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Abstract. In this paper we deeply analyze a family of vector valued pseudoconcave functions which extends to the vector case both scalar pseudoconcavity and scalar strictly pseudoconcavity as well as their optimality properties, such as the global optimality of local optima, of critical points and of points verifying Kuhn-Tucker conditions. This class of functions comes out to be particularly relevant since it is possible for it to determine several first and second order characterizations, thus offering a complete extension to the vector case of the well known scalar pseudoconcavity and giving the chance to work in multiobjective optimization with all the properties of the scalar case.

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1 Introduction

Pseudoconcave vector valued functions play a key role in multicriteria decision making and in multiobjective programming since their properties allow to recognize the efficient points.

In these very last years, several classes of vector valued pseudoconcave functions have been introduced and studied with the aim of extending to the vector case some properties of scalar pseudoconcavity [3–12,14–16].

In this paper we deeply analyze a family of vector valued pseudoconcave functions, among the whole proposed ones, which extends to the vector case both scalar pseudoconcavity and scalar strictly pseudoconcavity as well as their optimality properties, such as the global optimality of local optima, of critical points and of points verifying Kuhn-Tucker conditions.

This class of functions, introduced in [6,8], comes out to be particularly relevant since it is possible for it to determine both first and second order characterizations.

* The paper has been discussed jointly by the authors. In particular Sections 3,4,5,6 have been developed by R. Cambini.

The proposed class of vector valued pseudoconcave function thus offers a complete extension to the vector case of the well known scalar pseudoconcavity, giving the chance to work in multiobjective optimization with all the properties of the scalar case.

2 Definitions and preliminary results

From now on we will consider a vector valued function f, a cone C and some of its subcones, verifying the following properties:

i) $f : A \to \Re^m$, defined on the open convex set $A \subseteq \Re^n$, is a differentiable vector valued function,

ii) $C \subset \Re^m$ is a closed convex pointed cone with nonempty interior,

iii) $C^* \subset \Re^m$ is any cone such that $\text{Int}(C) \subseteq C^* \subseteq C$,

iv) $C^0 = C \setminus \{0\}$.

Some of the main results of this paper will be based on the following fundamental preliminary theorem which specifies the behaviour of function f when $J_f(x_0)(x - x_0) \in \text{Int}(C)$.

Theorem 1. *Let $x, x_0 \in A$, $x \neq x_0$. It results $J_f(x_0)(x - x_0) \in \text{Int}(C)$ if and only if the following condition (1) holds:*

$$\begin{gathered} \text{there exists a vector } \xi_{x,x_0} \in \text{Int}(C), \exists \lambda^* \in (0,1] \text{ such that:} \\ f(x_0 + \lambda(x - x_0)) \in f(x_0) + \lambda(1-\lambda)\xi_{x,x_0} + C \quad \forall \lambda \in (0, \lambda^*) \end{gathered} \tag{1}$$

Proof. $\Rightarrow$) Suppose by contradiction that $\forall \xi_{x,x_0} \in \text{Int}(C)$, $\forall \lambda^* \in (0,1]$, $\exists \lambda \in (0, \lambda^*)$ such that $f(x_0 + \lambda(x - x_0)) \notin f(x_0) + \lambda(1-\lambda)\xi_{x,x_0} + C$. Let $c \in \text{Int}(C)$ and consider the sequences $\xi_{x,x_0} = \frac{1}{n}c \in \text{Int}(C)$ and $\lambda^* = \frac{1}{n} \in (0,1]$, $n = 1, 2, \ldots$, then there exists a sequence $\lambda_n \in (0, \frac{1}{n})$ such that:

$$\frac{f(x_0 + \lambda_n(x - x_0)) - f(x_0)}{\lambda_n} \notin \frac{1}{n}(1 - \lambda_n)c + C.$$

Since $\lim_{n \to +\infty} \lambda_n = 0$ we then have that:

$$J_f(x_0)(x - x_0) = \lim_{n \to +\infty} \frac{f(x_0 + \lambda_n(x - x_0)) - f(x_0)}{\lambda_n} \notin \text{Int}(C)$$

which is a contradiction.

