Report n. 153

# On the supremum in fractional programming

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Pisa, November 1999

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**Abstract**

We consider a constrained maximization problem with a quadratic fractional function $f$ over any closed and unbounded set $X$. The behavior of feasible unbounded sequences is studied in order to derive conditions under which $f$ attains maximum, conditions guaranteeing its supremum is finite and finally conditions that ensure $\sup f(x) = +\infty$. We first consider a function $f$ where the quadratic form is semidefinite and then we specify our results for a particular fractional programming problem in a more general context. Our results cover the linear fractional case which can be seen as a particular case of the previous ones. Moreover we give a new characterization of pseudoconvexity for a quadratic fractional function.

*Keywords:* Fractional Programming, Generalized Convexity

*1991 Mathematics Subject Classification.* Primary 90C32; Secondary 90C26.

## 1 Introduction

The optimization problems with unbounded feasible regions have been exhaustively handled in recent literature. Theoretical and algorithmic aspects have been studied in order to find both constructive new methods and conditions under which the maximum value exists. See for example the survey proposed by Auslender [1] or the wide literature of fractional programming problem dealing with polyhedral feasible region (Schaible [8]).

Even though we can find many results for optimality conditions, there is almost nothing about the supremum of a function over an unbounded feasible

*The paper has been discussed jointly by the authors. In particular, sections 3 and 4 have been developed by Laura Carosi.

region. On the other hand, when a function does not attain maximum value it is important to know if its supremum is finite or not. For the remarkable role played both in optimization and in economic theory, we turn our attention to a nonlinear fractional problem where the objective function is a ratio between a quadratic and a linear function and the feasible region is any closed and unbounded set. We study the case where the quadratic function is convex or concave, or it is the product between two affine functions. For these classes of problems, several necessary and/or sufficient conditions for a finite supremum are established by means of the recession cone of the feasible set and suitable directions associated with such a cone. Our results cover the linear and the linear fractional case which can be seen as a particular case of the previous ones.

To simplify notation and to save words, we take advantage of the fact that, for any bounded (unbounded) sequence, there exists a convergent (divergent in norm) subsequence. Namely, when we say that a bounded sequence $\{x_n\}$ converges to $x$, we mean that if this sequence is not convergent, then it is replaced by an appropriate subsequence and this subsequence is again denoted by $\{x_n\}$. Similar abuse of language and notation is applied to unbounded sequence, when we say that $\{x_n\}$ diverges to infinity.

# 2 Pseudo concavity of a quadratic fractional function

Consider the function

$$f(x) = \frac{x^T Q x + a^T x + a_0}{b^T x + b_0}$$

where $Q$ is a $n \times n$ symmetric square matrix, $a, x, b \in \Re^n$ and $a_0, b_0 \in \Re$. We want to investigate the properties of this function. It is well known [2] that if the matrix $Q$ is negative (positive) semidefinite then the function $f$ is pseudo concave (pseudo convex). The following Theorem extends this kind of result establishing a necessary and sufficient condition for pseudo concavity (pseudo convexity) of $f$.

**Theorem 2.1** *Consider the function*

$$f(x) = \frac{x^T Q x + a^T x + a_0}{b^T x + b_0}, \quad x \in X = \{x \in \Re^n : b^T x + b_0 > 0\}$$

*i) $f$ is pseudo concave if and only if*

$$\forall u \in \Re^n, \forall x^0 \in X : u^T Q u > 0 \Longrightarrow u^T H(x^0) u \leq 0 \tag{2.1}$$

*ii) f is pseudo convex if and only if*

$$\forall u \in \Re^n, \forall x^0 \in X : u^T Q u < 0 \Longrightarrow u^T H(x^0) u \geq 0 \tag{2.2}$$

**Proof.** i) As it is well known, a function is pseudo concave if and only if any of its restrictions on a feasible line is pseudo concave too. For such a reason we study the pseudo concavity of the function $\varphi(t) = f(x_0 + tu)$, for every $x_0 \in X$, for every $u \in \Re^n$. We have

$$\varphi(t) = f\left(x^0 + tu\right) = \frac{\alpha t^2 + \beta t + \gamma}{\delta t + \delta_0}, \; \delta t + \delta_0 > 0$$

where $\alpha = u^T Q u$, $\beta = u^T\left(2Qx^0 + a\right)$, $\gamma = x^{0T} Q x^0 + a^T x^0 + a_0$, $\delta = b^T u$, $\delta_0 = b^T x^0 + b_0$.

Let us note that when $\alpha = 0$ the function $\varphi(t)$ is linear fractional so that it is both pseudo concave and pseudo convex and the thesis follows. Consider now the case $\alpha \neq 0$.

It is known (see for example [3]) that $\varphi$ is pseudo concave if and only if the following condition holds:

$$\begin{array}{c} \forall t^0 : \delta t^0 + \delta_0 > 0 \text{ with } \varphi'\left(t^0\right) = 0 \text{ we have either } \varphi''\left(t^0\right) < 0 \\ \text{or } \varphi''\left(t^0\right) = 0 \text{ and } t^0 \text{ is a relative maximum point for } \varphi(t) \end{array} \tag{2.3}$$

