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Diffusion Processes for Asset Prices under Bounded Rationality

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Abstract

We study an asset prices model under bounded rationality. In the economy there are rational traders and noise traders. If noise traders market behaviour is modeled as a pure noise (random walk) and rational traders compute the expected price as a geometric average of the observed prices (bounded rationality), then we show that in the limit, as the trade interval goes to zero, the asset price is described by a mean reverting process.

Keywords: Bounded Rationality, Diffusion processes

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1 Introduction

In a continuous time setting, the evolution of asset prices, interest rates, risk factors is described through stochastic differential equations. In many cases these stochastic differential equations are taken for granted, and no much attention is paid to their microeconomic foundation.

There are of course many attempts aiming to determine the most appropriate stochastic process to describe the evolution of asset prices and interest rates. They are mainly based on econometric arguments estimating and comparing the performances of different specifications of the processes with the corresponding time series. For a survey on this literature we refer to Campbell et al. [6]. There are also some pieces of theory well suited to characterize the evolution of such quantities. Following the seminal contribution of Lucas [21], general equilibrium theory provides us with a theoretical foundation for the short term interest rate and for the asset price dynamics, e.g. see [11]. In equilibrium, the Arrow-Debreu prices process contains many interesting pieces of information. In a continuous time setting, under some conditions, we have that the interest rate is the negative of the drift of the Arrow-Debreu prices process. In a complete market setting, choosing properly the utility function of the representative agent, and the dynamics of the state variables, some diffusion processes for the short term interest rate are obtained. In Cox et al. [7, 8] and in Goldstein and Zapatero [16], considering a power utility function we obtain a theoretical foundation of the Cox, Ingersoll and Ross (see [8]) and of the Vasicek interest rates dynamics (see [28]).

The general equilibrium approach can also be used to model the stock price dynamics. Indeed, in a stationary economy where all agents are fully rational and trust the model itself, the equilibrium stock price is given by the expected value of the discounted future dividends under an equivalent risk neutral probability measure. Under this measure the asset expected return is equal to the risk free rate. The theory provides us with restrictions for the asset prices under the risk neutral probability meaure. If the agents are risk neutral and are characterized by a discount factor which is the inverse of the risk free rate of return, then the risk neutral probability measure and the historical probability measure coincide. Exploiting these results, in Kreps [18] and in Bick [3] the classical random walk with drift process for the stock price (geometric brownian motion) has been obtained in a general equilibrium framework.

This asset pricing theory provides us with some testable implications. Mainly, assuming a stationary economy, the asset excess returns should be unpredictable and no sign of autocorrelation of the excess returns should be observed. Many studies have tested empirically these implications. An evaluation of the empirical literature allows us to establish that returns are predictable and that they are mainly characterized by mean reversion (excess returns are negatively autocorrelated, see [14]). Moreover some phenomena such as booms, crashes and excess volatility are difficult to be interpreted inside the classical asset price framework. Two well established schools of thinking can be identified in the literature about the interpretation of these facts: the *die hard* classical asset pricing school, and the so called *behavioural finance* school. People belonging to the first school explain these phenomena by relaxing the assumption of stationarity for the asset dividends-returns. As a consequence, we have that the equity premia are not costant over time and excess returns turn out to be autocorrelated, e.g. see [14]. The partizans of the behavioural school argue that the nonstationarity of the model is not enough to explain the phenomena observed in financial markets and invoke the presence of some elements of irrationality in the market, e.g. see [10] and [20]. On this debate we refer to Fama [12, 13].

This paper aims to contribute to this debate, by providing a bounded rationality microfoundation for asset prices in continuous time.

Over the last ten years we have a large literature on asset prices with heterogeneous agents, and in particular with the presence in the market of *rational* and *non rational* agents, e.g. see [9, 10]. In Föllmer and Schweizer [15], for instance, a microeconomic approach is developed to determine the stochastic differential equation for the stock prices as the equilibrium outcome in a market populated by heterogeneous agents: *rational* traders, *fundamentalist* traders and *noise* or *liquidity* traders. The agents of the first class aim to exploit all arbitrage opportunities in the market, the traders of the second class base their decisions on the comparison between the stock price and the *fundamentals* about the security. Finally noise traders are pure noise in the market demand, noise due traders' buying and selling stock for liquidity needs. In order to derive a diffusion process, in Föllmer and Schweizer [15] it is assumed that rational agents are myiopic, i.e. they foresee the future price as the price one period before.

In this paper, we also look for a microfoundation foundation of the evolution of stock prices in an equilibrium perspective with heterogeneous agents. The main feature of our analysis is that we assume the agents not to be fully rational, i.e. they are characterized by bounded rationality. To simplify the analysis we assume that there are two classes of agents, (boundedly) rational traders and noise traders. Bounded rationality is modeled by assuming that traders forecast the future price by updating their expectations through a first order autoregressive learning mechanism. The so called *(modified) adaptive expectations*: the today expectation for the tomorrow price is a convex combination of the yesterday expectation for the today price and of the yesterday price. This learning rule can also be interpreted as an extrapolative technical analysis trading strategy. The noise traders demand is described by a pure white noise component. The diffusion process for the asset price, obtained in the standard weak limit by means of a suitable time rescaling of the discrete modeling equations (see [22]), performs a mean reverting process around the level given by agent's expectation, which in turn is modeled by a recurrent Ornstein-Ulenbeck process. As a consequence, we obtain that the autocovariance of asset price increments is negative, and therefore the price process is characterized by mean reversion.