$\Leftarrow$) By means of the hypothesis we have:

$$J_f(x_0)(x - x_0) = \lim_{\lambda \to 0^+} \frac{f(x_0 + \lambda(x - x_0)) - f(x_0)}{\lambda} \in \xi_{x,x_0} + C$$

so that the thesis follows since $\xi_{x,x_0} + C \subset \text{Int}(C)$ being C a convex cone.

Another useful result, still based on the behaviour of function f with respect to its Jacobian and Hessians matrices, is the following.

Theorem 2. *If the following condition holds:*

$$\exists \lambda^* \in (0,1) \text{ such that: } \quad f(x_0 + \lambda(x - x_0)) \in f(x_0) + C \quad \forall \lambda \in (0, \lambda^*) \tag{2}$$

then $J_f(x_0)(x - x_0) \in C$. *If* f *is also twice differentiable then when* $J_f(x_0)(x - x_0) = 0$ *it is* $(x - x_0)^T H_f(x_0)(x - x_0) \in C$.

Proof. By means of the hypothesis $\exists \lambda^* \in (0,1)$ such that:

$$\frac{f(x_0 + \lambda(x - x_0)) - f(x_0)}{\lambda} \in C \quad \forall \lambda \in (0, \lambda^*)$$

Being C a closed cone it then results:

$$J_f(x_0)(x - x_0) = \lim_{\lambda \to 0} \frac{f(x_0 + \lambda(x - x_0)) - f(x_0)}{\lambda} \in C$$

and the result is proved.
Suppose now $J_f(x_0)(x - x_0) = 0$; by means of the hypothesis $\exists \lambda^* \in (0,1)$ such that:

$$\frac{f(x_0 + \lambda(x - x_0)) - f(x_0)}{\lambda^2} \in C \quad \forall \lambda \in (0, \lambda^*)$$

Let us consider the following second order Taylor expansion of f at x_0:

$$f(x_0 + \lambda(x - x_0)) = f(x_0) + \lambda J_f(x_0)(x - x_0) + \\ + \frac{\lambda^2}{2}(x - x_0)^T H_f(x_0)(x - x_0) + \lambda^2 \|x - x_0\|^2 \sigma(\lambda, 0)$$

where $\lim_{\lambda \to 0^+} \sigma(\lambda, 0) = 0$. Since $J_f(x_0)(x - x_0) = 0$ it results:

$$\frac{f(x_0 + \lambda(x - x_0)) - f(x_0)}{\lambda^2} = \frac{1}{2}(x - x_0)^T H_f(x_0)(x - x_0) + \|x - x_0\|^2 \sigma(\lambda, 0)$$

so that, being C a closed cone, it results:

$$\frac{1}{2}(x - x_0)^T H_f(x_0)(x - x_0) = \lim_{\lambda \to 0} \frac{f(x_0 + \lambda(x - x_0)) - f(x_0)}{\lambda^2} \in C$$

and the thesis is proved.

Note that the two previous preliminary results are not based on the generalized concavity of function f.

3 Pseudoconcavity and Quasiconcavity

The classes of pseudoconcave functions, whose properties are going to be deeped on in this paper, are defined as follows (see [6,8]).

Definition 1. Let $C \subset \Re^m$ be a closed convex pointed cone with nonempty interior and let C^* be a cone such that $\text{Int}(C) \subseteq C^* \subseteq C$. A differentiable function $f : A \to \Re^m$, where $A \subseteq \Re^n$ is an open convex set, is said to be a $(C^*, \text{Int}(C))$-*pseudoconcave function* if the following logical implication holds $\forall x, x_0 \in A$, $x \neq x_0$:

$$f(x) \in f(x_0) + C^* \quad \Longrightarrow \quad J_f(x_0)(x - x_0) \in \text{Int}(C)$$

Note that in the scalar case, where $C = \Re_+$ and $\text{Int}(C) = \Re_{++}$, the classes of $(C^*, \text{Int}(C))$-pseudoconcave functions coincide with the well know pseudoconcave and strictly pseudoconcave functions when $C^* = \Re_{++}$ and $C^* = \Re_+$ respectively.

In the vector case the following concept of quasiconcavity has been also studied (see [6,8]).