Note that $u^T H(x^0) u = \varphi''(0)$ so that (2.1) is equivalent to the following

$$\alpha > 0 \Rightarrow \varphi''(0) \leq 0 \tag{2.4}$$

Consequently i) is equivalent to prove that (2.3) implies (2.4) and viceversa. Taking the first and the second derivative of the function $\varphi$ we have

$$\varphi'(t) = \frac{\alpha\delta t^2 + 2\alpha\delta_0 t + \beta\delta_0 - \delta\gamma}{\left(\delta t + \delta_0\right)^2}$$

$$\varphi''(t) = \frac{2\left(\alpha\delta_0^2 - \beta\delta\delta_0 + \delta^2\gamma\right)}{\left(\delta t + \delta_0\right)^3}.$$

Setting

$$\Delta = \alpha\left(\alpha\delta_0^2 - \beta\delta\delta_0 + \delta^2\gamma\right)$$

we have

$$\varphi''(t) = \frac{2}{\alpha}\Delta\frac{1}{\left(\delta t + \delta_0\right)^3}.$$

When $\Delta = 0$ we have $\varphi''(t) = 0$ for every $t$, so that $\varphi$ is linear and in particular it is both pseudo concave and pseudo convex. It remains to analyze the case $\Delta \neq 0$.

(2.3) $\Longrightarrow$ (2.4)

Assume that $\alpha > 0$ and $\varphi''(0) > 0$, hence $\Delta > 0$. Let us note that $t_0 = \frac{-\alpha\delta_0 + \sqrt{\Delta}}{\alpha\delta}$ is feasible since $\delta t_0 + \delta_0 = \frac{1}{\alpha}\sqrt{\Delta} > 0$ and it verifies $\varphi'(t_0) = 0$. Since $\Delta > 0$ implies $\varphi''(t) > 0$ for every $t$ with $\delta t + \delta_0 > 0$, we have in particular that $\varphi''(t_0) > 0$ and this contradicts (2.3).

(2.4) $\Longrightarrow$ (2.3).

Let be $t_0$ such that $\delta t + \delta_0 > 0$ and $\varphi'(t_0) = 0$; necessarily we have $\Delta \geq 0$. The case $\Delta = 0$, $\alpha \neq 0$ implies $\varphi''(t) = 0$ for every $t$, so that $\varphi'$ is constant and since $\varphi'(t_0) = 0$, it results $\varphi'(t) = 0$ for every $t$, that is $\varphi$ is a constant function. Consequently (2.3) is verified. Taking into account (2.4), $\Delta > 0$ implies $\alpha < 0$ hence $\varphi''(t) < 0$ for every $t$ and (2.3) is verified. ■

The following example shows that Theorem 2.1 generalizes the classical result dealing with semidefinite matrices which can be recovered as a particular case.

**Example 2.1** *Consider the function* $f(x, y) = \frac{x^2 - y^2}{x}$ *and the set* $X = \{(x, y) \in \Re^2 : x > 0\}$. *The quadratic form is indefinite; nevertheless, it is easy to verify that* $f$ *is concave and in particular pseudo concave.*

# 3 On the supremum of a quadratic fractional problem.

Consider the following quadratic fractional program

$$\sup_{x \in X} \left[ f(x) = \frac{x^T Q x + a^T x + a_0}{b^T x + b_0} \right] \tag{3.1}$$

where $a, b, x \in \Re^n$, $a_0, b_0 \in \Re$. $Q$ is a symmetric matrix, $X$ is any closed and unbounded set and $b^T x + b_0 \geq \delta > 0$ for every $x \in X$.

In order to study the existence of finite supremum for Problem (3.1), it is necessary to analyze the behavior of $f$ along unbounded feasible sequences.

With this aim, we recall the following definitions and properties [5], [7].

**Definition 3.1** *Let* $X \subset \Re^n$ *be a closed set. The recession cone of* $X$ *is the set* $RecX = \{d \in \Re^n :$ *there exist* $\{\alpha_n\} \subset \Re_{++}$, $\{x_n\} \subset X$, *such that* $\alpha_n \to 0$, $\alpha_n x_n \to d.\}$

**Definition 3.2** *Let $X \subset \Re^n$ be a closed set. The set of recession direction is the set $O^+X = \{d \in RecX : \|d\| = 1\}$.*

The following properties are well known:

**Proposition 3.1** *Let $X$ be a closed set and $RecX$ its recession cone.*
*i) $RecX$ is closed cone.*
*ii) $X$ is bounded if and only if $RecX = \{0\}$.*
*iii)$O^+X$ is a compact set.*

The following Lemma holds.

**Lemma 3.1** *Let $\{x_n\}$ be a feasible unbounded sequence for Problem (3.1) and set $d = \lim_{n\to+\infty} \frac{x_n}{\|x_n\|}$. If $d^TQd \neq 0$, then $f(x_n) \to +\infty$ or $f(x_n) \to -\infty$ according to $d^TQd > 0$ or $d^TQd < 0$.*

**Proof.** It follows by calculating the limit of

$$f(x_n) = \frac{\frac{x_n}{\|x_n\|}^T Q \frac{x_n}{\|x_n\|} \|x_n\| + a^T \frac{x_n}{\|x_n\|} + \frac{a_0}{\|x_n\|}}{b^T \frac{x_n}{\|x_n\|} + \frac{b_0}{\|x_n\|}}$$

■

As a direct consequence of the previous Lemma we obtain a necessary and a sufficient condition for the supremum of Problem (3.1) to be finite.