This result calls for a discussion with those obtained in the bounded rationality literature. The analysis of financial markets under bounded rationality has been developed in several papers, e.g. see [1, 5, 25, 26, 27]. If dividends are autocorrelated, then bounded rationality generates excess return autocorrelation, and phenomena such as excess volatility and long swings from the fundamental value can be explained. Our paper does not rely upon the autocorrelation of the fundamentals process. Pure non-correlated noise in the market can generate mean reversion under bounded rationality. The second interesting feature of our analysis is that the stochastic process for the asset price performs a mean reverting around the level given by agent's expectation process. If agents use an extrapolative learning mechanism-technical analysis trading rule then the drift is determined by agents' expectation. This paper provides a theoretical analysis of the Shiller [23] noise traders model under bounded rationality.

The paper is organized as follows. In Section 2 we present the discrete time financial market model. In Section 3 we study the convergence of the asset price to a diffusion process.

2 Bounded Rationality in Financial Markets

Following Shiller [23], we consider the following forward-looking difference equation

$$S_k = v_k \hat{S}_k + \sigma_k Z_k, \quad k = 1, 2, \dots \tag{1}$$

where S_k is the asset price at time k, $\hat{S}_k$ denotes agent's expectation at time k of the asset price at time $k+1$, the coefficient v_k is a suitable discount factor, the sequence $(Z_k)_{k\geq 1}$, which models the noise in the market, is a sequence of independent and normally distributed real random variables such that $\mathbf{E}[Z_k] = 0$ and $\mathbf{D}^2[Z_k]$ and the coefficient σ_k is the noise traders component variance.

Equation (1) is the classical *no arbitrage equation* plus a noise component. In a market with two assets, a risky asset and a risk-free asset characterized by a interest rate r, setting $\hat{S}_k \equiv \mathbf{E}_k[S_{k+1}]$, where $\mathbf{E}_k$ denotes the conditional expectation at time k given the available information, $v_k \equiv (1+r)^{-1}$, and $\sigma_k \equiv 0$ we end up with the classical no arbitrage equation with respect to the risk neutral probability measure.

The random variable S_k in (1) can be interpreted as the equilibrium asset price in a market where there are two classes of traders: *rational traders* and *noise traders*. Agents belonging to the first class behave according to the no arbitrage principle looking at the expected rate of the return of the asset, when the expected return is larger or lower than the risk free rate they buy or sell short the risky asset. Agents belonging to the second class act for pure liquidity needs and therefore their effect on the market price is purely idiosyncratic and is described by the sequence of random variables $(Z_k)_{k\geq 1}$.

To simplify the analysis we assume that the asset does not deliver dividends. The asset can be interpreted as a future contract. The noise component does not affect the fundamentals of the contract, therefore if the agents are fully rational the price of the contract should be constant over time. Our choice of not considering dividends is motivated by the fact that we want to isolate the effect of pure non fundamental noise on the asset price when the agents are not fully rational. Being constant the asset price when the agents are fully rational, the comparison will be straightforward.

In what follows the agents are not characaterized by rational expectations. The rational expectations assumption is a mile stone in modern economic and finance theory, every other behavioural assumption is named *bounded rationality*. The rational expectations assumption is based on two main hypotheses: agents know the model and use all the available information in the best way. Bounded rationality requires to weaken these two assumptions. In our analysis, following among the others Barucci [2], we assume that *rational traders* update their expectations according to the *first order autoregressive learning mechanism*:

$$\hat{S}_k = \hat{S}_{k-1} + \alpha_k(S_{k-1} - \hat{S}_{k-1}), \quad k = 1, 2, \ldots \tag{2}$$

for a suitable learning coefficient α_k ($0 \leq \alpha_k \leq 1$). As mentioned above, the idea captured in (2) is that the today expectation for the tomorrow price is a convex combination of the yesterday expectation for the today price and of the yesterday price. Note that, to avoid simultaneity problems between the expectation formation and the determination of the equilibrium price, the asset price is not compared to the contemporaneous expectation, as it is done in the classical adaptive expectation framework.

3 Convergence to a Diffusion Process

The system of stochastic difference equations (1) and (2) can be rewritten in the following canonical innovation form:

$$\begin{aligned} S_k &= v_k \hat{S}_k + \sigma_k Z_k, \\ \hat{S}_{k+1} &= \hat{S}_k + \alpha_{k+1}(v_k - 1)\hat{S}_k + \alpha_{k+1}\sigma_k Z_k, \end{aligned} \tag{3}$$

and where

$$\hat{S}_1 = \hat{S}_0 + \alpha_1(S_0 - \hat{S}_0).$$

Since S_0 is the datum asset price at time $k = 0$, if we make the natural assumption that the random variables of the sequence $\left(\hat{S}_0, Z_1, \ldots Z_n, \ldots\right)$ are independent, then it is well known that the solution $\left(S_k, \hat{S}_k\right)_{k\geq 0}$ of (3) is a Markov chain with respect to the filtration $(\mathcal{F}_k)_{k\geq 0}$ generated by the sequence $\left(\hat{S}_0, Z_1, \ldots Z_n, \ldots\right)$ itself.