Definition 2. Let $C \subset \Re^m$ be a closed convex pointed cone with nonempty interior and let C^1 and C^2 be cones such that $\text{Int}(C) \subseteq C^1 \subseteq C$ and $\text{Int}(C) \subseteq C^2 \subseteq C$. A function $f : A \to \Re^m$, where $A \subseteq \Re^n$ is a convex set, is said to be a (C^1, C^2)-*quasiconcave function* if the following logical implication holds $\forall x, x_0 \in A$, $x \neq x_0$:

$$f(x) \in f(x_0) + C^1 \quad \Longrightarrow \quad \begin{array}{c} f(x_0 + \lambda(x - x_0)) \in f(x_0) + C^2 \\ \forall \lambda \in (0,1) \end{array}$$

Unlike the scalar case, the concept of vector pseudoconcavity does not imply the one of quasiconcavity introduced so far, as it is pointed out in the next example (see [6,8]).

Example 1. Consider the cone $C = \Re^3_+$ and the following differentiable function $f : [0,3] \to \Re^3$:

$$f(x) = \begin{cases} (-x^2 + 2x)[1/2, 1/2, 1]^T & \text{if } x \in [0,1] \\ [1/2, 1/2, 1]^T + (-2x^3 + 9x^2 - 12x + 5)[1, -1, 0]^T & \text{if } x \in (1,2) \\ [3/2, -1/2, 1]^T + (x-2)^2[-5/6, 7/6, -1/3]^T & \text{if } x \in [2,3] \end{cases}$$

this function is $(C^*, \text{Int}(C))$-pseudoconcave but it is not (C^1, C^2)-quasiconcave for any cones C^1 and C^2 such that $\text{Int}(C) \subseteq C^1 \subseteq C$ and $\text{Int}(C) \subseteq C^2 \subseteq C$ (see [6,8]) since $f(3) \in f(0) + \text{Int}(C)$ while $f(2) \notin f(0) + C$.

The relationship existing among pseudoconcave and quasiconcave functions in the scalar case can be restored in the vector case by means of the following new concept of vector quasiconcavity

Definition 3. Let $C \subset \Re^m$ be a closed convex pointed cone with nonempty interior and let C^1 and C^2 be cones such that $\text{Int}(C) \subseteq C^1 \subseteq C$ and $\text{Int}(C) \subseteq C^2 \subseteq C$. A function $f : A \to \Re^m$, with $A \subseteq \Re^n$ convex set, is said to be a *locally* (C^1, C^2)*-quasiconcave function* if the following logical implication holds $\forall x, x_0 \in A$, $x \neq x_0$:

$$f(x) \in f(x_0) + C^1 \implies \begin{array}{c} \exists \lambda^* \in (0,1] \text{ such that } \forall \lambda \in (0, \lambda^*) \\ f(x_0 + \lambda(x - x_0)) \in f(x_0) + C^2 \end{array} \tag{3}$$

Theorem 3. *Let $f : A \to \Re^m$, with A open convex set, be a differentiable $(C^*, \text{Int}(C))$-pseudoconcave function, then it is also locally (C^1, C^2)-quasiconcave for any cones C^1 and C^2 such that $\text{Int}(C) \subseteq C^1 \subseteq C^*$ and $\text{Int}(C) \subseteq C^2 \subseteq C$.*

Proof. The inclusion relationship follows directly from the definitions and Theorem 1, being C a convex cone.

Note that, as a particular case, any $(C^*, \text{Int}(C))$-pseudoconcave function is also locally (C^*, C^*)-quasiconcave and locally $(\text{Int}(C), C)$-quasiconcave. Note also that, by means of the definitions, the locally (C^1, C^2)-quasiconcave functions properly contain the (C^1, C^2)-quasiconcave functions studied in [6,8], as it is pointed out in Example 1.

In the scalar case the concepts of quasiconcavity and of locally quasiconcavity coincide under continuity hypothesis.

Theorem 4. *A continuous scalar function $f : A \to \Re$, $A \subseteq \Re^n$ convex set, is quasiconcave if and only if it is locally quasiconcave.*