**Theorem 3.1** *i) If $\sup_{x\in X} f(x) < +\infty$, then $d^TQd \leq 0$ for every $d \in O^+X$.*
*ii) If $d^TQd < 0$ for every $d \in O^+X$ then $\sup_{x\in X} f(x) = \max_{x\in X} f(x)$.*

**Proof.** i) If there exists $d \in O^+X$ such that $d^TQd > 0$, from Lemma 3.1 there exists a feasible unbounded sequence $\{x_n\}$ such that $\frac{x_n}{\|x_n\|} \to d$, $f(x_n) \to +\infty$ and this is absurd.

ii) Let $\{x_n\}$ be a feasible sequence such that $\lim_{n\to\infty} f(x_n) = \sup_{x\in X} f(x)$. If $\{x_n\}$ is unbounded then $\frac{x_n}{\|x_n\|} \to d \in O^+X$, so that, from Lemma 3.1, $f(x_n) \to -\infty$ and this is absurd. Consequently $\{x_n\}$ is bounded so that it converges to a point $x^* \in X$. From the continuity of the function $f$, we have

$$\lim_{n\to\infty} f(x_n) = f(x^*) = \sup_{x\in X} f(x).$$

■

The following example shows that the necessary condition i) of Theorem 3.1 is not sufficient to reach the finite supremum.

**Example 3.1** *Consider Problem (3.1) where* $f(x,y,z) = \dfrac{x^2+z^2}{x+y+1}$ *and* $X = \{(x,y,z) \in \Re^3 : x \geq 0, y = x^2, z = x\sqrt{x}\}$. *It results* $O^+X = \{d = (0,1,0)\}$, $d^TQd = 0$ *and* $\sup_{(x,y,z)\in X} f(x,y,z) = +\infty$, *since* $\lim_{x\to+\infty} f(x,x^2,\sqrt{x}) = +\infty$.

The previous example points out the necessity to deepen the analysis in the case $d^TQd = 0$. With this aim we will assume that $Q$ is semidefinite positive or negative; as it is well known, for such matrices it results $d^TQd = 0$ if and only if $Qd = 0$.

Let $KerQ = \{x \in \Re^n : Qx = 0\}$ and denote with $(KerQ)^\perp$ its orthogonal subspace. Any element $z \in \Re^n$ can be written as $z = z^* + z^\perp$ where $z^* \in KerQ$ and $z^\perp \in (KerQ)^\perp$.

The following Lemma holds.

**Lemma 3.2** *Consider Problem (3.1) where* $Q$ *is semidefinite and let* $\{x_n\}$ *be an unbounded feasible sequence with* $\frac{x_n}{\|x_n\|} \to d$. *Then i) and ii) hold.*

*i) If* $d \in KerQ$ *and* $\frac{x_n^\perp}{\sqrt{\|x_n\|}} \to w$, *with the convention* $w = 0$ *if* $x_n^\perp \equiv 0$, *then* $\lim_{n\to+\infty} f(x_n) = \dfrac{w^TQw + a^Td}{b^Td}$.

*ii) If* $d \in KerQ$ *and* $\left\{\frac{x_n^\perp}{\sqrt{\|x_n\|}}\right\}$ *is an unbounded sequence then* $f(x_n) \to +\infty$ *or* $f(x_n) \to -\infty$ *according to* $Q$ *is semidefinite positive or semidefinite negative.*

**Proof.** i) Taking into account that $x_n^TQx_n = x_n^{\perp T}Qx_n^\perp$ we have

$$f(x_n) = \frac{\frac{x_n^\perp}{\sqrt{\|x_n\|}}^T Q \frac{x_n^\perp}{\sqrt{\|x_n\|}} + a^T\frac{x_n}{\|x_n\|} + \frac{a_0}{\|x_n\|}}{b^T\frac{x_n}{\|x_n\|} + \frac{b_0}{\|x_n\|}}$$

and so i) holds.

ii) We have

$$f(x_n) = \frac{\frac{\|x_n^\perp\|^2}{\|x_n\|}\frac{x_n^\perp}{\|x_n^\perp\|}^T Q \frac{x_n^\perp}{\|x_n^\perp\|} + a^T\frac{x_n}{\|x_n\|} + \frac{a_0}{\|x_n\|}}{b^T\frac{x_n}{\|x_n\|} + \frac{b_0}{\|x_n\|}}$$

Since $\frac{x_n^\perp}{\|x_n^\perp\|} \to z \notin KerQ$, it results $z^TQz > 0$ or $z^TQz < 0$ according to $Q$ is semidefinite positive or semidefinite negative. The thesis follows by noting that $\frac{\|x_n^\perp\|^2}{\|x_n\|} = \left(\frac{\|x_n^\perp\|}{\sqrt{\|x_n\|}}\right)^2 \to +\infty$ since $\left\{\frac{x_n^\perp}{\sqrt{\|x_n\|}}\right\}$ is an unbounded sequence. ∎

Lemma 3.2 allows us to obtain conditions which ensure that the supremum of Problem (3.1) is finite.

With this regard, consider the following set associated with a recession direction $d$:

$$W_d = \{w \in \Re^n : \exists \{x_n\} \subset X, \frac{x_n}{\|x_n\|} \to d \in O^+X, \frac{x_n^\perp}{\sqrt{\|x_n\|}} \to w\}$$

with the convention $w = 0$ if $x_n^\perp \equiv 0$.

The following Theorem holds.

