightsquigarrow$	(11)	$\leftarrow$	$\Delta e_1^{\bullet 1}$		

(*) Since *Rank* $(R, q^m) = I+1$ we can adjust simultaneously row (3_H) and (4_H).

Case B

Without any loss of generality we assume that $\zeta_H^{\sharp J_H} \in \hat{J}_{H2}^{\setminus}$. From the Jacobian of $\bar{F}$ we have:

				G	$\widehat{S}$	I	1	$\sharp J^{*\backslash}_H$	$\sharp \hat{J}^{\backslash}_{H2}$	I	1	$G^{\backslash}$	$\widehat{S}$		$G^{\backslash}$	$\widehat{S}$	1	..	1	S	..	S	S	..	S
		ξ_1		x_H	λ_H	b_H	m^0_H	ζ^*_H	ζ^2_H	q	q^{m0}	$e^{\backslash}_1$	$e^{\bullet 1}_1$	...	$e^{\backslash}_H$	$e^{\bullet 1}_H$	e^{0m}_1		e^{0m}_H	$\tau^{\bullet 1}_1$	..	$\tau^{\bullet 1}_H$	e^{1m}_1	..	e^{1m}_H
$(1_1)\ G$		*																							
$(2_1)\ \widehat{S}$		*								*	*	$\Sigma^{\backslash}_1$	Σ^1_1				$-q^{0m}$ $\mathbf{0}$			$\Pi^{\bullet 1}_1$			0 I		
	.....																								
$(1_H)\ G$	$D_{x_H} u_H$...			D^2_H	$-\Phi^T$																				
$(2_H)\ \widehat{S}$	$-\Phi x_H -$ ..			$-\Phi$		R	q^m			b_H $\mathbf{0}$	z^{m0}_H $\mathbf{0}$				$\Sigma^{\backslash}_H$	Σ^1_H			$-q^{0m}$ $\mathbf{0}$			$\Pi^{\bullet 1}_H$			0 I
$(3_H)\ I$	$\lambda_H R +$ ..				R^T	$\overline{\overline{\Upsilon}}_{b_H}$	$\overline{\overline{\Upsilon}}_{b_H m^0_H}$	$\left[D_{b_H} a^{*\backslash}_H\right]^T$	$\left[D_{b_H} a^{2\backslash}_H\right]^T$	$-\lambda_0 I_I$															
$(4_H)\ 1$	$\lambda_H q^m +$ ...				$(q^m)^T$	$\left[\overline{\overline{\Upsilon}}_{b_H m^0_H}\right]^T$	$\overline{\overline{\Upsilon}}_{m^0_H}$	$\left[D_{m^0_H} a^{*\backslash}_H\right]^T$	$\left[D_{m^0_H} a^{2\backslash}_H\right]^T$		$-\lambda_0$														
$(5_H)\ \sharp J^{*\backslash}_H$	$l^{*\backslash}_H$					$D_{b_H} a^{2\backslash}_H$	$D_{m^0_H} a^{2\backslash}_H$																		
$(6_H)\ \sharp \hat{J}^{\backslash}_{H2}$	$l^{\backslash}_{H2}$					$D_{b_H} a^{*\backslash}_H$	$D_{m^0_H} a^{*\backslash}_H$																		
$(8)\ G^{\backslash}$	$\sum_{h=1}^{H} x^{\backslash}_h -$.	$0I$		$0I$																					
$(9)\ I$	$\sum_{h=1}^{H} b_h$	I				I						$-I$			$-I$										
$(10)\ 1$	$\sum_{h=1}^{H} m^0_h -$.	1					1										-1		-1						
$(11)\ S$	$\sum_{h=1}^{H} - m^0_h$..	-1					-1					*	$\Phi^{\cdot 1}$		$\widetilde{\Sigma}^{\backslash}_H$	$\widetilde{\Sigma}^1_H$				$\widetilde{\Pi}^{\bullet 1}_1$		$\widetilde{\Pi}^{\bullet 1}_H$	$-I$		$-I$
$(12)\ 1$	$\zeta^{\sharp J_H}$								0 1																

We first consider the submatrix obtained by erasing row (12). From case A we know this has full row rank, hence we are left to perturb only the last row.

$$\begin{array}{ccccccccc}
(12) & \longleftarrow & \Delta\zeta_H^{\sharp J_H} & \rightsquigarrow & \begin{array}{c}(3_H)\\(4_H)\end{array} & \leftarrow & \Delta\lambda_H & \rightsquigarrow & (1_H) \\
(1_H) & \longleftarrow & \Delta x_H^1 & \rightsquigarrow & \begin{array}{c}(2_H)\\(8)\end{array} & \longleftarrow & & & \\
(8) & \leftarrow & \Delta e_H^{\backslash} & \rightsquigarrow & \begin{array}{c}(2_H)\\(11)\end{array} & & & & \\
(2_H) & \leftarrow & \Delta e_H^{\bullet 1} & \rightsquigarrow & (11) & \leftarrow & \Delta e_1^{\bullet 1} & &
\end{array}$$

■

Definition 22 *B is the set of equilibria such that F is not differentiable , i.e $B = \{(\xi, \omega) \in E$: there exists at least a h and j such that $\zeta_h^j = 0$, $l_h^j(b_h, m_h^0) = 0\}$.*

Theorem 23 *There exists an open and full measure set $\widetilde{\Omega} \subseteq \Omega$ such that F_ω is differentiable for every $\xi \in F_\omega^{-1}(0)$.*

Proof. From Theorem 20 if $(\xi, \omega) \in E$ then there exists a set of indexes $\widehat{J}$ such that $pr(\xi, \omega) \in E^{\widehat{J}}$, i.e $pr(\xi, \omega)$ is an auxiliary equilibria. From Theorem 21 $\pi\left(B^{\widehat{J}}\right)$ is a set of measure zero on Ω.

If $\widehat{J} \neq \widehat{J}^*$, $B^{\widehat{J}}$ and $B^{\widehat{J}^*}$ are contained in different euclidean spaces. With this regard we introduce the function $in : \widehat{\Xi}^{\widehat{J}} \times \Omega \to \Xi \times \Omega$, $in : \left(\xi^{\widehat{J}}, \omega\right) \mapsto (\xi, \omega)$ where ξ has the same component of $\xi^{\widehat{J}}$ and 0 in correspondence of the missing ones. (Note that $in(\xi, \omega) = pr^{-1}(\xi, \omega)$ for every $(\xi, \omega) \in E$). The set $B \subseteq \bigcup_{\widehat{J} \in \widehat{\mathfrak{P}}} in\left(B^{\widehat{J}}\right) \subseteq \bigcup_{\widehat{J} \in \mathfrak{J}} in\left(B^{\widehat{J}}\right)$ where $\widehat{\mathfrak{P}} = \{\widehat{J} \in \mathfrak{J} : :pr(\xi, \omega) \in E^{\widehat{J}}, (\xi, \omega) \in E\}$.

That implies $\pi(B) \subseteq \pi\left(\bigcup_{\widehat{J} \in \widehat{\mathfrak{P}}} inB^{\widehat{J}}\right)$ and $\pi\left(\bigcup_{\widehat{J} \in \widehat{\mathfrak{P}}} inB^{\widehat{J}}\right) = \widehat{\pi}\left(\bigcup_{\widehat{J} \in \widehat{\mathfrak{P}}} B^{\widehat{J}}\right)$.Hence $\pi\left(\bigcup_{\widehat{J} \in \widehat{\mathfrak{P}}} inB^{\widehat{J}}\right)$ has zero measure since it is the union of zero measure sets. Hence $\pi(B)$ has zero measure. The closure of $\pi(B)$ follows from the continuity of function F and from the properness of the function π. ■

Since the function F is differentiable on the full measure, open set $\widetilde{\Omega}$, regularity analysis is restricted to $\widetilde{\Omega}$.

Theorem 24 *0 is a regular value of F_ω, for every $\omega \in \widetilde{\Omega}$.*

Proof. We claim that the Jacobian of F_ω has full row rank. Let us consider the following partition on J_h

$$\begin{array}{l} J_h^1 = \{j \in J_h : \zeta_h^j > 0, l_h^j\,(b_h, m_h^0) = 0\} \\ J_h^2 = \{j \in J_h : \zeta_h^j = 0,\ l_h^j\,(b_h, m_h^0) = 0\} \\ J_h^3 = \{j \in J_h : \zeta_h^j = 0, l_h^j\,(b_h, m_h^0) > 0\} \end{array} .$$

Hence we have the following matrix.

			G	$\widehat{S}$	I	1	$\sharp J_H^{1\backslash}$	$\sharp \hat{J}_H^{3\backslash}$	1		$\widehat{S}$		$G^\backslash$	$\widehat{S}$	1	..	1	S	..	S	..	S
	ξ_1	..	x_H	λ_H	b_H	m_H^0	ζ_H^1	ζ_H^3	$\zeta_H^{J_H+1}$	...	$e_1^{\bullet 1}$	...	$e_H^\backslash$	$e_H^{\bullet 1}$	e_1^{0m}		e_H^{0m}	$\tau_1^{\bullet 1}$	..	$\tau_H^{\bullet 1}$		e_H^{1m}
$(1_1)\ G$	$*$																					
$(2_1)\ \widehat{S}$	$*$										Σ_1^1				$-q^{0m}$ $\mathbf{0}$			$\Pi_1^{\bullet 1}$				
$(1_H)\ G$			D_H^2	$*$																		
$(2_H)\ \widehat{S}$			$-\Phi$		R	q^m							$\Sigma_H^\backslash$	Σ_H^1			$-q^{0m}$ $\mathbf{0}$			$\Pi_H^{\bullet 1}$		0 I
$(3_H)\ I$				$*$	$*$	$*$	$*$	$*$														
$(4_H)\ 1$				$*$	$*$	$*$	$*$	$*$	1													
$(5_H)\ \sharp J_H^{1\backslash}$					$*$	$*$																
$(6_H)\ \sharp \hat{J}_H^{3\backslash}$								I														
$(7_H)\ 1$						$*$			$*$													
$(8)\ G^\backslash$	$*$		$0I$										$-I$									
$(9)\ I$	$*$				I																	
$(10)\ 1$	$*$					1									-1		-1					
$(11)\ S$	$*$					$\mathbf{-1}$					$*$		$\widetilde{\Sigma}_H^\backslash$	$\widetilde{\Sigma}_H^1$				$\widetilde{\Pi}_1^{\bullet 1}$		$\widetilde{\Pi}_H^{\bullet 1}$		$-I$

where according to the partition on J_h we have considered $\zeta_h = \left[\mu_h^1, \mu_h^2, \mu_h^3, \zeta_h^{J_{H+1}}\right]$ and $l_h\left(b_{h,} m_h^0\right) = \left[l_h^1\left(b_{h,} m_h^0\right), l_h^2\left(b_{h,} m_h^0\right), l_h^3\left(b_{h,} m_h^0\right), l_h^{J_{H+1}}\left(b_{h,} m_h^0\right)\right]$,

$$\begin{array}{lcl} \zeta_h^1 \gg 0 & & l_h^1\left(b_{h,} m_h^0\right) = 0 \\ \zeta_h^2 = 0 & \text{and} & l_h^2\left(b_{h,} m_h^0\right) = 0 \\ \zeta_h^3 = 0 & & l_h^3\left(b_{h,} m_h^0\right) \gg 0 \end{array} ,$$