Proof. By means of the definitions every quasiconcave function is also locally quasiconcave. Suppose now by contradiction that f is locally quasiconcave but not quasiconcave, that is to say that $\exists x, y \in A$, $x \neq y$, $\exists \lambda_1 \in (0,1)$ such that $f(y) \geq f(x)$ and $f(x + \lambda_1(y - x)) < f(x)$; by means of the local quasiconcavity of f $\exists \lambda_2 \in (0,1]$ such that $\forall \lambda \in (0, \lambda_2)$ $f(x + \lambda(y - x)) \geq f(x)$, so that $\lambda_2 < \lambda_1$. Let us now define the following $\overline{\lambda} \in (\lambda_2, \lambda_1)$:

$$\overline{\lambda} = sup\{t \in [0,1] \text{ s.t. } f(x + \lambda(y - x)) \geq f(x) \quad \forall \lambda \in [0, t]\};$$

by means of the continuity of f we can easily prove that $f(x + \overline{\lambda}(y - x)) = f(x)$ and that $\exists \epsilon > 0$ such that:

$$f(x + \lambda(y - x)) < f(x + \overline{\lambda}(y - x)) = f(x) \quad \forall \lambda \in (\overline{\lambda}, \overline{\lambda} + \epsilon). \tag{4}$$

Being $f(y) \geq f(x + \overline{\lambda}(y - x)) = f(x)$ we have, by means of the local quasiconcavity of f, that $\exists \lambda^* \in (\overline{\lambda}, 1]$ such that $\forall \lambda \in (\overline{\lambda}, \lambda^*)$ $f(x + \lambda(y - x)) \geq f(x + \overline{\lambda}(y - x))$ and this contradicts (4).

Note that if the scalar function is not continuous these classes do not coincide, as it is pointed out in the next example.

Example 2. Let us consider the following scalar function $f : \Re \to \Re$:

$$f(x) = \begin{cases} 2 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$$

It comes out that f is locally quasiconcave even if it is not quasiconcave, being $f(2) = f(0)$ while $f(1) < f(0)$.

Note finally that Theorem 4 will allow us to show that the characterizations of pseudoconcave vector valued functions, we are going to state in the forthcoming sections, are a generalization of the known results of the scalar case.

4 Pseudoconcavity and Optimality

In this section we are going to point out that the family of $(C^*, \text{Int}(C))$-pseudoconcave functions extends to the vector case all the optimality properties of the scalar pseudoconcave functions, that is to say that they verify the global optimality of local optima, the global optimality of critical points and the sufficiency of the Kuhn-Tucker like optimality conditions. These are not new results and their proofs are given for a sake of completeness.

From now on we will use the following concepts of vector efficiency.

Definition 4. A point $x_0 \in A$ is said to be a *global C^*-efficient point* for f if:

$$\nexists x \in A, x \neq x_0, \text{ such that } f(x) \in f(x_0) + C^*;$$

A point $x_0 \in A$ is said to be a *local C^*-efficient point* for f if there exists a neighbourhood I_{x_0} of x_0 such that:

$$\nexists x \in A \cap I_{x_0}, x \neq x_0, \text{ with } f(x) \in f(x_0) + C^*$$

The first result which is extended to the vector case is the global efficiency of a local efficient point.

Theorem 5. *If $f : A \to \Re^m$, A convex set, is a locally (C^*, C^*)-quasiconcave function, then every local C^*-efficient point is also a global one.*

Proof. Suppose by contradiction that $x_0 \in A$ is a local C^*-efficient point for f but it is not also global, that is to say that $\exists x \in A$, $x \neq x_0$, such that $f(x) \in f(x_0) + C^*$; by means of the local (C^*, C^*)-quasiconcavity of f we then have that $\exists \lambda^* \in (0, 1]$ such that $f(x_0 + \lambda(x - x_0)) \in f(x_0) + C^*$ $\forall \lambda \in (0, \lambda^*)$ and this contradicts the local C^*-efficiency of x_0.

Being any $(C^*, \text{Int}(C))$-pseudoconcave funcion also locally (C^*, C^*)-quasiconcave, we have the following corollary.

Corollary 1. *If a differentiable function $f : A \to \Re^m$, A open convex set, is $(C^*, \mathrm{Int}(C))$-pseudoconcave then every local C^*-efficient point is also a global one.*

Another important property which is extended to the vector case is the optimality of all the critical point. Denoted with $C^+ = \{d \in \Re^m : d^T c \geq 0 \ \forall c \in C\} \subset \Re^m$ the positive polar cone of C, it is known that if $x_0 \in A$ is a local C^*-efficient point for f then (see for example [3,4]):

$$\exists \alpha \in C^+, \alpha \neq 0, \text{ such that } \alpha^T J_f(x_0) = 0$$

This suggests the following definition.