Following Nelson [22], we can show, by means of a standard stepwise time-rescaling and under suitable hypotheses on the coefficients, that it is possible to obtain the weak convergence of the solutions of the rescaled systems to the solution of a system of diffusive stochastic differential equations. To this end, first, we rewrite (3) in the following equivalent form

$$\begin{aligned} S_k - S_{k-1} &= -d_k \hat{S}_k + \hat{S}_k - S_{k-1} + \sigma_k Z_k, \\ \hat{S}_{k+1} - \hat{S}_k &= -\alpha_{k+1} d_k \hat{S}_k + \alpha_{k+1}\sigma_k Z_k, \end{aligned} \tag{4}$$

where we have introduced the discount rate $d_k \equiv 1 - v_k$. Then, for each $n \geq 1$, we consider the partition of the interval $[k-1, k[$ $(k \geq 1)$ by means of the n points $k - 1 \equiv t_{n(k-1)} < t_{n(k-1)+1} < \ldots < t_{nk-1} < t_{nk} \equiv k$, where $t_j - t_{j-1} \equiv \Delta t = 1/n$ for every $j \geq 1$, and we rescale the system accordingly by writing

$$\begin{aligned} S^{(n)}_{t_j} - S^{(n)}_{t_{j-1}} &= -d_{t_j}\hat{S}^{(n)}_{t_j} + \Delta t\left(\hat{S}^{(n)}_{t_j} - S^{(n)}_{t_{j-1}}\right) + \sigma_{t_j} Z^{(n)}_{t_j}, \\ \hat{S}^{(n)}_{t_{j+1}} - \hat{S}^{(n)}_{t_j} &= -\alpha_{t_{j+1}} d_{t_j}\hat{S}^{(n)}_{t_j} + \alpha_{t_{j+1}}\sigma_{t_j} Z^{(n)}_{t_j}, \end{aligned} \tag{5}$$

Notice that, since we want to make both the drift terms and the variance of the noise terms of the rescaled system (5) proportional to Δt, we are led to introduce the term $\Delta t\left(\hat{S}^{(n)}_{t_j} - S^{(n)}_{t_{j-1}}\right)$, and we are also led to require that $\left(Z^{(n)}_{t_j}\right)_{j\geq 1}$ is a sequence of independent and normally distributed real random variables having mean 0 and variance Δt. On the other hand, the discount rate d_{t_j} depends on Δt owing to its own nature. Actually $d_{t_j} \equiv d(t_j, t_{j+1}) \equiv d(t_j, \Delta t)$, and we assume

$$d_{t_j} = \delta_{t_j}\Delta t + o(\Delta t), \tag{6}$$

where δ_{t_j} is a suitable interest strenght at time t_j, for every $j \geq 0$.

Lkewise the solution of (3), the solution $\left(S_{t_j}^{(n)}, \hat{S}_{t_j}^{(n)}\right)_{j\geq 0}$ of (5) is a Markov chain with respect to the filtration $\left(\mathcal{F}_{t_j}^{(n)}\right)_{j\geq 0}$ generated by the sequence $\left(\hat{S}_0^{(n)}, Z_{t_1}^{(n)}, \ldots, Z_{t_n}^{(n)}, \ldots\right)$.

Now, we introduce the sequence $\left(W_{t_j}^{(n)}\right)_{j\geq 0}$ given by

$$W_{t_j}^{(n)} \overset{def}{=} \begin{cases} \sum_{i=1}^{j} Z_{t_i}^{(n)} & \text{if } j \geq 1 \\ 0 & \text{if } j = 0 \end{cases},$$

and we write

$$S_t^{(n)} \overset{def}{=} S_{t_j}^{(n)}, \quad \hat{S}_t^{(n)} \overset{def}{=} \hat{S}_{t_j}^{(n)}, \quad W_t^{(n)} \overset{def}{=} W_{t_j}^{(n)}, \quad \text{for } t_j \leq t < t_{j+1}.$$

The processess $\left(S_t^{(n)}\right)_{t\geq 0} \equiv S^{(n)}$, $\left(\hat{S}_t^{(n)}\right)_{t\geq 0} \equiv \hat{S}^{(n)}$, and $\left(W_t^{(n)}\right)_{t\geq 0} \equiv W^{(n)}$ have right continuous paths with finite left-hand limits (RCLL paths). Moreover, given the Polish space $D([0, +\infty[\,; \mathbb{R})$ of all RCLL paths endowed with the Skorohod distance, it is well known that the $D([0, +\infty[\,; \mathbb{R})$-valued sequence of random variables $\left(W^{(n)}\right)_{n\geq 1}$ converges weakly to the Wiener process starting at 0.

We want to show how, applying Nelson's criteria, it is possible to check the weak convergence of the sequence $\left(S^{(n)}, \hat{S}^{(n)}\right)_{n\geq 0}$, as n goes to infinity, to the solution of a system of stochastic differential equation. To simplify the analysis we assume a constant learning rate, market volatility and interest strenght:

$$\alpha_{t_j} \equiv \alpha, \quad \sigma_{t_j} \equiv \sigma, \quad \delta_{t_j} \equiv \delta, \quad \text{for } j \geq 1. \tag{7}$$

The results can be easily extended to time varying parameters. Our main result is the following.

Proposition 4 *As n goes to infinity, the sequence $\left(S^{(n)}, \hat{S}^{(n)}\right)_{n\geq 0}$ converges weakly to the solution of the system of stochastic differential equations*

$$\begin{cases} dS_t = \left((1-\delta)\hat{S}_t - S_t\right) dt + \sigma\, dW_t \\ d\hat{S}_t = -\alpha\delta\hat{S}_t\, dt + \alpha\sigma\, dW_t, \end{cases} \tag{8}$$

where $(W_t)_{t\geq 0}$ is a standard Wiener process.

Proof. The proof follows the guideline outlined in Nelson [22], and it is based on a classical existence result for stochastic differential equations (see [17, Chap. 5, Theor. 2.9]).