**Theorem 3.2** *Consider Problem (3.1) where $Q$ is semidefinite negative. Then the supremum is finite.*

**Proof.** Let $\{x_n\}$ be a feasible sequence such that $\lim_{n\to+\infty} f(x_n) = \sup_{x\in X} f(x)$. If $\{x_n\}$ is unbounded then $\frac{x_n}{\|x_n\|} \to d \in O^+X$. The following cases arise:

- $\left\{\frac{x_n^\perp}{\sqrt{\|x_n\|}}\right\}$ is bounded, then from i) of Lemma 3.2 $\lim_{n\to+\infty} f(x_n)$ is finite.
- $\left\{\frac{x_n^\perp}{\sqrt{\|x_n\|}}\right\}$ is unbounded, then from ii) of Lemma 3.2 $\lim_{n\to+\infty} f(x_n) = -\infty = \sup_{x\in X} f(x)$ and this is absurd. ∎

The following Theorem specifies when Problem (3.1) has maximum value and gives a characterization of the supremum when the maximum is not attained.

**Theorem 3.3** *Consider the Problem (3.1) where $Q$ is semidefinite negative. Then the following statements hold.*

*i) If $O^+X \cap KerQ = \emptyset$ then $\sup_{x\in X} f(x) = \max_{x\in X} f(x)$.*

*ii) If $Q$ is definite negative then $\sup_{x\in X} f(x) = \max_{x\in X} f(x)$.*

*iii) If $W_d = \emptyset$ for every $d \in O^+X \cap \ker Q$ then $\sup_{x\in X} f(x) = \max_{x\in X} f(x)$.*

*iv) If Problem (3.1) does not reach maximum then*

$$\sup_{x\in X} f(x) = \sup_{d\in O^+X\cap KerQ} \left( \sup_{w\in W_d} \frac{w^TQw + a^Td}{b^Td} \right).$$

**Proof.** i) and ii) follow directly from ii) of Theorem 3.1.
iii) follows from ii) of Lemma 3.2.
iv) follows from i) of Lemma 3.2. ∎

The following examples point out that when there exists $d \in O^+X \cap KerQ$ with $W_d \neq \emptyset$, we can have both maximum and finite supremum not attained.

**Example 3.2** *Consider Problem (3.1) where* $f(x,y) = \dfrac{-2x^2+y+1}{x+y+3}$ *and* $X = \{(x,y) \in \Re^2 : y = x^2\}$. *We have* $O^+X = \{d = (0,1)\}$. *Consider a feasible unbounded sequence of the kind* $\{z_n\} = \{(x_n, x_n^2)\}$. *It results* $z_n^\perp = (x_n, 0)$ *and* $\sqrt{\|z_n\|} = \|x_n\| \sqrt[4]{\frac{1}{x_n^2}+1}$. *Hence* $\frac{z_n^\perp}{\sqrt{\|z_n\|}} \to w$, $w = (1,0)$ *so that* $W_d \neq \emptyset$. *It this case we have* $\sup\limits_{w \in W_d} \frac{w^TQw+a^Td}{b^Td} = -1$ *while the maximum value of the function is* $1/3$ *attained at* $(0,0)$.

**Example 3.3** *Consider Problem (3.1) where* $f(x,y) = \dfrac{-x^2+y+1}{y+6}$,
$X = \{(x,y) \in \Re^2 : y = 2x^2\}$. *We have* $O^+X = \{d = (0,1)\}$. *Consider a feasible unbounded sequence of the kind* $z_n = (x_n, 2x_n^2)$. *It results* $z_n^\perp = (x_n, 0)$ *and* $\sqrt{\|z_n\|} = \|x_n\| \sqrt[4]{\frac{1}{x_n^2}+4}$. *Hence* $\frac{z_n^\perp}{\sqrt{\|z_n\|}} \to w$, $w = (\frac{1}{\sqrt{2}}, 0)$ *so that* $W_d \neq \emptyset$. *It easy to verify that* $\sup\limits_{w \in W_d} \frac{w^TQw+a^Td}{b^Td} = -\frac{1}{2}+1 = \frac{1}{2} = \sup\limits_{x \in X} f(x)$ *which is not attained as a maximum since it results* $f(x, 2x^2) < 1/2$ *for every* $x \in \Re$.

Now we will consider the case $Q$ is semidefinite positive. Let us note that for every $d \in O^+X$ we may have feasible sequences $\{x_n\}$ such that $\frac{x_n}{\|x_n\|} \to d$ and some of the corresponding sequences $\left\{\frac{x_n^\perp}{\sqrt{\|x_n\|}}\right\}$ are unbounded.

In the following we will denote $W_d^*$ the set of all unbounded sequence of this kind.

The following Theorem holds.

**Theorem 3.4** *Consider Problem (3.1) where* $Q$ *is semidefinite positive Then the following statements hold.*

*i) If there exists* $d \in O^+X$ *such that* $d \notin KerQ$ *then* $\sup\limits_{x \in X} f(x) = +\infty$.

*ii) If* $Q$ *is definite positive then* $\sup\limits_{x \in X} f(x) = +\infty$.

*iii) If* $O^+X \subset KerQ$ *and if there exists* $d \in O^+X$ *such that* $W_d^* \neq \emptyset$ *then* $\sup\limits_{x \in X} f(x) = +\infty$.

*iv) If $O^+X \subset KerQ$ and if $W_d^* = \emptyset$, then the supremum of Problem (3.1) is finite. Moreover if $f$ does not attain maximum then it results*

$$\sup_{x\in X} f(x) = \sup_{d\in O^+X\cap KerQ} \left( \sup_{w\in W_d} \frac{w^TQw + a^Td}{b^Td} \right).$$

**Proof.** i) and ii) follow from i) of Lemma 3.1.
iii) follows from ii) of Lemma 3.2.
iv) The desired result follows from i) of Lemma 3.2. ■

The following examples show that when $Q$ is semidefinite positive, $O^+X \subset KerQ$ with $W_d^* = \emptyset$ for all $d \in O^+X$, $f$ may admit maximum value or not.