Definition 5. A point $x_0 \in A$ is said to be a *critical point* for f if:

$$\exists \alpha \in C^+, \alpha \neq 0, \text{ such that } \alpha^T J_f(x_0) = 0$$

By means of this definition we can prove the following result.

Theorem 6. *If $f : A \to \Re^m$, A open convex set, is a $(C^*, \mathrm{Int}(C))$-pseudoconcave function then every critical point is also a global C^*-efficient one.*

Proof. Suppose by contradiction that $x_0 \in A$ is critical point for f but it is not also a global C^*-efficient one, that is to say that $\exists x \in A$, $x \neq x_0$, such that $f(x) \in f(x_0) + C^*$; by means of the $(C^*, \mathrm{Int}(C))$-pseudoconcavity of f we then have that $J_f(x_0)(x - x_0) \in \mathrm{Int}(C)$. Since x_0 is a critical point there exists an $\alpha \in C^+$, $\alpha \neq 0$, such that $\alpha^T J_f(x_0) = 0$; for a known property of polar cones condition $J_f(x_0)(x - x_0) \in \mathrm{Int}(C)$ implies that $\alpha^T J_f(x_0)(x - x_0) > 0$ which is a contradiction being $\alpha^T J_f(x_0) = 0$.

Let us now see how the defined classes of vector valued pseudoconcave and quasiconcave functions give us the chance to extend the sufficiency of the Kuhn-Tucker conditions.

With this aim let us consider the following vector optimization problem:

$$P : \begin{cases} C^*\text{-max } f(x) \\ g(x) \in V \end{cases}$$

where A is an open convex set, $g : A \to \Re^p$ is a vector valued differentiable function, $V \subset \Re^p$ is a convex cone and $x_0 \in A$ is such that $g(x_0) = 0$. With respect to function g we will use the concept of weakly (C^*, C)-quasiconcavity studied in [6,8] ([1]).

[1] Let $C \subset \Re^m$ be a closed convex pointed cone with nonempty interior and let $C^* \subset \Re^m$ be any cone such that $\mathrm{Int}(C) \subseteq C^* \subseteq C$. A function $f : A \to \Re^m$ is said to be a *weakly (C^*, C)-quasiconcave function* if the following logical implication holds $\forall x, x_0 \in \Re^n$, $x \neq x_0$:

$$f(x) \in f(x_0) + C^* \implies J_f(x_0)(x - x_0) \in C$$

Note that in the scalar case the weakly (C^*, C)-quasiconcave functions coincide with the quasiconcave scalar functions.

Theorem 7. *Let us consider problem P; if $f : A \to \Re^m$ is a $(C^*, \mathrm{Int}(C))$-pseudoconcave function, $g : A \to \Re^p$ is a weakly (V, V)-quasiconcave function and $x_0 \in A$ verifies the following condition:*

$$\begin{array}{c} \exists \alpha \in C^+, \alpha \neq 0, \exists \beta \in V^+ \\ \text{such that } \alpha^T J_f(x_0) + \beta^T J_g(x_0) = 0 \end{array} \tag{5}$$

then x_0 is a global C^*-efficient point.

Proof. Suppose by contradiction that x_0 is not a global C^*-efficient point, that is to say that $\exists x \in A$, $x \neq x_0$, such that:

$$f(x) \in f(x_0) + C^* \text{ and } g(x) \in V;$$

then by means of the $(C^*, \mathrm{Int}(C))$-pseudoconcavity of f and the weakly (V, V)-quasiconcavity of g we have:

$$J_f(x_0)(x - x_0) \in \mathrm{Int}(C) \text{ and } J_g(x_0)(x - x_0) \in V.$$

Using the multiplier vectors α and β of condition (5) we then have:

$$\alpha^T J_f(x_0)(x - x_0) > 0 \text{ and } \beta^T J_g(x_0)(x - x_0) \geq 0$$

so that $[\alpha^T J_f(x_0) + \beta^T J_g(x_0)](x - x_0) > 0$ which is a contradiction.