Let us consider first the matrix field

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^2 \otimes \mathbb{R}^2, \quad \sigma(x_1, x_2) \stackrel{def}{=} \begin{pmatrix} \sigma & 0 \\ \alpha\sigma & 0 \end{pmatrix},$$

and the vector field

$$b : \mathbb{R}^2 \to \mathbb{R}^2, \quad b(x_1, x_2) \equiv (b_1(x_1, x_2), b_2(x_1, x_2)),$$

where

$$b_1(x_1, x_2) \stackrel{def}{=} (1-\delta)x_2 - x_1, \quad b_2(x_1, x_2) \stackrel{def}{=} -\alpha\delta x_2.$$

Since it is easily seen that the conditions given in [17, Chap. 5, Theor. 2.9] hold true, we can conclude that (8) has a unique non-exploding strong solution for every given initial price S_0 and for every distribution of the expected price $\hat{S}_0$.

Now, following Nelson, to prove the weak convergence of $\left(S^{(n)}, \hat{S}^{(n)}\right)_{n\geq 0}$ to the solution of (8), as n goes to infinity, we want to show that the conditional variance-covariance matrix, and the conditioanl expectation vector per unit of time of $\left(S^{(n)}, \hat{S}^{(n)}\right)_{n\geq 0}$ converge uniformly on compact sets to the components of the symmetric non-negative definite matrix field

$$a : \mathbb{R}^2 \to \mathbb{R}^2, \quad a(x_1, x_2) \equiv \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix},$$

given by

$$a \stackrel{def}{=} \sigma\sigma^{\intercal},$$

and to the components of the vector field $b : \mathbb{R}^2 \to \mathbb{R}^2$ respectively.

To this task, observe that the hypotheses on the noise sequence $\left(Z_{t_j}^{(n)}\right)_{j\geq 1}$ and on the filtration $\left(\mathcal{F}_{t_j}^{(n)}\right)_{j\geq 0}$ give

$$\begin{aligned}
\mathbf{E}\left[S_{t_j}^{(n)} | \mathcal{F}_{t_j}^{(n)}\right] &= S_{t_j}^{(n)}, \\
\mathbf{E}\left[\hat{S}_{t_{j+1}}^{(n)} | \mathcal{F}_{t_j}^{(n)}\right] &= \hat{S}_{t_{j+1}}^{(n)}, \\
\mathbf{E}\left[Z_{t_{j+1}}^{(n)} | \mathcal{F}_{t_j}^{(n)}\right] &= \mathbf{E}\left[Z_{t_{j+1}}^{(n)}\right] = 0, \\
\mathbf{E}\left[\left(Z_{t_{j+1}}^{(n)}\right)^2 | \mathcal{F}_{t_j}^{(n)}\right] &= \mathbf{E}\left[\left(Z_{t_{j+1}}^{(n)}\right)^2\right] = \Delta t, \\
\mathbf{E}\left[\left(Z_{t_{j+1}}^{(n)}\right)^3 | \mathcal{F}_{t_j}^{(n)}\right] &= \mathbf{E}\left[\left(Z_{t_{j+1}}^{(n)}\right)^3\right] = 0, \\
\mathbf{E}\left[\left(Z_{t_{j+1}}^{(n)}\right)^3 | \mathcal{F}_{t_j}^{(n)}\right] &= \mathbf{E}\left[\left(Z_{t_{j+1}}^{(n)}\right)^3\right] = 3\Delta t^2.
\end{aligned}$$

Then, taking into account of (6) and (7), by straightforward computations, we obtain

$$\text{(9)}\qquad \mathbf{E}\left[\left(S_{t_j}^{(n)}-S_{t_{j-1}}^{(n)}\right)|\mathcal{F}_{t_{j-1}}^{(n)}\right] = -d\hat{S}_{t_j}^{(n)}+\Delta t\left(\hat{S}_{t_j}^{(n)}-S_{t_{j-1}}^{(n)}\right),$$

$$\text{(10)}\qquad \mathbf{E}\left[\left(\hat{S}_{t_{j+1}}^{(n)}-\hat{S}_{t_j}^{(n)}\right)|\mathcal{F}_{t_{j-1}}^{(n)}\right] = -\alpha d\hat{S}_{t_j}^{(n)},$$

$$\text{(11)}\qquad \mathbf{E}\left[\left(S_{t_j}^{(n)}-S_{t_{j-1}}^{(n)}\right)^2|\mathcal{F}_{t_{j-1}}^{(n)}\right] = d^2\left(\hat{S}_{t_j}^{(n)}\right)^2+\Delta t^2\left(\hat{S}_{t_j}^{(n)}-S_{t_{j-1}}^{(n)}\right)^2+\Delta t\sigma^2 -2\Delta t d\hat{S}_{t_j}^{(n)}\left(\hat{S}_{t_j}^{(n)}-S_{t_{j-1}}^{(n)}\right),$$

$$\text{(12)}\qquad \mathbf{E}\left[\left(S_{t_j}^{(n)}-S_{t_{j-1}}^{(n)}\right)\left(\hat{S}_{t_{j+1}}^{(n)}-\hat{S}_{t_j}^{(n)}\right)|\mathcal{F}_{t_{j-1}}^{(n)}\right] = \alpha d^2\left(\hat{S}_{t_j}^{(n)}\right)^2-\Delta t\alpha d\left(\hat{S}_{t_j}^{(n)}-S_{t_{j-1}}^{(n)}\right)\hat{S}_{t_j}^{(n)} +\Delta t\alpha\sigma^2,$$

$$\text{(13)}\qquad \mathbf{E}\left[\left(\hat{S}_{t_{j+1}}^{(n)}-\hat{S}_{t_j}^{(n)}\right)^2|\mathcal{F}_{t_{j-1}}^{(n)}\right] = \alpha^2 d^2\left(\hat{S}_{t_j}^{(n)}\right)^2+\Delta t\alpha^2\sigma^2,$$