$\Sigma_h^{\backslash}$ is a matrix $\widehat{S} \times G^{\backslash}$ such that

		$C - 1$			...	$C - 1$		
		e_h^{02}	...	e_h^{0C}	...	e_h^{S2}	...	e_h^{SC}
1	0	p^{02}	...	p^{0C}				
1	1	$p^{12}\left(1-\tau_h^{12}\right)$	...	$p^{1C}\left(1-\tau_h^{1C}\right)$				
..	...							
1	S					$p^{S2}\left(1-\tau_h^{S2}\right)$	...	$p^{SC}\left(1-\tau_h^{SC}\right)$

while Σ_h^1 is a diagonal matrix $\widehat{S} \times \widehat{S}$ such that

	e_h^{01}	...	e_h^{S1}
0	1	...	
1	$p^{11}\left(1-\tau_h^{11}\right)$	...	
...		$\ddots$	
S			$p^{S1}\left(1-\tau_h^{S1}\right)$

The matrix $\widetilde{\Sigma}_h^{\backslash}$ is a $S \times \left(G - \widehat{S}\right)$ matrix such that

	e_h^{02}	..	e_h^{0C}	...	e_h^{12}	..	e_h^{0C}		e_h^{S2}	..	e_h^{SC}
1	0	..	0		$p^{12}\tau_h^{12}$		$p^{1C}\tau_h^{1C}$				
...								$\ddots$			
S	0		0						$p^{S2}\tau_h^{S2}$	..	$p^{SC}\tau_h^{SC}$

and the matrix $\widetilde{\Sigma}_h^1$ has dimension $S \times \widehat{S}$ such that

	e_h^{01}	e_h^{11}	...	e_h^{S1}
1	0	$p^{11}\tau_h^{11}$	...	
...			$\ddots$	
S	0			$p^{S1}\tau_h^{S1}$

$\Pi_h^{\bullet 1}$ and $\widetilde{\Pi}_h^{\bullet 1}$ are respectively

$$\begin{array}{c|c|c|c|} & \tau_h^{11} & \cdots & \tau_h^{S1} \\ \hline 0 & 0 & & 0 \\ \hline 1 & -p^{11}e_h^{11} & \cdots & \\ \hline \cdots & & \ddots & \\ \hline S & & & -p^{S1}e_h^{S1} \\ \hline \end{array} \quad \text{and} \quad \begin{array}{c|c|c|c|} & \tau_h^{11} & \cdots & \tau_h^{S1} \\ \hline 1 & p^{11}e_h^{11} & \cdots & \\ \hline \cdots & & \ddots & \\ \hline S & & & p^{S1}e_h^{S1} \\ \hline \end{array}$$

Without any loss of generality we can assume that H verifies Assumption 12 ii) for good 1 and therefore the matrix $\Sigma_H^{\backslash 1}$ has full rank.

Consider the submatrix obtained by deleting the superrows $(8)-(12)$.

It is known that this submatrix has full rank for every h (see Carosi 1999a) and so the submatrix we have previously created has full row rank. In order to obtain the desired result we use the so called perturbation methods. We are left to perturb the last four superrows.

Perturbation of (8):

$(8) \longleftarrow \Delta e_H^{\backslash} \rightsquigarrow (2_H) \leftarrow \Delta e_H^{\bullet 1}$

Where " $(8) \longleftarrow \Delta e_H^{\backslash}$ " means "we use $\Delta e_H^{\backslash}$ in order to perturb the superrow (8)", while " $\rightsquigarrow (2_H)$ " says "this alter the superrow (2_H)".

Perturbation of (9) :

We recall that from Assumption 7 iv) and 7.v), for every asset i, there exists at least one consumer h' such that for every $(b_{h'}, m_{h'}^0) \in \mathbb{R}^{I+1}$ $D_{b_{h'}^i} a_{h'}(b_{h'}, m_{h'}^0) = 0$. Then if for every asset i, we perturb the corresponding row of (9) with $\Delta b_{h'}^{\backslash}$, this alter only the row $(2_{h'})$.

Hence we get:

$$\begin{array}{lllll} (9^i)_{i=1}^I \longleftarrow & \Delta b_{h'}^i & \rightsquigarrow & (2_{h'}^0) & \leftarrow \Delta e_{h'}^{0m} \\ & & \rightsquigarrow & (2_{h'}^s)_{s=1}^S & \leftarrow \Delta \tau_{h'} \end{array}$$

Perturbation of (10) :

$(10) \longleftarrow \Delta e_H^{m0} \rightsquigarrow (2_H^0) \leftarrow \Delta e_H^{01}$

Perturbation of (11) :

We consider a supercolumn corresponding to $e_{h^*}^{1m}$ such that h^* has not be used to perturb the previous columns. By using the columns $e_{h^*}^{1m}$ we erase the columns of $\Sigma_{h^*}^{\bullet 1}$ corresponding to $(e_{h^*}^{s1})_{s=1}^S$, at superrow (11). Without any loss of generality we assume $h^* = 1$.

Let us consider the submatrix obtained by erasing the last two superrows of the new matrix. We have previously stated that this submatrix has full row rank. We are left to perturb the last superrow.

Perturbation of (11) :

$(11) \longleftarrow (\Delta e_1^{\bullet s1})_{s=1}^S$ ■

Theorem 25 (Regularity) *There exists an open and full measure set* $\widetilde{\Omega}$ *such that*

i) $\sharp F^{-1}(0)$ *is finite*

ii) for every $(\xi,\omega) \in F^{-1}(0)$, *there exists an open set* $U \subseteq \Xi \times \Omega$ *of* (ξ,ω), *such that the restriction of the function* π *on* $U \cap (F^{-1}(0))$ *is a diffeomorphism*

Proof. i)- ii) From the previous Theorems, the hypothesis of the Transversality Theorem are satisfied and so the thesis follows as an easy application of Transversality Theorem. ■

5 Equilibria and Pareto Optima

As in the standard model with incomplete markets, Pareto Optimal equilibria are rare. We first recall a well known necessary condition in order to guarantee the efficiency of equilibrium allocations.

Lemma 26 *Let be* $\omega^* \in \Omega$ *an economy such that* $e^* \in \mathbb{R}^{GH}_{++}$ *is a Pareto Optima allocation and let be* $\xi \in \Xi$ *a vector such that* $F(\xi,\omega^*) = 0$.

Then for every states s, s' *and every* h, h' *we have:*

$$\frac{\lambda_h^s}{\lambda_h^{s'}} = \frac{\lambda_{h'}^s}{\lambda_{h'}^{s'}}$$

Take into account the above fact we have the following result.

Theorem 27 *There exists a full measure and open subset* $\Omega^{IN} \subseteq \Omega$, *such that the equilibrium allocations associated with every* $\omega \in \Omega^{IN}$ *are not Pareto Optima.*

Proof. From Theorem 24 we know there exists a full measure and open subset $\widetilde{\Omega} \subseteq \Omega$ such that JF has full row rank for every $(\omega,\xi) \in F^{-1}(0)$.

We show that the set Ω^{IN} has full measure by applying the Transversality Theorem to the following function

$$F^{IN} : \Xi \times \widetilde{\Omega} \to \mathbb{R}^{n'} \times \mathbb{R}$$

$$F^{IN} : (\xi,\omega) \mapsto \left(F_\omega(\xi), \lambda_H^s - \frac{\lambda_1^s}{\lambda_1^{s*}} \lambda_H^{s*} \right)$$

Where s is a state whose corresponding column in $[R, q^m]^T$ is linearly dependent from the others. (Without any loss of generality we can assume that $s = 1$ and $s^* = 0$). By using perturbation methods we get JF^{IN} has full row rank. (The interested reader can find a detailed proof in Step1.)

From Transversality Theorem we can conclude that there exists a full measure and open set $\Omega^{IN} \subseteq \widetilde{\Omega} \subseteq \Omega$ such that 0 is a regular value for F_ω^{IN} where $\omega \in \Omega^{IN}$. Since the number of equations in system $F_\omega^{IN}(\xi) = 0$ is greater then the number of unknowns, by the Preimage Theorem $0 \notin F_\omega^{IN}(\xi)$ for every $\omega \in \Omega^{IN}$. Then the optimality necessary condition is not satisfied and consequently for every $\omega \in \Omega^{IN}$ the associated equilibria are not Pareto Optima. Finally consider the set

$B^{IN} = \{(\xi, \omega) \in \Xi \times \Omega : F^{IN}(\xi, \omega) = 0, \text{ rank } JF^{IN} < n' + 1\}$. Since B^{IN} is a closed subset of $F^{-1}(0)$ the openness of Ω^{IN} follows from the properness of the function π.

Step1. JF^{IN} has full row rank.

We compute the rank of JF^{IN}. From Theorem 24, JF^{IN} if and only if the following matrix $\overline{JF}^{IN}$ has full row rank.

		ξ_1	G x_H	$\widehat{S}$ λ_H	I b_H	1 m^0_H	$\sharp J^{1\backslash}_H$ $\mu^{1\backslash}_H$	$\sharp j^{3\backslash}_H$ $\mu^{3\backslash}_H$	1 $\gamma^{(J_H+1)}_H$	I q	1 q^{m0}	$G^\backslash$ $e^\backslash_1$	$\widehat{S}$ $e^{\bullet 1}_1$	..	$G^\backslash$ $e^\backslash_H$	$\widehat{S}$ $e^{\bullet 1}_H$	1 e^{0m}_1	..	1 e^{0m}_H	S $\tau^{\bullet 1}_1$	..	S $\tau^{\bullet 1}_H$	S e^{1m}_1	S e^{1m}_H
$(1_1)\ G$		*																						
$(2_1)\ \widehat{S}$		*								*	*	$\Sigma^\backslash_1$	$\mathbf{1}$ $\mathbf{0}$				$-q^{0m}$ $\mathbf{0}$			$\Pi^{\bullet 1}_1$			0 I	
	.....																							
$(1_H)\ G$	$D_{x_H}u_H$..		D^2_H	$-\Phi^T$																				
$(2_H)\ \widehat{S}$	$-\Phi x_H$		$-\Phi$		R	q^m				b_H $\mathbf{0}$	z^{m0}_H $\mathbf{0}$				$\Sigma^\backslash_H$	Σ^1_H			$-q^{0m}$ $\mathbf{0}$			$\Pi^{\bullet 1}_H$		0 I
$(3_H)\ I$	$\lambda_H R+.$			$\tilde{R}^T$	β_{b_h}	β_{b_h,m^0_h}	$\left[D_{b_H}a^{1\backslash}_H\right]^T$	$\left[D_{b_H}a^{1\backslash}_H\right]^T$		$-\lambda_0 I$														
$(4_H)\ 1$	$\lambda_H q^m$			$(\bar{q}^m)^T$	$\left(\beta_{b_h,m^0_h}\right)^T$	$\beta_{m^0_h}$	$\left[D_{m^0_H}a^{3\backslash}_H\right]^T$	$\left[D_{m^0_H}a^{3\backslash}_H\right]^T$	$\mathbf{1}$		$-\lambda_0$													
$(5_H)\ \sharp J^{1\backslash}_H$	$\min a^{1\backslash}_H,$				$\left[D_{b_H}a^{1\backslash}_H\right]$	$\left[D_{b_H}a^{1\backslash}_H\right]$																		
$(6_H)\ \sharp \hat{j}^{3\backslash}_H$	$\min a^{3\backslash}_H,$							I																
$(7_H)\ \mathbf{1}$	$\min m^0_H,$					$\lambda_h[m^0_h=0]$			$\lambda_h[\gamma^0_h=0]$															
$(8)\ G^\backslash$	$\sum_{h=1}^{H} x^\backslash_h$	$0I$	$0I$									$-I$			$-I$									
$(9)\ I$	$\sum_{h=1}^{H} b_h$	I			I																			
$(10)\ 1$	$\sum_{h=1}^{H} m^0_h.$	1				1											$\mathbf{-1}$		$\mathbf{-1}$					
$(11)\ S$	$\sum_{h=1}^{H} -m^0_h$	$\mathbf{-1}$				$\mathbf{-1}$						$\tilde{\Sigma}^\backslash_1$	$\Phi^{\bullet 1}$		$\tilde{\Sigma}^\backslash_H$	$\tilde{\Sigma}^1_H$				$\tilde{\Pi}^{\bullet 1}_1$		$\tilde{\Pi}^{\bullet 1}_H$	$-I$	$-I$
$(12)\ 1$	λ^1_H..			$0,1,..$																				