5 First Order Characterizations

The well known result by Thompson and Parke [17] states that a scalar function f, defined on a convex set A, is [strictly] pseudoconcave if and only if the following logical implication holds $\forall x, x_0 \in A$, $x \neq x_0$:

$$f(x) > f(x_0) \ [\geq] \implies \begin{array}{c} \exists \xi_{x,x_0} > 0 \text{ such that } \forall \lambda \in (0,1) \\ f(x_0 + \lambda(x - x_0)) \geq f(x_0) + \lambda(1-\lambda)\xi_{x,x_0} \end{array}$$

In the vector case the following new result, which extends the one by Thompson and Parke, can be proved.

Theorem 8. *A differentiable function f, defined on an open convex set A, is a $(C^*, \mathrm{Int}(C))$-pseudoconcave function if and only if the following condition holds $\forall x, x_0 \in A$, $x \neq x_0$:*

$$f(x) \in f(x_0) + C^* \implies \begin{array}{c} \exists \xi_{x,x_0} \in \mathrm{Int}(C), \exists \lambda^* \in (0,1] \\ \text{such that } \forall \lambda \in (0, \lambda^*) \\ f(x_0 + \lambda(x - x_0)) \in f(x_0) + \lambda(1-\lambda)\xi_{x,x_0} + C \end{array} \tag{6}$$

Proof. The thesis follows directly by means of Theorem 1.

Note that in [6,8] a function f over a convex set A has been defined to be a *strictly* $(C^*, \mathrm{Int}(C))$*-pseudoconcave function* if the following logical implication holds $\forall x, x_0 \in A$, $x \neq x_0$:

$$f(x) \in f(x_0) + C^* \quad \Longrightarrow \quad \begin{array}{c} \exists \xi_{x,x_0} \in \mathrm{Int}(C) \text{ such that } \forall \lambda \in (0,1) \\ f(x_0 + \lambda(x - x_0)) \in f(x_0) + \lambda(1-\lambda)\xi_{x,x_0} + C \end{array}$$

This class of functions comes out, by means of Theorem 8, to be strictly included in the one of $(C^*, \mathrm{Int}(C))$-pseudoconcave functions, as it is focused on by Example 1 (see [6,8]). Example 1 points out also the importance for $(C^*, \mathrm{Int}(C))$-pseudoconcave functions of the existence of a line segment property in just the beginning ([2]) of the interval $(0,1)$, that is in $(0, \lambda^*)$, and points out also that (6) is a better definition for nonsmooth vector valued pseudoconcave function than the one given in [6,8].

In [13], see also [1], a continuously differentiable scalar function f, defined on an open convex set A, has been proved to be [strictly] pseudoconcave if and only if the following condition holds:

> $\forall x_0 \in A$ and $\forall d \in \Re^n$, $d \neq 0$, such that $\nabla f(x_0)^T d = 0$ the function $\phi(\lambda) = f(x_0 + \lambda d)$ attains a [strict] local maximum at $\lambda = 0$.

In the vector case it is possible to prove the following results.

Theorem 9. *Let f be a differentiable function defined over an open convex set A. If f is a $(C^*, \mathrm{Int}(C))$-pseudoconcave function then:*

> *$\forall x_0 \in A$ and $\forall d \in \Re^n$, $d \neq 0$, such that $J_f(x_0)d \in \mathrm{Fr}(C)$ the function $\phi(\lambda) = f(x_0 + \lambda d)$, $\lambda \geq 0$, attains a local C^*-efficient point at $\lambda = 0$.*

Proof. Suppose by contradiction that $\exists x_0 \in A$, $\exists d \in \Re^n$, $d \neq 0$, such that $J_f(x_0)d \in \mathrm{Fr}(C)$ and $\phi(\lambda)$ does not attain a local C^*-efficient point at $\lambda = 0$, that is to say that $\exists \overline{\lambda} > 0$ such that $f(x_0 + \overline{\lambda} d) \in f(x_0) + C^*$; by means of the $(C^*, \mathrm{Int}(C))$-pseudoconcavity of f it results $\overline{\lambda} J_f(x_0)d \in \mathrm{Int}(C)$ so that $J_f(x_0)d \in \mathrm{Int}(C)$ and this is a contradiction since $J_f(x_0)d \in \mathrm{Fr}(C)$.

Theorem 10. *Let f be a differentiable function defined over an open convex set A. If the two following conditions hold:*

i) f is a locally (C^*, C^*)-quasiconcave function,

ii) $\forall x_0 \in A$ and $\forall d \in \Re^n$, $d \neq 0$, such that $J_f(x_0)d \in \mathrm{Fr}(C)$ the function $\phi(\lambda) = f(x_0 + \lambda d)$, $\lambda \geq 0$, attains a local C^*-efficient point at $\lambda = 0$.