Now, writing $P_{t_{j-1},t_j}^{(n)}:\mathbb{R}^2\times\mathfrak{B}(\mathbb{R}^2)\to\mathbb{R}_+$ for the j-th transition probability of the Markov chain $\left(S_{t_j}^{(n)},\hat{S}_{t_j}^{(n)}\right)_{j\geq 0}$, where $\mathfrak{B}(\mathbb{R}^2)$ denotes the Borel σ-algebra on $\mathbb{R}^2$, for $k,l=1,2$, we define

$$\hat{a}_{k,l}^{(n)}(x_1,x_2)\overset{def}{=}\int_{\mathbb{R}^2}(y_k-x_k)(y_l-x_l)\,P_{t_{j-1},t_j}^{(n)}(x_1,x_2,dy_1,dy_2),$$

and

$$\hat{b}_k^{(n)}(x_1,x_2)\overset{def}{=}\int_{\mathbb{R}^2}(y_k-x_k)\,P_{t_{j-1},t_j}^{(n)}(x_1,x_2,dy_1,dy_2),$$

where we claim that the integrals on the right hand side of the above equalities exist and are finite.

Indeed, setting

$$c_k^{(n)}(x_1,x_2)\overset{def}{=}\Delta t^{-1}\int_{\mathbb{R}^2}(y_k-x_k)^4\,P_{t_{j-1},t_j}^{(n)}(x_1,x_2,dy_1,dy_2)$$

for $k=1,2$, and recalling that the Markov property gives

$$\mathbf{E}\left[\left(S_{t_j}^{(n)}-S_{t_{j-1}}^{(n)}\right)^4|\mathcal{F}_{t_{j-1}}^{(n)}\right]=\int_{\mathbb{R}^2}\left(y_k-S_{t_{j-1}}^{(n)}\right)^4P_{t_{j-1},t_j}^{(n)}\left(\hat{S}_{t_{j-1}}^{(n)},S_{t_{j-1}}^{(n)},dy_1,dy_2\right)$$

and

$$\mathbf{E}\left[\left(\hat{S}_{t_j}^{(n)}-\hat{S}_{t_{j-1}}^{(n)}\right)^4|\mathcal{F}_{t_{j-1}}^{(n)}\right]=\int_{\mathbb{R}^2}\left(y_k-\hat{S}_{t_{j-1}}^{(n)}\right)^4P_{t_{j-1},t_j}^{(n)}\left(\hat{S}_{t_{j-1}}^{(n)},S_{t_{j-1}}^{(n)},dy_1,dy_2\right),$$

computing

$$\mathbf{E}\left[\left(S_{t_j}^{(n)}-S_{t_{j-1}}^{(n)}\right)^4|\mathcal{F}_{t_{j-1}}^{(n)}\right]$$

$$
\begin{aligned}
= \; & d^4 \left(\hat{S}_{t_j}^{(n)}\right)^4 - 4\Delta t d^3 \left(\hat{S}_{t_j}^{(n)}\right)^3 \left(\hat{S}_{t_j}^{(n)} - S_{t_{j-1}}^{(n)}\right) \\
& +6\Delta t^2 d^2 \left(\hat{S}_{t_j}^{(n)}\right)^2 \left(\hat{S}_{t_j}^{(n)} - S_{t_{j-1}}^{(n)}\right)^2 + 6\Delta t d^2 \sigma^2 \left(\hat{S}_{t_j}^{(n)}\right)^2 \\
& -4\Delta t^3 d \hat{S}_{t_j}^{(n)} \left(\hat{S}_{t_j}^{(n)} - S_{t_{j-1}}^{(n)}\right)^3 - 12\Delta t^2 d\sigma^2 \hat{S}_{t_j}^{(n)} \left(\hat{S}_{t_j}^{(n)} - S_{t_{j-1}}^{(n)}\right) \\
& +\Delta t^3 \left(\hat{S}_{t_j}^{(n)} - S_{t_{j-1}}^{(n)}\right)^4 + 6\Delta t^3 \sigma^2 \left(\hat{S}_{t_j}^{(n)} - S_{t_{j-1}}^{(n)}\right)^2 + 3\Delta t^2 \sigma^4,
\end{aligned}
$$

and

$$
\mathbf{E}\left[\left(\hat{S}_{t_j}^{(n)} - \hat{S}_{t_{j-1}}^{(n)}\right)^4 | \mathcal{F}_{t_{j-1}}^{(n)}\right] = \alpha^4 d^4 \left(\hat{S}_{t_j}^{(n)}\right)^4 + 6\Delta t \alpha^4 d^2 \left(\hat{S}_{t_j}^{(n)}\right)^2 + 3\Delta t^2 \alpha^4 \sigma^4,
$$

we obtain

$$
\begin{aligned}
c_1^{(n)}(x_1, x_2) \; = \; & \Delta t^{-1} d^4 x_2^4 - 4d^3 x_2^3 (x_2 - x_1) + 6\Delta t d^2 x_2^2 (x_2 - x_1)^2 + 6d^2 \sigma^2 x_2^2 \\
& -4\Delta t^2 d x_2 (x_2 - x_1)^3 - 12\Delta t d\sigma^2 x_2 (x_2 - x_1) + \Delta t^3 (x_2 - x_1)^3 \\
& +6\Delta t^2 \sigma^2 (x_2 - x_1)^2 + 3\Delta t \sigma^4,
\end{aligned}
$$

and

$$
c_2^{(n)}(x_1, x_2) = \Delta t^{-1} \alpha^4 d^4 x_2^4 + 6\alpha^4 d^2 x_2^2 + 3\Delta t \alpha^4 \sigma^4.
$$

Taking again into account of (6) and (7), it follows that for $k = 1, 2$

$$
\lim_{n \to \infty} c_k^{(n)}(x_1, x_2) = 0,
$$

uniformly on compact sets of $\mathbb{R}^2$, which gives our claim (see [22, sect. 2.2]).