where $\tilde{\Phi}$, $\begin{bmatrix} \tilde{R}^T \\ (\tilde{q}^m)^T \end{bmatrix}$ are Φ and $\begin{bmatrix} R^T \\ (q^m)^T \end{bmatrix}$ after elementary row operation performed in order to eliminate $\frac{\lambda_1^1}{\lambda_1^0}$ from row (12). Note that $rank \begin{bmatrix} \tilde{R}^T \\ (\tilde{q}^m)^T \end{bmatrix} = rank \begin{bmatrix} R^T \\ (q^m)^T \end{bmatrix}$.

We consider the submatrix $\overline{JF}^{PO}$ obtained by erasing the last row. From Theorem 24 we know that $\overline{JF}^{PO}$ has full row rank. We are left to perturb the last row:

$$\begin{array}{lllll}
(12) & \leftarrow & (\Delta\lambda_H^1) & \rightsquigarrow & (1_H^1) \\
 & & & \rightsquigarrow & \begin{array}{c}(3_H)\\(4_H)\end{array} \leftarrow (\Delta\lambda_H^s)_{s\neq 1} \rightsquigarrow (1_H) \\
(1_H) & \leftarrow & (\Delta x_H) & \rightsquigarrow & \begin{array}{c}(2_H)\\(8)\end{array} \\
(8) & \leftarrow & \left(\Delta e_H^{\backslash}\right) & \rightsquigarrow & (2_H) \\
(2H) & \leftarrow & (\Delta e_H^1) & \rightsquigarrow & (11) \\
(11) & \leftarrow & (\Delta e_1^{\bullet 1}) & &
\end{array}$$

Hence JF^{IN} has full row rank. By applying the Transversality Theorem we can conclude that there exists a full measure and open set $\Omega^{IN} \subseteq \widetilde{\Omega} \subseteq \Omega$ such that 0 is a regular value for F_ω^{IN} where $\omega \in \Omega^{IN}$. Since the number of equations in system $F_\omega^{IN}(\xi) = 0$ is greater then the number of unknowns, by the Preimage Theorem $0 \notin F_\omega^{IN}(\xi)$ for every $\omega \in \Omega^{IN}$. Then the optimality necessary condition is not satisfied and consequently for every $\omega \in \Omega^{IN}$ the associated equilibria are not Pareto Optima.

Step b). Ω^{IN} is an open set

Let $B^{IN} = \{(\xi,\omega) \in \Xi \times \Omega : F^{IN}(\xi,\omega) = 0, \text{ rank } JF^{IN} < n' + 1\}$. From the rank condition we get that the determinant of every square submatrix of JF^{IN} whose dimension is $n' + 1$, is zero. Since the function determinant is continuous, B^{IN} is a closed subset of $F^{-1}(0)$. Since the function π is proper, the thesis follows immediately.

■

6 Monetary policy and Pareto Improvability.

We suppose that the policy maker can modify only the amount of money endowments of one consumer, say consumer H, in period 1. Let T_1 be the set of independent instruments of monetary policy whose generic element is $t = (t_1^s)_{s=1}^S$. The planner does not have to respect any constraint and so the space of policy instruments collapses with the space of independent instruments.

We perform a quadratic perturbation the function u_h. So we construct the a finite dimensional subset $\mathcal{A}_u \equiv \prod \mathcal{A}_{u_h}$ of $\mathcal{U}$. With this regard, the reader can see Citanna, Kajii and Villanacci (1998).

The monetary intervention modifies both the maximization problem of household 1 and the market clearing conditions. We can then define a new equilibria that takes into account the planner's intervention as follows.

Definition 28 *Given an economy $(\omega, u) \in \Omega \times \mathcal{U}$ ξ is a vector of equilibrium endogenous variables with respect to "an economy with planner intervention t " if and only if $F_{pl}(\xi, t, \omega, u) = 0$ where F_{pl} is defined as follows:*

$$F_{pl} : \Xi \times T \times \Omega \times \mathcal{U} \to \mathbb{R}^n \qquad F_{pl} : (\xi, t, \omega, u) \mapsto$$

$$\left(\begin{array}{lr} Foc_H & \left(\begin{array}{r} D_{x_1} u_1(x_1) - \lambda_1 \Phi \\ -\Phi(x_1 - e_1) + q^m m_1^0 + \hat{U}\left(e_1^m + [0, t_1]^T\right) - \Psi(\tau_1, p) e_1 + R b_1 \\ \lambda_1 R + \mu_1 D_{b_1} a_1(b_1, m_1^0) \\ (\forall j \in J_1) \qquad \min\left[\mu_1^j, a_1^j(b_1, m_1^0)\right] \\ \lambda_1 q^m + \mu_1 D_m a_1(b_1, m_1^0) + \gamma_1 \\ \min\left[\gamma_1, m_1^0\right] \end{array} \right) \\ (Foc_h)_{h \neq H} = (\text{left side of equations } 5)_{h \neq H} & \\ (M1) & \sum_{h=1}^{H} \left(x_h^\backslash - e_h^\backslash\right) \\ (M2) & \sum_{h=1}^{H} b_h \\ (M3) & \sum_{h=1}^{H} (m_h^0 - e_h^{m0}) \\ (M4) & \sum_{h=1}^{H} \left(-m_h^0 - e_h^{ms} + \sum_{c=1}^{C} \tau_h^{sc} p^{sc} e_h^{sc} + t_1^s\right) (s > 0) \end{array} \right)$$

F_{pl} is a variation of F. More precisely the function $\psi : \Xi \to \mathbb{R}^S$ is the constant function $\psi : \xi \mapsto 0$. Hence $F_{eq}(\xi, \omega, u) = 0 \Leftrightarrow F^{pl}(\xi, 0, \omega, u) = 0$.

From the regularity result we have that there exists an open and dense set $\mathcal{E}_r \subseteq \Omega \times \mathcal{U}$ that verifies the following properties :

for every $(\omega, u) \in \mathcal{E}_r$, $F_{eq}(\xi, \omega, u) = 0 \Rightarrow rank D_\xi \tilde{F}(\xi, \omega, u) = n$.

Finally the properness of the projection function $\pi^{eq} : \Xi \times \Omega \times \mathcal{U} \to \Omega \times \mathcal{U}$, $\pi^{eq} : (\xi, \omega, u) \mapsto (\omega, u)$ can be easily checked.

We recall a very well known result about pareto Improvability. The reader can find the proof in Citanna, Kajii and Villanacci (1998). and Cass (1995)

.

Proposition 29 *There exists an open and dense set $\mathcal{E}_I$ in $\Omega \times \mathcal{U}$ such that for any $(\omega, u) \in \mathcal{E}_I$, every associated equilibrium ξ is Pareto Improvable if one of the following conditions holds:*

i) The following system has no solution

$$\left\{ \begin{array}{lll} F(\xi, \omega, u) & = 0 & (1) \\ \left[D_{\xi,t}\left(F_{pl}\left(\xi, \psi(\xi), \omega, u\right), U\left(\xi, \psi(\xi), \omega, u\right)\right)\right]^T \kappa & = 0 & (2) \\ \frac{1}{2}\kappa^T \kappa - 1 & = 0 & (3) \end{array} \right. \qquad (10)$$

ii) There exists a subset D^* which is dense in $\Omega \times \mathcal{U}$ and such that for every $(\omega, t) \in D^*$, the matrix $D\widetilde{F}_{\mathcal{A}_u}$

	ξ	κ	A	ω
$F\left(\xi, \rho, \left(u\left(.; A_h\right)\right)_{h=1}^H\right)$	$D_\xi F$	0	0	$*$
$\left[D_{\xi,t}\left(F_{pl}, U\right)\right]\kappa$	$*$	$\left[D_{\xi,t}\left(F_{pl}, U\right)\right]^T$	$N(\kappa_x)$	$*$
$1\backslash 2\kappa^T\kappa - 1$	0	κ	0	0

(11)

has full row rank.

iii) There exists a subset D^* which is dense in $\Omega \times \mathcal{U}$ and such that for every $(\omega, t) \in D^*$, the matrix

$$M\left(\xi, \omega, \left(u\left(.; A_h\right)\right)_{h=1}^H\right) \equiv \left[\begin{array}{cc} \left[D_{\xi,t}\left(F_{pl}, U\right)\right]^T & N(\kappa_x) \\ \kappa & 0 \end{array} \right] \qquad (12)$$

has full rank

The following result allows us to state that there exists an open and dense set of economies such that equilibria are Pareto Improvable. That has a remarkable consequences: even a limited monetary policy has real effects, i.e. money is not neutral.

Theorem 30 (Pareto Improvability) *Suppose that the policy maker can modify only the amount of money endowments of one consumer, in period 1. Then there exists an open and dense set $\mathcal{E}_I$ in $\Omega \times \mathcal{U}$ such that for any $(\omega, u) \in \mathcal{E}_I$, every associated equilibrium ξ is Pareto Improvable.*

Proof. We are going to show one of the conditions of previous theorem is verified. The proof is quite long and not so easy to read. Hence we split it in several different parts. We describe the strategy of the proof and we give details in the following theorems. Note that we are dealing even with endogenous variables and that enforced us to distinguish several cases. The submatrix $N\kappa_x$ depends on κ_{x_h} $\forall h$ (see also Citanna, Kajii and Villanacci (1998).) and according to this, we consider the following :

i) CASE 1. $\kappa_{x_h} \neq 0$ for every h, that is $N\kappa_{x_h}$ has full row rank for every h. In this case, we have the desired result by showing matrix $D\widetilde{F}_{A_u}$ (see 11) has full rank. This is proved in Theorem 31.

ii) CASE 2. $\kappa_{x_h} = 0$ for every h, that is $N\kappa_{x_h}$ does not have full row rank for every h. In theorem 33 we show that system 10 has no solution and then we get the desired result

iii)CASE 3. There exists at least an h such that $\kappa_{x_h} = 0$, that is $N\kappa_{x_h}$ does not have full rank. If there exists at least a consumer such that $N\kappa_{x_h}$ has full rank and $\kappa_{U_h} \neq 0$ we can follows the same procedure we have seen for case 1.