[2] Function f of Example 1 is $(C^*, \mathrm{Int}(C))$-pseudoconcave but it is not strictly $(C^*, \mathrm{Int}(C))$-pseudoconcave (see [6,8]) since the line segment property holds in $(0,1)$ but not in $(0,3)$, even if $f(3) \in f(0) + \mathrm{Int}(C)$.

then f is a $(C^*, \mathrm{Int}(C))$*-pseudoconcave function.*

Proof. Suppose by contradiction that there exist $x, x_0 \in \Re^n$, $x \neq x_0$, such that $f(x) \in f(x_0) + C^*$ and $J_f(x_0)(x - x_0) \notin \mathrm{Int}(C)$. By means of hypothesis *i)* it results $f(x_0 + \lambda(x - x_0)) \in f(x_0) + C^*$ $\forall \lambda \in (0, \lambda^*)$; this implies that $\lambda = 0$ is not a local C^*-efficient point for $\phi(\lambda)$ where $d = x - x_0$ and, by means of Theorem 2, that $J_f(x_0)(x - x_0) \in C$; we then have that $J_f(x_0)(x - x_0) \in \mathrm{Fr}(C)$ and this contradicts hypothesis *ii)*.

The following example points out the importance of condition i) in Theorem 10.

Example 3. Consider the cone $C = \Re^2_+$ and the following twice differentiable function $f : [0, 2] \to \Re^2$:

$$f(x) = \begin{cases} [3, -1]^T - (x-1)^4[3, -1]^T & \text{if } x \in [0, 1] \\ [3, -1]^T + (x-1)^4[-2, 2]^T & \text{if } x \in (1, 2] \end{cases}$$

It results $J_f(x_0)d \in \mathrm{Fr}(C)$ only for $x = 1$ which is a global C^*-efficient point for f, so that condition ii) holds in Theorem 10; we also have that f is neither locally (C^*, C^*)-quasiconcave nor weakly (C^*, C)-quasiconcave nor $(C^*, \mathrm{Int}(C))$-pseudoconcave since $f(2) \in f(0) + \mathrm{Int}(C)$, $f(x) \notin f(0) + C$ $\forall x \in (0, 1)$, and $J_f(0)(x - 0) = 4(x - 0)[3, -1]^T \notin C$ $\forall x > 0$.

We finally provide the following result which characterize a $(C^*, \mathrm{Int}(C))$-pseudoconcave function with no quasiconcavity requirement.

Theorem 11. *Let f be a differentiable function defined over an open convex set A. Function f is a* $(C^*, \mathrm{Int}(C))$*-pseudoconcave function if and only if the following condition holds:*

> $\forall x_0 \in A$ *and* $\forall d \in \Re^n$, $d \neq 0$, *such that* $J_f(x_0)d \notin \mathrm{Int}(C)$ *the function* $\phi(\lambda) = f(x_0 + \lambda d)$, $\lambda \geq 0$, *attains a global* C^**-efficient point at* $\lambda = 0$.

Proof. $\Rightarrow$) Suppose by contradiction that $\exists x_0 \in A$, $\exists d \in \Re^n$, $d \neq 0$, $\exists \overline{\lambda} > 0$ such that $J_f(x_0)d \notin \mathrm{Int}(C)$ and $f(x_0 + \overline{\lambda} d) \in f(x_0) + C^*$; by means of the $(C^*, \mathrm{Int}(C))$-pseudoconcavity of f it results $\overline{\lambda} J_f(x_0)d \in \mathrm{Int}(C)$ so that $J_f(x_0)d \in \mathrm{Int}(C)$ which is a contradiction.
$\Leftarrow$) Suppose by contradiction that there exist $x, x_0 \in \Re^n$, $x \neq x_0$, such that $f(x) \in f(x_0) + C^*$ and $J_f(x_0)(x - x_0) \notin \mathrm{Int}(C)$. Setting $d = x - x_0$, it results $J_f(x_0)d \notin \mathrm{Int}(C)$ and $\phi(1) \in \phi(0) + C^*$ which is a contradiction.