This existence and finiteness result allows us to combine the relations

$$
\mathbf{E}\left[\left(S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)}\right) | \mathcal{F}_{t_{j-1}}^{(n)}\right] = \int_{\mathbb{R}^2} \left(y_1 - S_{t_{j-1}}^{(n)}\right) P_{t_{j-1},t_j}^{(n)} \left(S_{t_{j-1}}^{(n)}, \hat{S}_{t_j}^{(n)}, dy_1, dy_2\right),
$$

$$
\mathbf{E}\left[\left(\hat{S}_{t_{j+1}}^{(n)} - \hat{S}_{t_j}^{(n)}\right) | \mathcal{F}_{t_{j-1}}^{(n)}\right] = \int_{\mathbb{R}^2} \left(y_2 - \hat{S}_{t_{j+1}}^{(n)}\right) P_{t_{j-1},t_j}^{(n)} \left(S_{t_{j-1}}^{(n)}, \hat{S}_{t_j}^{(n)}, dy_1, dy_2\right),
$$

$$
\mathbf{E}\left[\left(S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)}\right)^2 | \mathcal{F}_{t_{j-1}}^{(n)}\right] = \int_{\mathbb{R}^2} \left(y_1 - S_{t_{j-1}}^{(n)}\right)^2 P_{t_{j-1},t_j}^{(n)} \left(S_{t_{j-1}}^{(n)}, \hat{S}_{t_j}^{(n)}, dy_1, dy_2\right),
$$

$$
\begin{aligned}
& \mathbf{E}\left[\left(S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)}\right)\left(\hat{S}_{t_{j+1}}^{(n)} - \hat{S}_{t_j}^{(n)}\right) | \mathcal{F}_{t_{j-1}}^{(n)}\right] \\
& \quad = \int_{\mathbb{R}^2} \left(y_1 - S_{t_{j-1}}^{(n)}\right)\left(y_2 - \hat{S}_{t_j}^{(n)}\right) P_{t_{j-1},t_j}^{(n)} \left(S_{t_{j-1}}^{(n)}, \hat{S}_{t_j}^{(n)}, dy_1, dy_2\right),
\end{aligned}
$$

$$
\mathbf{E}\left[\left(\hat{S}_{t_{j+1}}^{(n)} - \hat{S}_{t_j}^{(n)}\right)^2 | \mathcal{F}_{t_{j-1}}^{(n)}\right] = \int_{\mathbb{R}^2} \left(y_2 - \hat{S}_{t_j}^{(n)}\right)^2 P_{t_{j-1},t_j}^{(n)} \left(S_{t_{j-1}}^{(n)}, \hat{S}_{t_j}^{(n)}, dy_1, dy_2\right),
$$

with (9)-(13), to obtain

$$\begin{aligned}
\hat{a}_{1,1}^{(n)}(x_1,x_2) &= d^2x_2^2+\Delta t^2(x_2-x_1)^2+\sigma^2\Delta t-2\Delta t d x_2(x_2-x_1),\\
\hat{a}_{1,2}^{(n)}(x_1,x_2) &= \hat{a}_{2,1}^{(n)}(x_1,x_2)=\alpha d^2x_2^2-\Delta t\alpha d(x_2-x_1)x_2+\Delta t\alpha\sigma^2,\\
\hat{a}_{2,2}^{(n)}(x_1,x_2) &= \alpha^2d^2x_2^2+\Delta t\alpha^2\sigma^2,\\
\hat{b}_{1}^{(n)}(x_1,x_2) &= -dx_2+\Delta t(x_2-x_1),\\
\hat{b}_{2}^{(n)}(x_1,x_2) &= -\alpha d x_2.
\end{aligned}$$

Thererefore, writing

$$a_{k,l}^{(n)}(x_1,x_2)\overset{def}{=}\Delta t^{-1}\left(\hat{a}_{k,l}^{(n)}(x_1,x_2)-\hat{b}_{k}^{(n)}(x_1,x_2)\,\hat{b}_{l}^{(n)}(x_1,x_2)\right),$$

and

$$b_{k}^{(n)}(x_1,x_2)\overset{def}{=}\Delta t^{-1}\hat{b}_{k}^{(n)}(x_1,x_2)$$

for $k,l=1,2$, we have

$$a_{1,1}^{(n)}(x_1,x_2)=\sigma^2,\qquad a_{1,2}^{(n)}(x_1,x_2)=a_{2,1}^{(n)}(x_1,x_2)=\alpha\sigma^2,\qquad a_{2,2}^{(n)}(x_1,x_2)=\alpha^2\sigma^2$$

and

$$b_{1}^{(n)}(x_1,x_2)=-\Delta t^{-1}dx_2+x_2-x_1,\qquad b_{2}^{(n)}(x_1,x_2)=-\Delta t^{-1}\alpha d x_2.$$

Hence it is immediately seen that, for all $k,l=1,2$, we have

$$\lim_{n\to\infty}a_{k,l}^{(n)}(x_1,x_2)=a_{k,l}\quad\text{and}\quad\lim_{n\to\infty}b_{k}^{(n)}(x_1,x_2)=b_k(x_1,x_2)$$

uniformly on compact sets of $\mathbb{R}^2$.