Otherwise the result can be obtained by combining the two strategies we have already presented in CASE 1 and CASE 2. This is proved in Theorem 34. ■

Theorem 31 *If $\kappa_{x_h} \neq 0$ for every h, then the matrix 11 has full row rank*

Proof. We consider the submatrix

	ξ	κ	A	τ, e^{m1}
$F\left(\xi, \rho, (u(.;A_h))_{h=1}^{H}\right)$	$*$	0	0	$*$
$[D_{\xi,t}(F_{pl}, U)]\kappa$	$*$	$[D_{\xi,t}(F_{pl}, U)]^T$	$N(\kappa_x)$	$*$
$1\backslash 2\kappa^T\kappa - 1$	0	κ	0	0

(13)

We know that there exists at least a $\kappa_{U_h} \neq 0$, without any loss of generality we assume $\kappa_{U_H} \neq 0$. (this will allow us to find an easy perturbation of the last row). It is easy to check that if the matrix (13) has full row rank, then the matrix (11) has full row rank. We write the matrix (13) extensively and we get :

	$n\ldots$ / $\xi\ldots$	$\kappa_{x_H}\ldots\ldots\kappa_{\gamma_H}$	$G^{\backslash}$ / $\kappa_{p^{\backslash}}$	I / κ_q	1 / $\kappa_{q^{m0}}$	S / $\kappa_{p'}$	$\ldots 1$ / $\ldots\kappa_{U_H}$	$\frac{G(G+1)}{2}$ / A_H	.	$\widehat{G}$ / $\widehat{\tau}_H$	.	S / $\tau_H^{\bullet 1}$	.	S / e_H^{1m}
$(1_1^F)\ G$	$*$													
.....														
$(1_H^F)\ G$	$*$							0^*						
$(2_H^F)\ \widehat{S}$	$*$									$\widehat{\Pi}_H$		$\Pi_H^{\bullet 1}$		$\begin{smallmatrix}0\\I\end{smallmatrix}$
....														
(10^F)	$*$													
$(11^F)\ S$	$*$									$\widetilde{\Pi}_H$		$\widetilde{\Pi}_H^{\bullet 1}$		$-I$
$(1_H)\ G$	$*$	$*$	$[0I]^T$				$(D_x u_H)^T$	$N(\kappa_{x_H})$						
$(2_H)\ \widehat{S}$	$*$	$*$												
$(3_H)\ I$	$*$	$*$		I										
$(4_H)\ 1$	$*$	$*$			1	-1^T								
$(5_H)\ \sharp J_H^{1\backslash}$	$*$	$*$												
$(6_H)\ \sharp \hat{J}_H^{3\backslash}$	$*$	$*$												
$(7_H)\ 1$	$*$	$*$												
$(8)\ G^{\backslash}$	$*$	$\left[\Lambda_H^{\backslash}\right]^T\ \widetilde{Z}_H^{\backslash}$				$\mathbb{T}^{\backslash}$				$\widehat{O}_H$				
$(9)\ I$	$*$	$b_H\quad \mathbf{0}\quad -\lambda_H^0 I_I$												
$(10)\ 1$	$*$	$z_H^{m0}\quad \mathbf{0}\quad -\lambda_H^0$												
$(11)\ S$	$*$	$[\Lambda_H^1]^T\ \widetilde{Z}_H^1$				$\mathbb{T}^1$						$O_H^{\bullet 1}$		
$(12)\ S$		$0\quad I$				$-I$								
$(13)\ 1$		$\kappa_{x_H},\ldots,\kappa_{\lambda_H}$	$\kappa_{p^{\backslash}}$	κ_q	$\kappa_{q^{0m}}$	$\kappa_{p'}$	$\ldots\kappa_{U_h}$							

(14)

$p' = (p^{s1})_{s=1}^{S}$, $\left(\Lambda_h^{\backslash}\right)^T$ and $(\Lambda_h^1)^T$ are a $G^{\backslash} \times G$ and $S \times G$ matrix respectively such that

$$\left(\Lambda_h^{\backslash}\right)^T = \begin{array}{|c|c|c|c|c|}\hline 0 & \lambda_h^0 I_{C-1} & & & \\ \hline & & \ddots & & \\ \hline & & & 0 & \lambda_h^S I_{C-1} \\ \hline \end{array}, \quad (\Lambda_h^1)^T = \begin{array}{|c|c|c|c|}\hline 0 & & & \\ \hline & \ddots & & \\ \hline & & \lambda_h^S & \\ \hline & & & 0 \\ \hline \end{array}$$

$\widetilde{Z}_h^{\backslash}$ is a $G^{\backslash} \times S+1$ matrix and $\widetilde{Z}_h^1$ is a $S \times S+1$ matrix such that

$$\widetilde{Z}_h^{\backslash} = \begin{array}{|c|c|c|}\hline \widetilde{z}_h^{02} & & \\ \hline & & \vdots \\ \hline & & \widetilde{z}_h^{SC} \\ \hline \end{array} \text{ and } \widetilde{Z}_h^1 = \begin{array}{|c|c|c|c|}\hline 0 & & & \\ \hline \vdots & \widetilde{z}_h^{11} & & \\ \hline & & \ddots & \\ \hline & & & \widetilde{z}_h^{S1} \\ \hline \end{array}$$

where $\widetilde{z}_h^{sc} = -x_h^{sc} + (1 - \tau_h^{sc})\, e_h^{sc}$ for $s > 0$ and $\widetilde{z}_h^{0c} = -x_h^{sc} + e_h^{sc}$
$\mathbb{T}^{\backslash}$ is a $G^{\backslash} \times S$ matrix such that

	1	..	S
p^{02}	0		0
...			
p^{0C}	0		0
p^{12}	$\sum_{h \in H} \tau_h^{12} e_h^{12}$		
...		$\ddots$	
p^{SC}			$\sum_{h \in H} \tau_h^{SC} e_h^{SC}$

$\mathbb{T}^1$ is a $S \times S$ matrix such that

p^{11}	$\sum_{h \in H} \tau_h^{11} e_h^{11}$		
...			
p^{S1}			$\sum_{h \in H} \tau_h^{S1} e_h^{S1}$

$\widehat{O}_h$ is a $\widehat{G} \times \widehat{G}$ matrix such that

	τ_h^{12}	...	τ_h^{SC}
$(p^{0c})_{c=2}^{C}$	0		0
p^{12}	$\left(\kappa_{p'}^1 - \kappa_{\lambda_h}^1\right) e_h^{12}$		
$\vdots$		$\ddots$	
p^{SC}			$\left(\kappa_{p'}^S - \kappa_{\lambda_h}^S\right) e_h^{SC}$

$O_h^{\bullet 1}$ is a $S \times S$ matrix such that

	τ_h^{11}	$\dots$	τ_h^{S1}
p^{11}	$\left(\kappa_{p'}^1 - \kappa_{\lambda_h}^1\right) e_h^{11}$		
....		$\ddots$	
p^{S1}			$\left(\kappa_{p'}^S - \kappa_{\lambda_h}^S\right) e_h^{S1}$

Observe that the submatrix obtained by erasing rows (12),(13) has full row rank in a dense and open set of economies. That follows from regularity result. Then we are left to perturb the last two superrows.

Perturbation of row (12). We consider two different cases

CASE A: there exists at least a consumer h such that $\left(\kappa_{p'}^s - \kappa_{\lambda_h}^s\right) \neq 0$ for every s.

Previous condition implies that the block matrices $\left[\widehat{O}_h\right]_{h \in H}$ and $[O_h^{\bullet 1}]_{h \in H}$ has full rank and that allows us to use these submatrix in order to perturb superrow (8) and (11) respectively. Without any loss of generality we can assume that $\left(\kappa_{p'}^s - \kappa_{\lambda_H}^s\right) \neq 0$

(12)	$\leftarrow$	$(\Delta\kappa_{p'})$	$\rightsquigarrow$	(11)	$\leftarrow$	$(\Delta\tau_H^{\bullet 1})$	$\rightsquigarrow$	$\binom{2_H^F}{(11^F)}$	
			$\rightsquigarrow$	(8)	$\leftarrow$	$(\Delta\widehat{\tau}_H)$	$\rightsquigarrow$	$\binom{2_H^F}{(11^F)}$	
			$\rightsquigarrow$	$(4_h)_{h=1}^H$	$\leftarrow$	$(\Delta\kappa_{q^{0m}})$			
$\binom{2_H^F}{(11^F)}$	$\leftarrow$	(Δe_H^{m1})							

CASE B: There exists at least an s $(s > 0)$ such that $\left(\kappa_{p'}^s - \kappa_{\lambda_h}^s\right) =$ **0 for every h**. We come back to system (10). If $\left(\kappa_{p'}^s - \kappa_{\lambda_h}^s\right) = 0$ for every h then we substitute rows (8^s), (11^s) and (12^s) with following:

	$\sum_{h=1}^{H} \left(\lambda_h^s \kappa_{x_h}^s + \left[-x_h^{s1} + (1-\tau_h^{s1}) e_h^{s1}\right] \kappa_{\lambda_h}^s + \tau_h^{s1} e_h^{s1} \kappa_{p'}^s\right) =$
	$= \sum_{h=1}^{H} \left(\lambda_h^s \kappa_{x_h}^{s1} + (e_h^{s1} - x_h^{s1}) \kappa_{\lambda_h}^s - \tau_h^{s1} e_h^{s1} \left(\kappa_{\lambda_h}^s - \kappa_{p'}^s\right)\right) =$
(11^s)	$= \sum_{h=1}^{H} \left(\lambda_h^s \kappa_{x_h}^{s1} + (e_h^{s1} - x_h^{s1}) \kappa_{x_h}^s\right) = 0$
(8^s)	$\sum_{h=1}^{H} \left(\lambda_h^s \kappa_{x_h}^{sc} + (e_h^{sc} - x_h^{sc}) \kappa_{\lambda_h}^s\right) = 0$ (for every $c > 1$)
(12^s)	$\kappa_{\lambda_h}^s - \kappa_{p'}^s = 0$

We study the rank of the matrix which corresponds to the new system. Note that the columns of $\mathbb{T}^1$ and $\mathbb{T}^\backslash$ corresponding to the state s are zero. That means the perturbation of row (12^s) does not alter the row (8),(11).

Perturbation of row (13) | (13) | $\leftarrow$ | $\Delta\kappa_{U_H}$ | $\rightsquigarrow$ | (1_H) | $\leftarrow$ | ΔA_H | .

■

Before handling CASE 2, we present the following useful result which shows that there exist an open and dense set of economies such that $\left[\left[\left(\widetilde{Z}_h^1 \lambda_h\right)_{h \neq H} \quad \widetilde{Z}_H^1 \lambda_H \right]\right.$ − has full rank.