**Example 3.4** *Consider Problem (3.1) where* $f(x,y) = \dfrac{y^2+1}{x+1}$ *and* $X = \{(x,y) \in \Re^2 : x \geq 0,\ y = \frac{1}{x+1}\}$.
*We have* $O^+X = \{d = (1,0)\}$, $O^+X \subset KerQ$ *and* $W_d^* = \emptyset$. *It results* $\max f(x,y) = 1$ *attained in* $(0,1)$.

**Example 3.5** *Consider Problem (3.1) where* $f(x,y) = \dfrac{x^2-4+y}{x+y+3}$ *and* $X = \{(x,y) \in \Re^2 : 0 \leq x \leq 1,\ y \geq 1\}$. *We have* $O^+X = \{d = (0,1)\}$, *and* $O^+X \subset KerQ$. *For every unbounded feasible sequence* $\{(x_n, y_n)\}$ *the corresponding sequence* $\frac{(x_n^\perp, y_n^\perp)}{\sqrt{\|(x_n,y_n)\|}} = \left(\frac{x_n^\perp}{\sqrt{\|(x_n,y_n)\|}}, 0\right)$ *is convergent to* $(0,0)$, *so that* $W_d^* = \emptyset$.
*Moreover the supremum* $\sup f(x,y) = \sup_{w\in W_d} \dfrac{w^TQw+1}{1} = 1$ *is not attained.*

# 4 On the supremum of the product of an affine function and a linear fractional one.

In this section we consider a particular quadratic fractional function for which we can study the behavior of any feasible unbounded sequence

$$\sup_{x\in X} f(x) = \left[ \frac{(a^Tx + a_0)\,(c^Tx + c_0)}{b^Tx + b_0} \right] \tag{4.1}$$

where $a, b, c, x \in \Re^n$, $a_0, b_0, c_0 \in \Re$, $b^Tx + b_0 > 0$ and $X$ is any closed and unbounded set.

Note that in this case we remove the Assumption $b^T x + b_0 \geq \delta > 0$ for every $x \in X$ therefore, now it may happen that $b^T d = 0$ for some recession directions. Consequently we establish results about $\sup f(x)$ in a more general context with respect to the previous case.

The following Lemma holds.

**Lemma 4.1** *Let $\{x_n\}$ be a feasible unbounded sequence for Problem (4.1) and set $d = \lim \frac{x_n}{\|x_n\|}$ such that $\left(a^T d\right)\left(c^T d\right) \neq 0$. Then $f(x_n) \longrightarrow +\infty$ or $f(x_n) \longrightarrow -\infty$ according to $\left(a^T d\right)\left(c^T d\right) > 0$ or $\left(a^T d\right)\left(c^T d\right) < 0$.*

**Proof.** It follows by noting that

$$f(x_n) = \frac{\|x_n\| \left(a^T \frac{x_n}{\|x_n\|} + \frac{a_0}{\|x_n\|}\right)\left(c^T \frac{x_n}{\|x_n\|} + \frac{c_0}{\|x_n\|}\right)}{b^T \frac{x_n}{\|x_n\|} + \frac{b_0}{\|x_n\|}}$$

■

As a direct consequence of the previous Lemma we obtain a necessary and a sufficient condition for the supremum of Problem (4.1) to be finite.

**Theorem 4.1** *i) If $\sup_{x \in X} f(x) < +\infty$, then $\left(a^T d\right)\left(c^T d\right) \leq 0$ , $\forall\, d \in O^+X$.*

*ii) If $\left(a^T d\right)\left(c^T d\right) < 0$ for every $d \in O^+X$ then $\sup_{x \in X} f(x) = \max_{x \in X} f(x)$.*

**Proof.** i) If there exists $d \in O^+X$ such that $\left(a^T d\right)\left(c^T d\right) > 0$, then from Lemma 4.1 there exists an unbounded sequence $\{x_n\}$ such that $\frac{x_n}{\|x_n\|} \longrightarrow d$, $f(x_n) \longrightarrow +\infty$ and this is absurd.

ii) Let $\{x_n\}$ be a feasible sequence such that $\lim_{n \to \infty} f(x_n) = \sup_{x \in X} f(x)$. If $\{x_n\}$ is unbounded then $\frac{x_n}{\|x_n\|} \longrightarrow d \in O^+X$, so that, from Lemma 4.1, $f(x_n) \longrightarrow -\infty$ and this is absurd. Since $\{x_n\}$ is bounded, it converges to a point $x^* \in X$. From the continuity of the function $f$, we have $\lim_{n \to \infty} f(x_n) = f(x^*) = \sup_{x \in X} f(x)$.

■

The following example shows that the necessary condition stated in i) of Theorem 4.1 is not sufficient to have finite supremum.

**Example 4.1** *Consider Problem (4.1) where $f(x, y, z) = \dfrac{(x+1)(y+z+2)}{x+y+5}$, $X = \{(x, y, z\} : x \geq 0, y = x^2, z = \sqrt{x}\}$. We get $O^+X = \{d = (0, 1, 0)\}$, $\left(a^T d\right)\left(c^T d\right) = 0$ and $\sup_{x \in X} f(x, y, z) = +\infty$.*

Theorem 4.1 points out the necessity to deepen the case $(a^T d)(c^T d) = 0$, since in this situation, we can have finite supremum as it is shown in following example.