What shown above implies that we are in a position to apply Nelson's criteria (see [22, 2.2 - 2.3]), and the desired result easily follows. □

System (8) can be integrated by means of a standard procedure (see [17, 5.6, p. 354]) and the solution $\left(S_t,\hat{S}_t\right)_{j\geq 0}$ is given by

$$S_t = \frac{1-\delta}{1-\alpha\delta}\hat{S}_t+\left(S_0-\frac{1-\delta}{1-\alpha\delta}\hat{S}_0\right)e^{-t}+\frac{1-\alpha}{1-\alpha\delta}\sigma e^{-t}\int_0^t e^s\,dW_s \tag{14}$$

$$\hat{S}_t = \hat{S}_0e^{-\alpha\delta t}+\alpha\sigma e^{-\alpha\delta t}\int_0^t e^{\alpha\delta s}\,dW_s. \tag{15}$$

Therefore, in our model, the limiting price process looks like a mean-reverting Ornstein-Ulenbeck process around the level given by agent's expectation process.

Having obtained an explicit form (14) for the limiting price process, we can apply Itô calculus to compute the main features of $(S_t)_{j\geq 0}$. In particular, it is matter of straightforward computations to prove the following result

Proposition 5 *For all* $t, \Delta t \geq 0$ *we have:*

$$
\begin{aligned}
(16)\ & Cov\left(S_{t+\Delta t} - S_t, S_t - S_{t-\Delta t}\right) \\
= & \frac{(1-\delta)^2}{(1-\alpha\delta)^2}\left(\mathbf{D}^2\left[\hat{S}_0\right] - \frac{1}{2}\sigma^2\frac{\alpha}{\delta}\right) e^{-\alpha\delta(2t-\Delta t)}\left(1-e^{-\alpha\delta\Delta t}\right)^2 \\
& + \frac{(1-\delta)^2}{(1-\alpha\delta)^2}\left(\mathbf{D}^2\left[\hat{S}_0\right] - \frac{1}{2}\sigma^2\frac{(1-\alpha)^2}{(1-\delta)^2}\right) e^{-(2t-\Delta t)}\left(1-e^{-\Delta t}\right)^2 \\
& - \frac{(1-\delta)^2}{(1-\alpha\delta)^2}\mathbf{D}^2\left[\hat{S}_0\right] e^{-(1+\alpha\delta)t}\left(e^{\Delta t}\left(1-e^{-\Delta t}\right)^2 - e^{(1-\alpha\delta)\Delta t}\left(1-e^{-(1-\alpha\delta)\Delta t}\right)^2\right. \\
& \qquad \left. + e^{\alpha\delta\Delta t}\left(1-e^{-\alpha\delta\Delta t}\right)^2\right) \\
& - \frac{1}{2}\frac{\sigma^2\alpha(1-\delta)^2}{\delta(1-\alpha\delta)^2}\left(1-e^{-\alpha\delta\Delta t}\right)^2 \\
& - \frac{\sigma^2\alpha(1-\alpha)(1-\delta)}{(1+\alpha\delta)(1-\alpha\delta)^2}\left(\left(1-e^{\alpha\delta\Delta t}\right)^2 + \left(1-e^{-\Delta t}\right)^2\right. \\
& \qquad \left. + e^{-(1+\alpha\delta)t}\left(\left(e^{\Delta t}-1\right)\left(1-e^{-\alpha\delta\Delta t}\right) + \left(e^{\alpha\delta\Delta t}-1\right)\left(1-e^{-\Delta t}\right)\right)\right) \\
& - \frac{1}{2}\frac{\sigma^2(1-\alpha)^2}{(1-\alpha\delta)^2}\left(1-e^{-\Delta t}\right)^2
\end{aligned}
$$

Proof. Indeed, from (14) it follows that for all $0 \leq s \leq t$ we have:

$$
(17) \qquad \mathbf{E}\left[S_t\right] = \frac{1}{1-\alpha\delta}\mathbf{E}\left[\hat{S}_0\right] e^{-\alpha\delta t} + \left(S_0 - \frac{1-\delta}{1-\alpha\delta}\mathbf{E}\left[\hat{S}_0\right]\right) e^{-t}
$$

and

$$
\begin{aligned}
(18)\ & Cov\left(S_s, S_t\right) \\
= & \frac{(1-\delta)^2}{(1-\alpha\delta)^2}\mathbf{D}^2\left[\hat{S}_0\right]\left(e^{-\alpha\delta s} - e^{-s}\right)\left(e^{-\alpha\delta t} - e^{-t}\right) \\
& + \frac{1}{2}\frac{\sigma^2\alpha(1-\delta)^2}{\delta(1-\alpha\delta)^2} e^{-\alpha\delta(s+t)}\left(e^{2\alpha\delta s} - 1\right). \\
& + \frac{\sigma^2\alpha(1-\alpha)(1-\delta)}{(1+\alpha\delta)(1-\alpha\delta)^2}\left(e^{-(s+\alpha\delta t)}\left(e^{(1+\alpha\delta)s} - 1\right) + e^{-(\alpha\delta s+t)}\left(e^{(1+\alpha\delta)s} - 1\right)\right) \\
& + \frac{1}{2}\frac{\sigma^2(1-\alpha)^2}{(1-\alpha\delta)^2} e^{-(s+t)}\left(e^{2s} - 1\right).
\end{aligned}
$$