Lemma 32

$$\begin{array}{ll} F_{eq}(\xi,\omega) & = 0 \\ A^T \varrho & = 0 \\ \rho^T \varrho - 1 & = 0 \end{array} \tag{15a}$$

where A is the following $S \times H$ matrix such that

$$\begin{bmatrix} \lambda_1^1\left(x_1^{11} - e_1^{11}\left(1 - \tau_1^{11}\right)\right) & \lambda_2^1\left(x_2^{11} - e_2^{11}\left(1 - \tau_2^{11}\right)\right) & \ldots\ldots & \lambda_H^1\left(x_H^{11} - e_H^{11}\right) \\ \lambda_1^2\left(x_1^{21} - e_1^{21}\left(1 - \tau_1^{21}\right)\right) & \lambda_2^2\left(x_2^{21} - e_2^{21}\left(1 - \tau_1^{21}\right)\right) & & \lambda_2^2\left(x_2^{21} - e_2^{21}\right) \\ \ldots & & \ddots & \\ \lambda_1^S\left(x_1^{S1} - e_1^{S1}\left(1 - \tau_1^{S1}\right)\right) & \lambda_2^2\left(x_2^{21} - e_2^{21}\left(1 - \tau_1^{21}\right)\right) & & \lambda_H^2\left(x_H^{21} - e_H^{21}\right) \end{bmatrix}$$

Proof. We consider the function $G : \Xi \times \Omega \times \mathbb{R}^H \rightarrow \mathbb{R}^n \times \mathbb{R}^S \times \mathbb{R}$ such that

$G : (\xi, \omega, \rho) \mapsto$ (left side of system (15a)). From the Jacobian of G we have the following matrix :

		n	1	1		1	$\widehat{G}$	.	$\widehat{G}$	S	.	S	S	.	S
		ξ	ρ_1	ρ_2	...	ρ_H	$\widehat{\tau}_1$		$\widehat{\tau}_H$	$\tau_1^{\bullet 1}$		$\tau_H^{\bullet 1}$	e_1^{1m}		e_H^{1m}
(1_1^F) G	$D_{x_1} u_{1..}$	*													
(2_1^F) $\widehat{S}$	$-\Phi x_{1..}$	*					$\widehat{\Pi}_1$			$\Pi_1^{\bullet 1}$			$\begin{matrix} 0 \\ I \end{matrix}$		
.....															
(1_H^F) G	$D_{x_H} u_H$	*													
(2_H^F) $\widehat{S}$	$-\Phi x_{H..}$	*							$\widehat{\Pi}_H$			$\Pi_H^{\bullet 1}$			$\begin{matrix} 0 \\ I \end{matrix}$
....															
(8^F)	$\sum_{h=1}^{H} x_h^{\backslash}..$	*													
(9^F)	$\sum_{h=1}^{H} b_h$	*													
(10^F)	$\sum_{h=1}^{H} m_h^0..$	*													
(11^F) S	$\sum_{h=1}^{H} - m_h^0..$	*					$\widetilde{\Pi}_1$		$\widetilde{\Pi}_H$	$\widetilde{\Pi}_1^{\bullet 1}$		$\widetilde{\Pi}_H^{\bullet 1}$	-I		-I
(12^1) 1	$\sum_{h=1}^{H} \widetilde{z}_h^{11} \lambda_h^1 \rho_h + \mathbb{T}^1 \left(\rho_{12_H} \lambda_H^1\right)$	*	$\lambda_1^1 \widetilde{z}_1^{11}$	$\lambda_2^1 \widetilde{z}_2^{11}$		$\lambda_H^1 \left(x_H^{11} - e_H^{11}\right)$				$\lambda_1^1 e_1^{11}$ $\mathbf{0}$					
...		*													
(12^S) 1	$\sum_{h=1}^{H} \widetilde{z}_h^{S1} \lambda_h^S \rho_h + \mathbb{T}^1 \left(\rho_{12_H} \lambda_H^S\right)$	*	$\lambda_1^S \widetilde{z}_1^{S1}$	$\lambda_2^S \widetilde{z}_2^{S1}$		$\lambda_H^S \left(x_H^{S1} - e_H^{S1}\right)$				$\mathbf{0}$ $\lambda_1^S e_1^{S1}$					
(13) 1	$\rho^T \rho$		ρ_1	ρ_2	...	ρ_H									

We first study the submatrix obtained by erasing the rows $(12^s)_{s=1}^S$ and (13). From regularity result we know that this submatrix has full row rank in a dense and open set of economies. We are left to perturb the remaining rows.

$$\begin{array}{llllll} s=1..S & (12^s) & \leftarrow & (\Delta\tau_1^{s1}) & \rightsquigarrow & \begin{array}{c}(11^F)\\(2_1^F)\end{array} & \leftarrow & (\Delta e_1^{ms})_{s=1}^S \\ & (13) & \leftarrow & (\Delta\rho_H) & \rightsquigarrow & (12^s)_{s=1}^S & & \\ & (12^s)_{s=1}^S & \leftarrow & (\Delta\tau_1^{s1})_{s=1}^S & \rightsquigarrow & \begin{array}{c}(11^F)\\(2_1^F)\end{array} & \leftarrow & (\Delta e_1^{ms})_{s=1}^S \end{array}$$

Then the Jacobian of G has full row rank in a dense and open set of economies Ω^G. Then from Transversality theorem there exists a dense and open set of economies such that for every $\omega \in \Omega^G$ 0 is a regular value for G_ω which is the restriction of G on ω. Since $n+S+1>n+H$, from Preimage Theorem $G_\omega^{-1}(0)=\emptyset$ for every $\omega \in \Omega^G$. Then we have the desired result

■

We are now ready to claim that if Case 2 occurs, then System (10) has no solution.

Theorem 33 *If Case2 occurs, then System (10) has no solution.*

Proof. System (10) can be written as follows

$$
\begin{array}{lll}
(1_h) & \left(D_h^2\right)^T \kappa_{x_h} - \Phi\kappa_{\lambda_h} + [0I]^T \kappa_{p\backslash} + \kappa_{U_h} D_x U_h & = 0 \\
(2_h) & -\Phi^T \kappa_{x_h} + R\kappa_{b_h} + q^m \kappa_{m_h^0} & = 0 \\
(3_h) & R^T \kappa_{\lambda_h} + (\beta_{b_h})^T \kappa_{b_h} + \left[\beta_{b_h m_h^0}\right]^T \kappa_{m_h^0} + \left[D_{b_h} a_h^{1\backslash}\right]^T \kappa_{\zeta_h^1} + I\kappa_q & = 0 \\
(4_h) & \begin{array}{l} (q^m)^T \kappa_{\lambda_h} + \beta_{b_h m_h^0}\kappa_{b_h} + \beta_{m_h^0}\kappa_{m_h^0} + \left[D_{m_h^0} a_h^{1\backslash}\right]^T \zeta_h^1 + \\ + \chi_{h[m_h^0=0]}\kappa_{\gamma_h} + \kappa_{q^{0m}} - (\mathbf{1})^T \kappa_{p'} \end{array} & = 0 \\
(5_h) & \left[D_{b_h} a_h^{1\backslash}\right] \kappa_{b_h} + \left[D_{m_h^0} a_h^{1\backslash}\right] \kappa_{m_h^0} & = 0 \\
(6_h) & \left[D_{b_h} a_h^{3\backslash}\right] \kappa_{b_h} + \left[D_{m_h^0} a_h^{3\backslash}\right] \kappa_{m_h^0} + I\kappa_{\zeta_h^3} & = 0 \\
(7_h) & \kappa_{\kappa_{m_h^0}} + \chi_{h[\gamma_h^0=0]}\kappa_{\gamma_h} & = 0 \\
(8) & \sum_{h=1}^{H} \left(\left[\Lambda_h^{\backslash}\right]^T \kappa_{x_h} + \widetilde{Z}_h^{\backslash}\kappa_{\lambda_h} \right) + \mathbb{T}^{\backslash}\kappa_{p'} & = 0 \\
(9) & \sum_{h=1}^{H} \left(\left(\begin{array}{cc} b_h & \mathbf{0} \end{array} \right) \kappa_{\lambda_h} - \lambda_0 I_I \kappa_{b_h} \right) & = 0 \\
(10) & \sum_{h=1}^{H} \left(\left(\begin{array}{cc} z_h^{m0} & \mathbf{0} \end{array} \right) \kappa_{\lambda_h} - \lambda_0 \kappa_{m_h^0} \right) & = 0 \\
(11) & \sum_{h=1}^{H} \left(\left[\Lambda_h^1\right]^T \kappa_{x_h} + \widetilde{Z}_h^1 \kappa_{\lambda_h} \right) + \mathbb{T}^1 \kappa_{p'} & = 0 \\
(12) & \left(\begin{array}{cc} 0 & I \end{array} \right) \kappa_{\lambda_H} + I\kappa_{p'} & = 0 \\
(13) & \begin{array}{l} \sum_{h=1}^{H} \kappa_{x_h}^T \kappa_{x_h} + \ldots + \sum_{h=1}^{H} \kappa_{\gamma_h}^T \kappa_{\gamma_h} + \kappa_{p\backslash}^T \kappa_{p\backslash} + \kappa_q^T \kappa_q + \\ + \kappa_{q^{0m}}^T \kappa_{q^{0m}} + \kappa_{p'}^T \kappa_{p'} + \kappa_U^T \kappa_U + 1 \end{array} & = 0
\end{array}
\tag{16}
$$

Step 1. $\kappa_{\lambda_h}^s = \kappa_{U_h}\lambda_h^s$ for every s. $\kappa_{p\backslash} = 0$.

Since $\kappa_{x_h} = 0$ for every h, from (1_h) we have $-p^{s1}\kappa_{\lambda_h}^s + \kappa_{U_h} D_{x_h^{s1}} = 0$.

From First Order Condition we have $-\lambda_h^s p^{s1} + D_{x_h^{s1}} = 0$ and so $p^{s1} = \frac{D_{x_h^{s1}}}{\lambda_h^s}$, hence $\frac{D_{x_h^{s1}}}{\lambda_h^s}\kappa_{\lambda_h}^s = \kappa_{U_h} D_{x_h^{s1}}$. We get $\kappa_{\lambda_h}^s = \kappa_{U_h}\lambda_h^s$.

Since $\kappa_{\lambda_h}^s = \kappa_{U_h}\lambda_h^s$, from (1_h) we have $\kappa_{p\backslash} = 0$.

Step 2. $\kappa_{b_h} = 0$. $\kappa_{m_h^0} = 0$.

Taking into account $\kappa_{x_h} = 0$ and the rank condition on (R, q^m) from (2_h) we get $\kappa_{b_h} = 0$. $\kappa_{m_h^0} = 0$..

Step 3. $\kappa_{p'}^s = -\lambda_H^s \kappa_{U_H}$.

From (13) we have $\kappa_{\lambda_H}^s - \kappa_{p'}^s = 0$ for every $s > 0$. Hence from step 1 we get $\kappa_{p'}^s = -\lambda_H^s \kappa_{U_H}$.

Step 4. $\kappa_{\zeta_h^3} = 0$, $\kappa_{\gamma_h} = 0$.

It follows respectively from (6_h) and (7_h) and from step 2.