**Example 4.2** *Consider Problem (4.1) where* $f(x,y) = \dfrac{(x+1)(y+2)}{x+y+5}$ *and* $X = \{(x,y) : x \geq 1, y = \frac{1}{x}\}$*. It results* $O^+X = \{d = (1,0)\}$, $(a^T d)(c^T d) = 0$. *Simple calculations show that* $\sup_{x \in X} f(x,y) = 2$ *which is not attained as a maximum.*

In order to study what happens in the case $(a^T d)(c^T d) = 0$ we consider the following sets: $a^\perp = \{x \in \Re^n : a^T x = 0\}$, $b^\perp = \{x \in \Re^n : b^T x = 0\}$, $c^\perp = \{x \in \Re^n : c^T x = 0\}$, $D = a^\perp \cup c^\perp = \{x \in \Re^n : (a^T x)(c^T x) = 0\}$.

According with the previous definitions any element of a feasible sequence $x_n$ can be written as follows:

$$\begin{array}{ll} (1) & x_n = a'_n + \alpha_n a \\ (2) & x_n = c'_n + \gamma_n c \\ (3) & x_n = b'_n + \beta_n b \end{array} \tag{4.2}$$

where $a'_n \in a^\perp, \beta'_n \in b^\perp, c'_n \in c^\perp$, $\alpha_n, \beta_n, \gamma_n$ are scalars.

Hence we have

$$\begin{array}{ll} (1) & a^T x_n = \alpha_n \|a\|^2 \\ (2) & c^T x_n = \gamma_n \|c\|^2 \\ (3) & b^T x_n = \beta_n \|b\|^2 \end{array} \tag{4.3}$$

Take into account the above fact, in the general case when $a, b, c \neq 0$, we can define the following sequences

$$\begin{array}{ll} (1) & \alpha_n^* = \alpha_n + \frac{a_0}{\|a\|^2} \\ (2) & \gamma_n^* = \gamma_n + \frac{c_0}{\|c\|^2} \\ (3) & \beta_n^* = \beta_n + \frac{b_0}{\|b\|^2} \end{array} \tag{4.4}$$

Consequently we can write

$$f(x_n) = \frac{(\alpha_n \|a\|^2 + a_0)(\gamma_n \|c\|^2 + c_0)}{\beta_n \|b\|^2 + b_0} = \frac{\alpha_n^* \gamma_n^*}{\beta_n^*} \frac{\|a\|^2 \|c\|^2}{\|b\|^2} \tag{4.5}$$

The following Lemma holds.

**Lemma 4.2** *Consider Problem (4.1) and let $\{x_n\}$ be an unbounded feasible sequence with $\frac{x_n}{\|x_n\|} \to d$ . Then i) and ii) hold.*

*i) If $d \in D$ and $\frac{\alpha_n^* \gamma_n^*}{\beta_n^*} \to \delta_d$, then $\lim_{n\to+\infty} f(x_n) = \delta_d \frac{\|a\|^2\|c\|^2}{\|b\|^2}$*

*ii) If $d \in D$ and $\frac{\alpha_n^* \gamma_n^*}{\beta_n^*} \to \infty$ then $f(x_n) \to +\infty$ or $f(x_n) \to -\infty$ according to $\frac{\alpha_n^* \gamma_n^*}{\beta_n^*} > 0$ or $\frac{\alpha_n^* \gamma_n^*}{\beta_n^*} < 0$ definitively.*

**Proof.** i) -ii) The result follows immediately by taking the limit in (4.5). ■

Lemma 4.2 allows us to obtain conditions which ensure that the supremum of Problem (4.1) is finite.

With this regard for every $d \in O^+X \cap D$, set

$$\Delta_d = \{\delta_d \in \Re : \exists \{x_n\} \subset X, \frac{x_n}{\|x_n\|} \to d \in O^+X, \frac{\alpha_n^* \gamma_n^*}{\beta_n^*} \to \delta_d\}$$

and define $\Delta_d^*$ the set of all unbounded, definitively positive sequences $\left\{\frac{\alpha_n^* \gamma_n^*}{\beta_n^*}\right\}$.

The following Theorem specifies when Problem (4.1) has maximum value and gives a characterization of the supremum when it is not attained.

**Theorem 4.2** *Consider Problem (4.1). Then the following statements hold.*

*i) If there exists $d \in O^+X$ such that $(a^T d)(c^T d) > 0$ then $\sup_{x\in X} f(x) = +\infty$.*

*ii) If there exists $d \in O^+X \cap D$ such that $\Delta_d^* \neq \emptyset$ then $\sup_{x\in X} f(x) = +\infty$.*

*iii) Assume $(a^T d)(c^T d) \leq 0$ for every $d \in O^+X$ . If we have $\Delta_d^* = \emptyset$, for every $d \in O^+X \cap D$, then $\sup_{x\in X} f(x)$ is finite and if it is not attained it results*

$$\sup_{x\in X} f(x) = \frac{\|a\|^2 \|c\|^2}{\|b\|^2} \sup_{d\in O^+X\cap D} \left( \sup_{\delta_d\in\Delta_d} \delta_d \right).$$

*Furthermore if $\Delta_d = \emptyset$ for every $d \in O^+X \cap D$ then $\sup_{x\in X} f(x) = \max_{x\in X} f(x)$.*

**Proof.** i) follows from Lemma 4.1

ii) follows from ii) of Lemma 4.2.