From the latter, thanks to the bilinearity property of the covariance functional, we obtain the stated result. □

Equation (16) shows clearly that if the variance of the expected initial price $\hat{S}_0$ is small enough, more precisely if

$$\mathbf{D}^2\left[\hat{S}_0\right] \leq \frac{1}{2}\frac{\sigma^2\alpha}{\delta} \wedge \frac{1}{2}\sigma^2\frac{(1-\alpha)^2}{(1-\delta)^2},$$

then the price process increments are negatively correlated. Moreover, (16) shows that for any value of $\mathbf{D}^2\left[\hat{S}_0\right]$ and for any time step Δt, the price process increments become negatively correlated as times flows. In particular, if the expected price $\hat{S}_0$ is a datum, then the price process increments are always negatively correlated. This result shows that a pure noise in a bounded rationality economy produces a mean reverting effect.

References

[1] Barsky R., De Long, B. (1993): Why does the stock market fluctuate? *Quarterly Journal of Economics*, CVIII: 291-311.

[2] Barucci, E. (1998): *Exponentially Fading Memory Learning in Forward Looking Economic Models.* to appear in *Journal of Economic Dynamics and Control.*

[3] Bick, A. (1987): On the Consistency of the Black-Scholes Model with a General Equilibrium Framework. *Journal of Financial and Quantitative Analysis*, 22: 259-275.

[4] Billingsley, P. (1968): *Convergenge of Probability Measures.* Wiley, New York.

[5] Buckley, G., Tonks, I. (1989): Are UK Stock Prices Excessively Volatile?: Trading Rules and Variance Bounds Tests, *Economic Journal*, 99: 1083-1098.

[6] Campbell, J. Lo, A., MacKinlay, C. (1997): *The Econometrics of Financial Markets.* Princeton University Press, Princeton.

[7] Cox, J., Ingersoll, J., Ross, S. (1985): An Intertemporal General Equilibrium Model of Asset Prices. *Econometrica*, 53: 363-384.

[8] Cox, J., Ingersoll, J., Ross, S. (1985): A Theory of the Term Structure of Interest Rates. *Econometrica*, 53: 385-407.

[9] Cutler, D., Poterba, J., Summers, L. (1990): Speculative Dynamics and the Role of Feedback Traders *The American Economic Review*, Proceedings: 63-68.

[10] Cutler, D., Poterba, J., Summers, L. (1991): Speculative Dynamics *Review of Economic Studies* 529-546.

[11] Duffie, D. (1996): *Dynamic Asset Pricing Theory.* Princeton University Press, Princeton.

[12] Fama, E. (1991): Efficient Capital Markets: II. *Journal of Finance*, 46: 1575-1618.

[13] Fama, E. (1998): Market Efficiency, Long-Term Returns, and Behavioural Finance *Journal of Financial Economics*, 49: 283-306.

[14] Fama, E., French, K. (1988): Permanent and Temporary Components of Stock Prices *Journal of Political Economy*, 96: 246-273.

[15] Follmer, H., Schweizer, M. (1993): A Microeconomic Approach to Diffusion Models for Stock Prices. *Mathematical Finance*, 3: 1-23.

[16] Goldstein, R., Zapatero, F. (1996): General Equilibrium with Constant Relative Risk Adversion and Vasicek Interest Rates. *Mathematical Finance*, 331-340.

[17] Karatzas, I., Sheevre, S.E. (1988): Brownian Motion and Stochastic Calculus. Springer-Verlag, New York.

[18] Kreps, D. (1982): Multiperiod Securities and the Efficient Allocation of Risk: A comment on the Black and Scholes Option Pricing Model. *The Economics of Uncertainty and Information*, McCall ed.. University of Chicago press. Chicago.

[19] Kushner H.J. (1984): *Approximation and Weak Convergence Methods for Random Processes, with Applications to Stochastic Systems Theory.* M.I.T. Press, Cambridge (MA).

[20] Lakonishok, J., Schleifer, A., Vishny, R. (1994): Contrarian Invetsment, Extrapolation and Risk. *Journal of Finance*, 49: 1541-1578.

[21] Lucas, R. (1978): Asset Prices in an Exchange Economy. *Econometrica*,¡46: 1429-45.

[22] Nelson, E. (1998): ARCH Models as Diffusion Approximations. *Journal of Econometrics*, 45: 7-38.

[23] Shiller, R. (1984): Stock Prices and Social Dynamics. Brookings Papers on Economic Activity, 2: 457, 498

[24] Stroock, D. W., Varadhan S.R.R. (1979): Multidimensional Diffusion Processes. Springer-Verlag, Berlin.

[25] Timmermann, A. (1993): How Learning in Financial Markets Generates Excess Volatility and Predictability in Stock Prices. *Quarterly Journal of Economics*, 108: 1135-1145.

[26] Timmermann, A. (1994): Can Agents Learn to Form Rational Expectations? Some Results on Convergence and Stability of Learning in the UK Stock Market. *Economic Journal*, 104: 777-797.

[27] Timmermann, A. (1996): Excess Volatility and Predictability of Stock Prices in Autoregressive Dividend Models with Learning. *Review of Economic Studies*, 63: 523-557.

[28] Vasicek, O. (1977): An Equilibrium Characterization of the Term Structure. *Journal of Financial Economics*, 5: 177-188.