Due to step 1-4, in order to get the desired result we can study the following reduced system:

$$\begin{array}{lll} (3_h) & R^T(\kappa_{U_h}\lambda_h) + \left[D_{b_h}a_h^{1\backslash}\right]^T \kappa_{\zeta_h^1} + I\kappa_q & = 0 \\ (11) & \sum\limits_{h=1}^{H} \widetilde{Z}_h^1(\lambda_h\kappa_{U_h}) + \mathbb{T}^1(\kappa_{U_H}\lambda_H) & = 0 \end{array} \qquad (17)$$

If we can prove that (17) has only the solution $\left(\left(\kappa_{\zeta_h^1}, \kappa_{U_h}\right)_{h=1}^{H}, \kappa_q\right) = 0$, from equation (4_h) of system (16) it follows $\kappa_{q^{0m}} = 0$ and then system (16) has no solution.

From Lemma 32 there exists an open and dense set of economies such that the matrix $\left[\left[\left(\widetilde{Z}_h^1\lambda_h\right)_{h\neq H} \quad \widetilde{Z}_H^1\lambda_H\right] - \left[\begin{array}{cc} 0 & \mathbb{T}^1 \end{array}\right]\right]$ has full rank. Then (11) implies $\kappa_{U_h} = 0$ for every h in a dense and open set of economies. Consequently we deal with the following system :

$$\begin{array}{lll} (3_1)\, I & \left[D_{b_1}a_1^{1\backslash}\right]^T \kappa_{\zeta_1^1} + \kappa_q I & = 0 \\ & \cdots\cdots & \\ (3_H)\, I & \left[D_{b_H}a_H^{1\backslash}\right]^T \kappa_{\zeta_H^1} + I\kappa_q & = 0 \end{array}$$

Hence we are left to study the rank of the following matrix

$$\begin{array}{cc|c|c|c|c|} & & \sharp J_1^{1\backslash} & & \sharp J_h^{1\backslash} & I \\ & & \kappa_{\zeta_1^1} & \cdots & \kappa_{\zeta_H^1} & \kappa_q \\ \hline I & (3_1) & \left[D_{b_1}a_1^{1\backslash}\right]^T & & & I \\ \hline & \cdots & & \ddots & & \\ \hline I & (3_H) & & & \left[D_{b_H}a_H^{1\backslash}\right]^T & I \\ \hline \end{array} \qquad (18)$$

By Assumption on Participation Constraints we know that for every asset i, there exists at least one consumer h' such that $D_{b_{h'}^i}a_{h'}(b_{h'}, m_{h'}^0) = 0$; consequently we know that at least I rows of the following submatrix are zero. By using these rows we perform some elementary row operations in order to obtain a submatrix (which can be the matrix itself) of matrix (18) such that:

	$\sharp J_1^{1\backslash}$ $\kappa_{\varsigma_1^1}$		$\sharp J_H^{1\backslash}$ $\kappa_{\varsigma_H^1}$	I κ_q
	$\left[D_{b_1} a_1^{1\backslash}\right]^T$			0
$\sum_{h=1}^{H} \sharp J_h^{1\backslash}$		$\ddots$		$\vdots$
			$\left[D_{b_H} a_H^{1\backslash}\right]^T$	0
I				I

It is easy to show that this matrix has full rank (It follows from Assumption iii) 12) Hence $\left(\kappa_{\varsigma_h^1}\right)_{h=1}^{H} = 0$ and $\kappa_q = 0$.

That implies system 16 has no solution. ∎

Theorem 34 *If for every h, either $\kappa_{x_h} = 0$ or $N\kappa_{x_h}$ has no full rank, then system 16 has no solution.*

Proof. The strategy of the theorem is the following: as in theorem 33 we eliminate redundant equations and we consider a reduced system where the number of equations are greater than the number of variables. Then as in Theorem 31, by a Trasversality argument on the reduced system we get the desired result. If for every h, either $\kappa_{x_h} = 0$ or $N\kappa_{x_h}$ has no full rank, then system 16 has no solution.

Take an arbitrary h. Note that one of the following case is verified:

(1) $N\kappa_{x_h}$ **has full row rank** **and** $\kappa_{U_h} = 0$

(2) $N\kappa_{x_h}$ **does not have full row rank** **and** $\kappa_{U_h} \neq 0$

(3) $N\kappa_{x_h}$ **does not have full row rank** **and** $\kappa_{U_h} = 0$

We make a partition on the set of consumer $\{1..H\}$ such that :

$A = \{h \in \{1..H\} : N\kappa_{x_h}$ has full row rank and $\kappa_{U_h} = 0\}$, $B = \{h \in \{1..H\}$: does not have full row rank and $\kappa_{U_h} \neq 0\}$ and $C = \{h \in \{1..H\}$: does not have full row rank and $\kappa_{U_h} = 0\}$

We first observe that if $A = \emptyset$ then we are in case 2, and we can immediately conclude that system (16) has no solution. Suppose on the contrary $B = \emptyset$ and consider again system (10). We can write equations (10.2) and (10.3) in the following way:

(1) ... (11)	$\left[D_\xi \left(F_{pl}, U\right)\right]^T \kappa$	$= 0$
(12)	$\left[D_t \left(F_{pl}, U\right)\right]^T \kappa$	$= 0$
(13)	$\kappa^T \kappa - 1$	$= 0$

. (19)

Since $[D_{\xi,t}(F_{pl})]^T$ has full rank, when $\kappa_{U_h} = 0$ for every h,the only solution to the previous system is $\kappa = 0$ and that contradicts equation 13 of system (19).

We are left to consider the case where$A \neq \emptyset$ and $B \neq \emptyset$. Without any loss of generality we can assume that $1 \in A$ and $H \in B$[2].

We write system (10).

$$
\begin{array}{lll}
(1_h) & (D_h^2)^T \kappa_{x_h} - \Phi\kappa_{\lambda_h} + [0I]^T \kappa_{p\backslash} + \kappa_{U_h} D_x U_h & = 0 \\
(2_h) & -\Phi^T \kappa_{x_h} + R\kappa_{b_h} + q^m \kappa_{m_h^0} & = 0 \\
(3_h) & R^T \kappa_{\lambda_h} + (\beta_{b_h})^T \kappa_{b_h} + \left[\beta_{b_h m_h^0}\right]^T \kappa_{m_h^0} + \left[D_{b_h} a_h^{1\backslash}\right]^T \kappa_{\zeta_h^1} + I\kappa_q & = 0 \\
(4_h) & (q^m)^T \kappa_{\lambda_h} + \beta_{b_h m_h^0}\kappa_{b_h} + \beta_{m_h^0}\kappa_{m_h^0} + \left[D_{m_h^0} a_h^{1\backslash}\right]^T \zeta_h^1 + \chi_{h[m_h^0=0]}\kappa_{\gamma_h} + \kappa_{q^{0m}} + \left(\widetilde{1}\right)^T \kappa_{p'} & = 0 \\
(5_h) & \left[D_{b_h} a_h^{1\backslash}\right] \kappa_{b_h} + \left[D_{m_h^0} a_h^{1\backslash}\right] \kappa_{m_h^0} & = 0 \\
(6_h) & \left[D_{b_h} a_h^{3\backslash}\right] \kappa_{b_h} + \left[D_{m_h^0} a_h^{3\backslash}\right] \kappa_{m_h^0} + I\kappa_{\zeta_h^3} & = 0 \\
(7_h) & \kappa_{\kappa_{m_h^0}} + \chi_{h[\gamma_h^0=0]}\kappa_{\gamma_h} & = 0 \\
(8) & \sum_{h=1}^{H} \left(\left[\Lambda_h^{\backslash}\right]^T \kappa_{x_h} + \widetilde{Z}_h^{\backslash}\kappa_{\lambda_h}\right) + \mathbb{T}^{\backslash}\kappa_{p'} & = 0 \\
(9) & \sum_{h=1}^{H} \left(\left(b_h \;\; \mathbf{0} \right) \kappa_{\lambda_h} - \lambda_0 I_I \kappa_{b_h}\right) & = 0 \\
(10) & \sum_{h=1}^{H} \left(\left(z_h^{m0} \;\; \mathbf{0} \right) \kappa_{\lambda_h} - \lambda_0 \kappa_{m_h^0}\right) & = 0 \\
(11) & \sum_{h=1}^{H} \left([\Lambda_h^1]^T \kappa_{x_h} + \widetilde{Z}_h^1 \kappa_{\lambda_h}\right) + \mathbb{T}^1 \kappa_{p'} & = 0 \\
(12) & \left(0 \;\; I \right) \kappa_{\lambda_H} + I\kappa_{p'} & = 0 \\
(13) & \sum_{h=1}^{H} \kappa_{x_h}^T \kappa_{x_h} + \ldots + \sum_{h=1}^{H} \kappa_{\gamma_h}^T \kappa_{\gamma_h} + \kappa_{p\backslash}^T \kappa_{p\backslash} + \kappa_q^T \kappa_q + \kappa_{q^{0m}}^T \kappa_{q^{0m}} + \kappa_{p'}^T \kappa_{p'} + \kappa_U^T \kappa_U & = 0
\end{array}
\tag{20}
$$

According with the procedure we have presented in chapter two, we rewrite system taking into account the conditions on κ_{x_h} , $N(\kappa_{x_h})$ and κ_{U_h}.

Step 1. For any $h \in A, C$, $\kappa_{\lambda_h}^s = \kappa_{U_h}\lambda_h^s$ for every s. $\kappa_{p\backslash} = 0$.

Since $\kappa_{x_h} = 0$ for every $h \in A$, from (1_h) we have

$$
\begin{array}{c}
(D_h^2)^T \kappa_{x_h} - \Phi\kappa_{\lambda_h} + [0I]^T \kappa_{p\backslash} = 0 \\
-\Phi^T \kappa_{x_h} + R\kappa_{b_h} + q^m \kappa_{m_h^0} = 0
\end{array}
$$

[2]We have no loss of generality even if household H is the one who is directly involved in planner's intervention. That is true because equations $(1_h), \ldots (7_h)$ for $h \in B$ are the same both for household H and for the other households which belong to B. Analogously, we would have no loss of generality if we have assumed $1 \in B$ and $H \in A$.

and $-p^{s1}\kappa^s_{\lambda_h} + \kappa_{U_h} D_{x^{s1}_h} = 0$.

From First Order Condition we have $-\lambda^s_h p^{s1} + D_{x^{s1}_h} = 0$ and so $p^{s1} = \frac{D_{x^{s1}_h}}{\lambda^s_h}$, hence $\frac{D_{x^{s1}_h}}{\lambda^s_h} \kappa^s_{\lambda_h} = \kappa_{U_h} D_{x^{s1}_h}$. We get $\kappa^s_{\lambda_h} = \kappa_{U_h} \lambda^s_h$.

Since $\kappa^s_{\lambda_h} = \kappa_{U_h} \lambda^s_h$, from (1_h) we have $\kappa_{p\backslash} = 0$.

Step 2. For $h \in A, C$, $\kappa_{b_h} = 0$ and $\kappa_{m^0_h} = 0$.

Taking into account $\kappa_{x_h} = 0$ and the rank condition on (R, q^m) from (2_h) we get $\kappa_{b_h} = 0$. $\kappa_{m^0_h} = 0$.

Step 3. For $h \in A, C$, $\kappa_{\zeta^3_h} = 0$, $\kappa_{\gamma_h} = 0$.