iii) Let us note that the supremum is finite if and only if, for every unbounded feasible sequence $\{x_n\}$, we have $\lim_{n\to+\infty} f(x_n) \neq +\infty$. The thesis follows by noting that $(a^T d)(c^T d) < 0$ implies $\lim_{n\to+\infty} f(x_n) = -\infty$,from Lemma 4.1, and that $(a^T d)(c^T d) = 0$ with $\Delta_d^* = \emptyset$ implies , from Lemma 4.2, that

$\lim_{n\to+\infty} f(x_n)$ is finite or $-\infty$. If in addition $\Delta_d = \emptyset$ for every $d \in O^+X \cap D$, necessarily we have $\lim_{n\to+\infty} f(x_n) = -\infty$ for every unbounded sequence, so that the supremum is attained as a maximum. ■

The following examples point out that when there exists $d \in O^+X \cap D$ with $\Delta_d^* = \emptyset$ and $\Delta_d \neq \emptyset$ we can have both maximum and finite supremum not attained.

**Example 4.3** *Consider Problem (4.1) where* $f(x,y,z) = \frac{(x+2)(y+z+2)}{y+2}$ *and* $X = \{(x,y,z) \in \Re^3 : 0 \le x \le 1, y \ge 0, z = \sqrt{y}\}$. *It is easy to verify that* $O^+X = \{d = (0,1,0)\}$ *so that* $d \in D$. *Consider any unbounded feasible sequence* $h_n = \{x_n, y_n, \sqrt{y_n}\}$. *Since* $x_n$ *is bounded, without any loss of generality we can assume* $\lim_{n\to+\infty} x_n = \bar{x}$ *with* $\bar{x} \in [0,1]$. *It results* $\alpha_n^* = x_n + 2$, $\gamma_n^* = \frac{1}{2}\left(y_n + \sqrt{y_n}\right) + 1, \beta_n^* = y_n + 2, \frac{\alpha_n^*\gamma_n^*}{\beta_n^*} \to \frac{1}{2}\bar{x} + 1$, *so that* $\Delta_d^* = \emptyset$, $\Delta_d \neq \emptyset$ *and* $\frac{\|a\|^2\|c\|^2}{\|b\|^2} \sup_{d\in O^+X\cap D}\left(\sup_{\delta_d\in\Delta_d} \delta_d\right) = 2\frac{3}{2} = 3$. *Simple calculations show that the function* $f$ *assume maximum value* $3 + \frac{3}{4}\sqrt{2}$ *attained at the point* $(1, 2, \sqrt{2})$.

**Example 4.4** *Consider Problem (4.1) where* $f(x,y,z) = \frac{(3x+y+2)(z+1)}{x+y+10}$, $X = \{(x,y,z) \in \Re^3 : x \ge 1, y = \sqrt{x}, 0 \le z \le 1\}$. *It is easy to verify that* $O^+X = \{d = (1,0,0)\}$ *so that* $d \in D$. *Consider any unbounded feasible sequence* $h_n = \{x_n, \sqrt{x_n}, z_n\}$. *Since* $z_n$ *is bounded, without any loss of generality we can assume* $\lim_{n\to+\infty} z_n = \bar{z}$ *with* $\bar{z} \in [0,1]$. *It results* $\alpha_n^* = \frac{1}{10}\left(3x_n + \sqrt{x_n}\right) + \frac{1}{5}$, $\gamma_n^* = z_n + 1$, $\beta_n^* = \frac{1}{2}\left(x_n + \sqrt{x_n}\right) + 5$, $\frac{\alpha_n^*\gamma_n^*}{\beta_n^*} \to \frac{1}{5}3(\bar{z}+1)$ *and* $\delta_d = \frac{3}{5}(\bar{z}+1)$, *so that* $\Delta_d^* = \emptyset$, $\Delta_d \neq \emptyset$ *and* $f(x_n, y_n, z_n) \to \frac{3}{5}(\bar{z}+1)$.
*Simple calculations show that function* $f$ *does not attain maximum value so that* $\sup_{x\in X} f(x) = \frac{\|a\|^2\|c\|^2}{\|b\|^2} \sup_{\delta_d\in\Delta_d} \delta_d = 5\frac{6}{5} = 6$.

## 4.1 Particular cases.

Consider the following Problem

$$\sup_{x\in X} f(x) = \left(a^T x + a_0\right)\left(c^T x + c_0\right) \tag{4.6}$$

obtained from Problem (4.1) setting $b = 0$ and $b_0 = 1$.
With similar argument, for any unbounded sequence $\{x_n\}$, we have

$$f(x_n) = \alpha_n^*\gamma_n^* \|a\|^2 \|c\|^2.$$

Setting

$$\Delta_d = \{\delta_d \in \Re : \exists \{x_n\} \subset X, \frac{x_n}{\|x_n\|} \to d \in O^+X, \alpha_n^* \gamma_n^* \to \delta_d\}$$

and $\Delta_d^*$ the set of all unbounded, definitively positive sequences $\{\alpha_n^* \gamma_n^*\}$, we have the following corollary.