It follows respectively from (6_h) and (7_h) and from step 2.

Due to steps 1-3, we can rewrite system (10) in the following way:

$$
\begin{array}{lll}
(1_h) & \left(D_h^2\right)^T \kappa_{x_h} - \Phi \kappa_{\lambda_h} & = 0 \\
(2_h) & -\Phi^T \kappa_{x_h} + R\kappa_{b_h} + q^m \kappa_{m_h^0} & = 0 \\
(3_h) & R^T \kappa_{\lambda_h} + \left(\beta_{b_h}\right)^T \kappa_{b_h} + \left[\beta_{b_h m_h^0}\right]^T \kappa_{m_h^0} + \left[D_{b_h} a_h^{1\backslash}\right]^T \kappa_{\zeta_h^1} + I\kappa_q & = 0 \\
(4_h) & \left(q^m\right)^T \kappa_{\lambda_h} + \beta_{b_h m_h^0} \kappa_{b_h} + \beta_{m_h^0} \kappa_{m_h^0} + \left[D_{m_h^0} a_h^{1\backslash}\right]^T \zeta_h^1 + & \\
 & +\chi_{h[m_h^0=0]} \kappa_{\gamma_h} + \kappa_{q^{0m}} - (\mathbf{1})^T \kappa_{p'} & = 0 \\
(5_h) & \left[D_{b_h} a_h^{1\backslash}\right] \kappa_{b_h} + \left[D_{m_h^0} a_h^{1\backslash}\right] \kappa_{m_h^0} & = 0 \\
(6_h) & \left[D_{b_h} a_h^{3\backslash}\right] \kappa_{b_h} + \left[D_{m_h^0} a_h^{3\backslash}\right] \kappa_{m_h^0} + I\kappa_{\zeta_h^3} & = 0 \\
(7_h)_{\ h\in B} & \kappa_{m_h^0} + \chi_{h[\gamma_h^0=0]} \kappa_{\gamma_h} & = 0
\end{array}
$$

$$
\begin{array}{lll}
(1_h) & \kappa_{x_h} & = 0 \\
(2_h) & \kappa_{\lambda_h} = \kappa_{U_h} \lambda_h & \\
(3_h) & R^T \left(\kappa_{U_h} \lambda_h\right) + \left[D_{b_h} a_h^{1\backslash}\right]^T \kappa_{\zeta_h^1} + I\kappa_q & = 0 \\
(4_h) & \left(q^m\right)^T \left(\kappa_{U_h} \lambda_h\right) + \left[D_{m_h^0} a_h^{1\backslash}\right]^T \kappa_{\zeta_h^1} + \kappa_{q^{0m}} - (\mathbf{1})^T \kappa_{p'} & = 0 \\
(4_h') & \kappa_{m_h^0} & = 0 \\
(5_h) & \kappa_{b_h} & = 0 \\
(6_h) & \kappa_{\zeta_h^3} & = 0 \\
(7_h)_{\ h\in A\cup C} & \kappa_{\gamma_h} & = 0
\end{array}
$$

$$
\begin{array}{lll}
(8) & \kappa_{p\backslash} & = 0 \\
(9) & \sum\limits_{h\in A} \left(\begin{pmatrix} b_h & 0 \end{pmatrix} \kappa_{U_h} \lambda_h\right) + \sum\limits_{h\notin A} \left(\begin{pmatrix} b_h & 0 \end{pmatrix} \kappa_{\lambda_h} - \lambda_h^0 I_I \kappa_{b_h}\right) & = 0 \\
(10) & \sum\limits_{h\in A} \left(\begin{pmatrix} z_h^{m0} & 0 \end{pmatrix} \kappa_{U_h} \lambda_h\right) + \sum\limits_{h\notin A} \left(\begin{pmatrix} z_h^{m0} & 0 \end{pmatrix} \kappa_{\lambda_h}\right) & = 0 \\
(11) & \sum\limits_{h\in A} \left(\widetilde{Z}_h^1 \kappa_{U_h} \lambda_h\right) + \sum\limits_{h\notin A} \left(\widetilde{Z}_h^1 \kappa_{\lambda_h}\right) + \mathbb{T}^1 \left(\kappa_{\lambda_h}\right) & = 0 \\
(12) & \begin{pmatrix} 0 & I \end{pmatrix} \kappa_{\lambda_H} + I\kappa_{p'} & = 0 \\
(13) & \kappa^T \kappa & = 0
\end{array}
\qquad (21)
$$

From Assumption on restricted participation we have that :

for every asset i, there exists at least one consumer h' such that for every $(b_{h'}, m_{h'}^0) \in \mathbb{R}^{I+1}$ the following condition holds :$D_{b_{h'}^i} a_{h'} (b_{h'}, m_{h'}^0) = 0$.

We can have three different cases:

a) $h' \in A \cup C$

b) $h' \in B$

In case a) from (3_h) we have

$$\sum_{s=0}^{S} r^{si} \lambda_{h'}^{s} \left(\kappa_{U_{h'}}\right) + \kappa_q^i = 0.$$

and so

$$\kappa_q^i = -\sum_{s=0}^{S} r^{si} \lambda_{h'}^{s} \left(\kappa_{U_{h'}}\right).$$

Moreover if $h' \in C$ we have $\kappa_q^i = 0$.

Then we can erase the redundant equations and the corresponding variables. We get:

$$\begin{array}{lll}
(1_h) & \left(D_h^2\right)^T \kappa_{x_h} - \Phi \kappa_{\lambda_h} & = 0 \\
(2_h) & -\Phi^T \kappa_{x_h} + R\kappa_{b_h} + q^m \kappa_{m_h^0} & = 0 \\
(3_h) & R^T \kappa_{\lambda_h} + \left(\beta_{b_h}\right)^T \kappa_{b_h} + \left[\beta_{b_h m_h^0}\right]^T \kappa_{m_h^0} + \left[D_{b_h} a_h^{1\backslash}\right]^T \kappa_{\zeta_h^1} + I\kappa_q & = 0 \\
(4_h) & \left(q^m\right)^T \kappa_{\lambda_h} + \beta_{b_h m_h^0} \kappa_{b_h} + \beta_{m_h^0} \kappa_{m_h^0} + \left[D_{m_h^0} a_h^{1\backslash}\right]^T \zeta_h^1 + & \\
 & + \chi_{h\left[m_h^0 = 0\right]} \kappa_{\gamma_h} + \kappa_{q^{0m}} - (\mathbf{1})^T \kappa_{p'} & = 0 \\
(5_h) & \left[D_{b_h} a_h^{1\backslash}\right] \kappa_{b_h} + \left[D_{m_h^0} a_h^{1\backslash}\right] \kappa_{m_h^0} & = 0 \\
(6_h) & \left[D_{b_h} a_h^{3\backslash}\right] \kappa_{b_h} + \left[D_{m_h^0} a_h^{3\backslash}\right] \kappa_{m_h^0} + I\kappa_{\zeta_h^3} & = 0 \\
(7_h)\ _{h \in B} & \kappa_{m_h^0} + \chi_{h\left[\gamma_h^0 = 0\right]} \kappa_{\gamma_h} & = 0 \\
 & & \\
(3_h)_{h \in A} & \widetilde{R}^T \left(\kappa_{U_h} \lambda_h\right) + \left[D_{b_h} \tilde{a}_h^{1\backslash}\right]^T \kappa_{\zeta_h^1} + \widetilde{I}\tilde{\kappa}_q & = 0 \\
(4_{h'})_{h' \in A} & q^{mT} \left(\kappa_{U_h} \lambda_h\right) + \left[D_{m_h^0} a_h^{1\backslash}\right]^T \kappa_{\zeta_h^1} + \kappa_{q^{0m}} - (\mathbf{1})^T \kappa_{p'} & = 0 \\
(9_i)^* & \sum_{h \in A} \left(\begin{pmatrix} b_h^i & \mathbf{0} \end{pmatrix} \kappa_{U_h} \lambda_h\right) + \sum_{h \notin A} \left(\begin{pmatrix} b_h^i & \mathbf{0} \end{pmatrix} \kappa_{\lambda_h} - \lambda_0 I_I \kappa_{b_h}\right) & = 0 \\
(11) & \sum_{h \in A} \left(\widetilde{Z}_h^1 \kappa_{U_h} \lambda_h\right) + \sum_{h \notin A} \left(\widetilde{Z}_h^1 \kappa_{\lambda_h}\right) + \mathbb{T}^1 \left(\kappa_{\lambda_h}\right) & = 0 \\
(12) & \begin{pmatrix} 0 & I \end{pmatrix} \kappa_{\lambda_H} + I\kappa_{p'} & = 0 \\
(13) & \kappa^T \kappa - 1 & = 0
\end{array} \tag{22}$$

where $\left[D_{b_h} \tilde{a}_h^{1\backslash}\right]^T$ is the square submatrix of $\left[D_{b_h} a_h^{1\backslash}\right]^T$ whose rank is $\sharp J_h^1$. $\tilde{R}$ is a $S \times \sharp J_h^1$ submatrix of R while $\tilde{I}$ is the identity matrix $\sharp J_h^1 \times \sharp J_h^1$ and $\tilde{\kappa}_q$ is a $\sharp J_h^1$ vector. Obviously $\tilde{R}, \tilde{\kappa}_q$ and $\left[D_{b_h} \tilde{a}_h^{1\backslash}\right]^T$ corresponds to the same assets.

$*$. We write equation $(9_i)^*$ corresponding to those assets such that :

the corresponding h' such that $D_{b_{h'}} a_{h'} = 0$, belongs to the set B. We denote I^* the number of equations in $(9_i)^*$.

Note that system (22)has

$$\sharp \text{ of equations} = \quad (\sharp J_h^1)_{h \in A} + \Big(G + \hat{S} + I + 1 + 1\Big) \sharp B + (\sharp J_h^1 + \sharp J_h^3)_{h \in B} + I^* + S + S + 1$$

$$\sharp \text{ of variables} = \quad (\sharp J_h^1)_{h \in A} + \Big(G + \hat{S} + I + 1 + 1\Big) \sharp B + (\sharp J_h^1 + \sharp J_h^3)_{h \in B} + I^* + S + H$$

Since $\sharp J_1^1 \leq I$ and $S \geq H$, $\sharp$ of equations is greater than the $\sharp$ of variables.

If we prove that the only solution of the subsystem $((22.1), (22.12))$is the zero solution, then we get the desired result because in this case the whole system has no solution.

With this regard we are left to show the following matrix has full row rank.