**Corollary 4.1** *Consider Problem (4.6). Then the following statements hold.*

*i) If there exists $d \in O^+X$ such that $(a^T d)(c^T d) > 0$ then $\sup_{x\in X} f(x) = +\infty$.*

*ii) If there exists $d \in O^+X \cap D$ such that $\Delta_d^* \neq \emptyset$ then $\sup_{x \in X} f(x) = +\infty$.*

*iii) Assume $(a^T d)(c^T d) \leq 0$ for every $d \in O^+X$. If we have $\Delta_d^* = \emptyset$, for every $d \in O^+X \cap D$, then $\sup_{x\in X} f(x)$ is finite and if it is not attained it results*

$$\sup_{x\in X} f(x) = \|a\|^2 \|c\|^2 \sup_{d \in O^+X \cap D} \left( \sup_{\delta_d \in \Delta_d} \delta_d \right).$$

*Furthermore if $\Delta_d = \emptyset$ for every $d \in O^+X \cap D$ then $\sup_{x\in X} f(x) = \max_{x\in X} f(x)$.*

As we have already seen in the general case, when $d \in O^+X \cap D$ with $\Delta_d^* = \emptyset$ and $\Delta_d \neq \emptyset$ we can have both maximum and finite supremum not attained.

**Example 4.5** *Consider Problem (4.6) where $f(x,y) = (x+1)\,y$ and $X = \{(x,y) \in \Re^2 : x \geq 1, y = \frac{1}{x}\}$. It is easy to verify that $O^+X = \{d = (1,0)\}$ so that $d \in D$. Consider any unbounded feasible sequence $\{(x_n, y_n)\}$. It results $\alpha_n^* = x_n + 1$, $\gamma_n^* = \frac{1}{x_n}$, $\alpha_n^* \gamma_n^* = 1 + \frac{1}{x_n} \to 1$, $\delta_d = 1$, so that $\Delta_d^* = \emptyset$, $\Delta_d \neq \emptyset$ and $\|a\|^2 \|c\|^2 \sup_{d\in O^+X\cap D} \left( \sup_{\delta_d \in \Delta_d} \delta_d \right) = 1$. Simple calculations show that the function $f$ assumes maximum value 2 attained at the point $(1,1)$.*

**Example 4.6** *Consider Problem (4.6) where $f(x,y) = (x-1)\,y$ and $X = \{(x,y) \in \Re^2 : x \geq 1, y = \frac{1}{x}\}$. It is easy to verify that $O^+X = \{d = (1,0)\}$ so that $d \in D$. Consider any unbounded feasible sequence $\{(x_n, y_n)\}$. It results $\alpha_n^* = x_n - 1$, $\gamma_n^* = \frac{1}{x_n}$, $\alpha_n^* \gamma_n^* = 1 - \frac{1}{x_n} \to 1$, $\delta_d = 1$, so that $\Delta_d^* = \emptyset$, $\Delta_d \neq \emptyset$ and $\|a\|^2 \|c\|^2 \sup_{d\in O^+X\cap D} \left( \sup_{\delta_d \in \Delta_d} \delta_d \right) = 1$. Simple calculations show that function $f$ does not attain maximum value.*

When $c = a$ and $c_0 = a_0$, Problem (4.1) becomes

$$\sup_{x \in X} \left[ f(x) = \frac{\left(a^T x + a_0\right)^2}{b^T x + b_0} \right] \tag{4.7}$$

where $a, b, x \in \Re^n$, $a_0, b_0 \in \Re$ and $X$ is any unbounded closed set.
We assume that $b^T x + b_0 > 0$ for every $x \in X$.
It is well known that $f$ is a convex function (see [3]).
We have the following results.

**Corollary 4.2** *Consider Problem (4.7). The following statements hold.*
*i) If* $\exists d \in O^+X : a^T d \neq 0$ *then* $\sup f(x) = +\infty$.
*ii) If* $\exists d \in O^+X : a^T d = 0$ *and* $\Delta_d^* \neq \emptyset$ *then* $\sup f(x) = +\infty$.
*iii)* $f$ *has finite supremum if and only if* $O^+X \subset a^\perp$ *and* $\Delta_d^* = \emptyset$. *Moreover if* $f$ *does not have maximum value then*

$$\sup_{x \in X} f(x) = \frac{\|a\|^4}{\|b\|^2} \sup_{d \in O^+X \cap D} \left( \sup_{\delta_d \in \Delta_d} \delta_d \right).$$

The following example shows that even in Problem (4.7), when $O^+X \subset a^\perp$ and $\Delta_d^* = \emptyset$, we can have both maximum and supremum not attained.

**Example 4.7** *Consider Problem (4.7) where* $f(x,y) = \frac{(y+k)^2}{x+y+2}$ *and* $X = \{(x,y) \in \Re^2 : x \geq 0, y = \sqrt{x}\}$, $O^+X = \{d = (1,0)\}$ *with* $d \in a^\perp$. $\alpha_n^* = \sqrt{x_n} + k$, $\beta_n^* = \frac{1}{2}\left(\sqrt{x_n} + x_n\right) + 1$. $\frac{(\alpha_n^*)^2}{\beta_n^*} = \frac{\left(\sqrt{x_n}+k\right)^2}{\frac{1}{2}(\sqrt{x_n}+x_n)+1} \to 2$. *Hence for any unbounded feasible sequence we have* $f(x_n, y_n) \to \frac{1}{2}2 = 1$. *Studying the inequality* $\frac{\left(\sqrt{x}+k\right)^2}{x+\sqrt{x}+2} \geq 1$, *it easy to verify that for* $0 \leq k \leq 1/2$ *the supremum of* $f$ *is not attained, while for* $k > 1/2$, *function* $f$ *admits maximum value.*

**Remark 4.1** *When* $c = 0$ *and* $c_0 = 1$ *the problem (4.1) reduces to a linear fractional problem whose properties regarding the existence of finite supremum are studied in [4].*

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