	ξ	$\sharp J_h^{1\backslash}\ldots$ $\kappa_{\zeta_h^1}$	$\sharp J_1^{1\backslash}$ $\kappa_{\zeta_1^1}$	G κ_{x_H} $D_{x_h}u_h.$	$\widehat{S}$ κ_{λ_H} $-\Phi x_h$	I κ_{b_H} $\lambda_h R+$	1 κ_{m_H} $\lambda_h q^m$	$\sharp J_h^{1\backslash}$ $\kappa_{\zeta_H^1}$ $\min a_h^{1\backslash}$	$\sharp \hat{J}_h^{3\backslash}$ $\kappa_{\zeta_H^3}$ $\min a_h^{3\backslash}$	1 κ_{γ_H} $\min m_h^0$	I κ_q $\sum_{h=1}^H b_h$	1 $\kappa_{q^{m0}}$ $\sum_{h=1}^H m_h^0$	S $\kappa_{p'}$ $\sum_{h=1}^H m_h^0.$	1 κ_{U_1} $J_x u_1$		1 κ_{U_H} $J_x u_H$	$\frac{G(G+1)}{2}$ A_H
$(1_1^F)\ G$	*																0^*
$(2_1^F)\ \widehat{S}$	*																
.....																	
$(1_H^F)\ G$	*																
$(2_H^F)\ \widehat{S}$	*																
....																	
(8^F)	*																
(9^F)	*																
(10^F)	*																
$(11^F)\ S$	*																
$(3_h)_{h\in A}\ \sharp J_h^{1\backslash}$	*	$\left[D_{b_1}\tilde{a}_h^{1\backslash}\right]^T$									I			$R^T\lambda_1$			
$(4_h)_{h=h'}^{h\in A}\ 1$		$\left[D_{m_{0_h}}\tilde{a}_h^{1\backslash}\right]^T$										1	$(\tilde{1})^T$		$q^{mT}\lambda_h$		
.....																	
$(1_H)\ G$	*			$(D_H^2)^T$	$-\Phi$											$(D_x u_H)^T$	$N(\kappa_{x_H})$
$(2_H)\ \widehat{S}$	*			$-\Phi^T$		R	q^m										
$(3_H)\ I$	*				R^T	$(\beta_{b_H})^T$	$\left[\beta_{b_H m_H^0}\right]^T$	$\left[D_{b_H}a_H^{1\backslash}\right]^T$			I						
$(4_H)\ 1$	*				$(q^m)^T$	$\beta_{b_H m_H^0}$	$\beta_{m_H^0}$	$\left[D_{m_H^0}a_H^{1\backslash}\right]^T$		$\chi_H[m_H^0=0]$		1	$-(1)^T$				
$(5_H)\ \sharp J_H^{1\backslash}$	*					$D_{b_H}a_H^{1\backslash}$	$D_{m_H^0}a_H^{1\backslash}$										
$(6_H)\ \sharp \hat{J}_H^{3\backslash}$	*					$D_{b_H}a_H^{3\backslash}$	$D_{m_H^0}a_H^{3\backslash}$		I								
$(7_H)\ 1$	*						1			$\chi_H[\gamma_H^0=0]$							
$(9_i)^*\ I^*$	*				$b_H^i\quad \mathbf{0}$	$-\lambda_H^0$								$\lambda_1^0 b_1^i\quad \mathbf{0}$			
$(11)\ S$	*			$[\Lambda_H^1]^T$	Z_H^1								$\mathbf{T}^1$				
$(12)\ S$	*				$0\quad I$								I				

(23)

From regularity result we have that the first (1^F) .. (11^F) rows are linearly independent in a open and dense subset of economies.

Perturbation of (3_h) $h \in A \cup C$ $(3_h)_{h \in A} \leftarrow \Delta\kappa_{\zeta_h^1}$.

Perturbation of $(4_{h'})$

$(4_{h'}) \leftarrow \Delta\kappa_{q^{0m}}$.

Perturbation of (1_h) $h \in B$.

$(1_h) \leftarrow \Delta\kappa_{x_h}$.

Perturbation of (2_h) $h \in B$.

$(2_h) \leftarrow \Delta\kappa_{x_h} \rightsquigarrow (1_h) \leftarrow \Delta A_h$.

Perturbation of (3_h) and (4_h) $h \in B$.

$$\begin{matrix}(3_h)\\(4_h)\end{matrix} \leftarrow \Delta\kappa_{\lambda_h} \rightsquigarrow (1_h) \leftarrow \Delta A_h .$$

Perturbation of (5_h) $h \in B$.

$$\begin{array}{llllllll} (5_h) & \leftarrow & \Delta\kappa_{\zeta_h^1} & \rightsquigarrow & \begin{matrix}(3_h)\\(4_h)\end{matrix} & \leftarrow & \Delta\kappa_{\lambda_h} & \rightsquigarrow (1_h) \\ & & & & (2_h) & \leftarrow & \Delta\kappa_{x_h} & \rightsquigarrow (1_h) \\ (1_h) & \leftarrow & \Delta A_h & & & & & \end{array}$$

Perturbation of (6_h) $h \in B$.

$(6_h) \leftarrow \Delta\kappa_{\zeta_h^3}$.

Perturbation of (7_h) $h \in B$.

If $\gamma_h = 0$ then

$(7_h) \leftarrow \Delta\kappa_{\gamma_h}$.

If $m_h^o = 0$ then

$$\begin{array}{lllll} (7_h) & \leftarrow & \Delta\kappa_{m_h^o} & \rightsquigarrow & (6_h) \\ & & & \rightsquigarrow & (5_h) \\ & & & \rightsquigarrow & \begin{matrix}(3_h)\\(4_h)\end{matrix} \\ & & & \rightsquigarrow & (2_h) \\ (6_h) & \leftarrow & \Delta\kappa_{\zeta_h^3} & & \\ (5_h) & \leftarrow & \Delta\kappa_{\zeta_h^1} & \rightsquigarrow & \begin{matrix}(3_h)\\(4_h)\end{matrix} \\ & & & \rightsquigarrow & (2_h) \\ \begin{matrix}(3_h)\\(4_h)\end{matrix} & \leftarrow & \Delta\kappa_{\lambda_h} & \rightsquigarrow & (1_h) \\ (2_h) & \leftarrow & \Delta\kappa_{x_h} & \rightsquigarrow & (1_h) \\ (1_h) & \leftarrow & \Delta A_h & & \end{array}$$

Perturbation of $(9_i)^*$.

For any i, we take the consumer h' such that $D_{b_{h'}} a_{h'} = 0$.

$(9_i) \leftarrow \Delta\kappa_{b_{h'}} \rightsquigarrow (2_h) \leftarrow \Delta\kappa_{x_h} \rightsquigarrow (1_h) \leftarrow \Delta A_h$.

Perturbation of (11).

$$\begin{array}{lllllll}
(11) & \leftarrow & \Delta\kappa_{x_h^1} & \rightsquigarrow & (2_h) & \leftarrow & \Delta\kappa_{x_h^\backslash} \rightsquigarrow (1_h) \\
 & & & \rightsquigarrow & (1_h) & & \\
(1_h) & \leftarrow & \Delta A_h & & & &
\end{array}.$$

Perturbation of (12).

$$\begin{array}{lllllll}
(12) & \leftarrow & \Delta\kappa_{p'} & \rightsquigarrow & (11) & & \\
 & & & \rightsquigarrow & (4_h) & & \\
(11) & \leftarrow & \Delta\kappa_{x_h^1} & \rightsquigarrow & (2_h) & \leftarrow & \Delta\kappa_{x_h^\backslash} \rightsquigarrow (1_h) \\
 & & & \rightsquigarrow & (1_h) & & \\
(4_h) & \leftarrow & \Delta\kappa_{q^{0m}} & & & & \\
(1_h) & \leftarrow & \Delta A_h & & & &
\end{array}.$$

So the matrix (23) has full row rank and due to this there exists an open and dense set on economies such that system (22.1-22.12) has only the solution $\kappa = 0$. Consequently system (22) has no solution and so we get the desired result. ■

7 References

Y. Balasko (1988), *"Foundations of the theory of General Equilibrium"* (Boston Academic Press)

Y. Balasko and D. Cass (1989), "The structure of financial equilibrium with exogenous yields: I. Unrestricted participation", *Econometrica* 57, 135-162

Y. Balasko, D. Cass and P.Siconolfi (1990), "The structure of financial equilibrium with exogenous yields: the case of restricted participations", *Journal of Mathematical Economic* 19, 195-216.

L. Carosi (1999a), "Competitive equilibria with money and restricted participation", *Report* n° 142, Diartimento di Statistica e Matematica applicata all'Economia, Università di Pisa.

D. Cass (1995), "Notes on Pareto Improvement in Incomplete Financial Markets", *Rivista di Matematica per le Scienze Economiche e Sociali 18*, I° Semestre, 3-14.

D. Cass and K. Shell (1980), "In defence of basic approach", in: Kareken and Wallace eds., *Models of Monetary economies*, Minneapolis, Federal Reserve Bank of Minneapolis, (Minneapolis, MN).

D. Cass, P. Siconolfi and A.Villanacci (1991), "A note on Generalizing The Model of Competitive Equilibrium with Restricted Participation on Financial Markets", CARESS working paper 91-13, (University of Pennsylvania, Philadelphia, PA).

A. Citanna, A. Kajii and A. Villanacci (1998), "Constrained suboptimality in incomplete markets: a general approach and two applications", Economic Theory, Mini-Symposium on "Pareto Improvement in Incomplete Financial Markets", *Economic Theory* 11, 495-521.

P. Dubey and J. Geanakoplos (1992), "The value of Money in a Finite-Horizon Economy: A Role for Banks", in: P. Dasgupta et al. eds., *Economic Analysis of Markets and Games: Essays in honor of Frank Hahn*, Cambridge, (The MIT Press, Massachusetts).

J. M. Grandmont and Y. Younes (1975), "On the role of Money and the Existence of Monetary Equilibrium", *Review of Economic Studies* 39, 355-372.

V. Guillemin and A. Pollack (1974), *Differential Topology*, (Prentice-Hall Inc., New Jersey).

F. Hahn (1965), "On Some Problems of Proving the Existence of an Equilibrium in a Monetary Economy", in: F. H. Hahn and F.P.R. Brechling eds., *The Theory of Interest Rates*, 126-135, (Macmillam, London).

A.P. Lerner (1947), "Money as a Creature of the State", *American Economic Review* 37, 312-317.

M. Magill and M.Quinzii (1992), "Real Effects of Money in General Equilibrium", *Journal of Mathematical Economics* 21, 301-342.

H. M. Polemarchakis and P. Siconolfi, "Generic Existence of Competitive Equilibria with Restricted Participation", *Journal of Mathematical Economic* 28, 1997, 289-311.

P. Siconolfi (1988), "Equilibrium with Asymmetric constraints on Portfolio" in: M. Galeotti, L. Geronazzo, F. Gori, eds., *Non-linear Dynamics in Economics and Social Science*, (Società Pitagora, Bologna).

R.M. Starr (1974), "The price of money in a pure exchange monetary economy with taxation", *Econometrica* 42, 42-54.

R.M. Starr (1989), "Introduction", in: R.M.Starr eds., *General Equilibrium models of monetary economies. Studies in Static Foundations of Monetary Theory*, (Academic Press Inc., San Diego).

A. Villanacci (1991), "Real Indeterminacy, Taxes and Outside Money in Incomplete Financial Market Economies: I. The case of Lump Sum Taxes", in M. Galeotti, L. Geronazzo, F. Gori, eds., *Proceedings of the Second Meeting on Non-linear Dynamics in Economics*, (Springer Verlag)

A. Villanacci, (1993), "Real Indeterminacy, Taxes and Outside Money in Incomplete Financial Market Economies: II. The Case of Taxes as Function of Wealth", DIMADEFAS, Ricerche n.6.