6 Second Order Characterizations

In [13], see also [1], a twice continuously differentiable scalar function f, defined on an open convex set A, has been proved to be [strictly] pseudoconcave if and only if the following condition holds:

$\forall x_0 \in A$ and $\forall d \in \Re^n$, $d \neq 0$, such that $\nabla f(x_0)^T d = 0$ either $d^T H_f(x_0) d < 0$ or $d^T H_f(x_0) d = 0$ and the function $\phi(\lambda) = f(x_0 + \lambda d)$ attains a [strict] local maximum at $\lambda = 0$.

In order to extend to the vector case this result, we firstly prove the following second order necessary condition which holds for the class of $(C^*, \text{Int}(C))$-pseudoconcave functions, for the class of (C^*, C^0)-pseudoconcave functions (class studied in [6,8] ([3]) which contains the $(C^*, \text{Int}(C))$-pseudoconcave functions), and for the class of locally $(\text{Int}(C), C)$-quasiconcave functions.

Theorem 12. *If function f verifies at least one of the following properties:*
i) f is a $(C^*, \text{Int}(C))$-pseudoconcave function,
ii) f is a (C^*, C^0)-pseudoconcave function,
iii) f is a locally $(\text{Int}(C), C)$-quasiconcave function,
then the following condition holds:

$$\forall x_0 \in A,\ \forall d \in \Re^n,\ d \neq 0,\ s.t.\ J_f(x_0) d = 0 \text{ it results } d^T H_f(x_0) d \notin \text{Int}(C)$$

Proof. Suppose by contradiction that there exists $w \in \Re^n$, $w \neq 0$, such that $J_f(x_0) w = 0$ and $w^T H_f(x_0) w \in \text{Int}(C)$. By means of the following second order Taylor expansion of f at x_0 it results:

$$\frac{f(x_0 + tw) - f(x_0)}{t^2} = \frac{1}{t} J_f(x_0) w + \frac{1}{2} w^T H_f(x_0) w + \|w\|^2 \sigma(t, 0)$$

where $\lim_{t \to 0} \sigma(t, 0) = 0$. Since $J_f(x_0) w = 0$ it results:

$$\begin{aligned} \lim_{t \to 0} \frac{f(x_0 + tw) - f(x_0)}{t^2} &= \lim_{t \to 0} \frac{1}{2} w^T H_f(x_0) w + \|w\|^2 \sigma(t, 0) \\ &= \frac{1}{2} w^T H_f(x_0) w \in \text{Int}(C) \end{aligned}$$

Then $\exists \epsilon > 0$ such that $f(x_0 + tw) \in f(x_0) + \text{Int}(C)$ $\forall t \in (-\epsilon, \epsilon)$. Let $t_1 \in (0, \epsilon)$ and let $x_1 = x_0 + t_1 w$ so that $f(x_1) \in f(x_0) + \text{Int}(C)$. If f is a (C^*, C^0)-pseudoconcave function [$(C^*, \text{Int}(C))$-pseudoconcave] it results $J_f(x_0)(x_1 - x_0) = t_1 J_f(x_0) w \in C^0$ [$\in \text{Int}(C)$] and this is a contradiction since $J_f(x_0) w = 0$; function f must then be, by means of the hypothesis, a locally $(\text{Int}(C), C)$-quasiconcave function.
By means of the continuity of f $\exists \bar{\epsilon}$, $0 < \bar{\epsilon} \leq \epsilon$, such that $\forall t \in (-\bar{\epsilon}, 0)$ it is $f(x_0 + tw) \in f(x_0) + \text{Int}(C)$ and $f(x_0 + tw) \in f(x_1) - \text{Int}(C)$. By means of a well known result on the existence of the maximal points (see [2]) then $\exists t_2 \in (-\bar{\epsilon}, 0)$ such that, defined $x_2 = x_0 + t_2 w$, it is $f(x_0 + tw) \notin f(x_2) + C$

[3] A function $f : A \to \Re^m$ is said to be a (C^*, C^0)-*pseudoconcave function* if the following logical implication holds $\forall x, x_0 \in \Re^n$, $x \neq x_0$:

$$f(x) \in f(x_0) + C^* \implies J_f(x_0)(x - x_0) \in C^0$$

$\forall t \in (t_2, 0)$; note also that it results $f(x_2) \in f(x_0) + \text{Int}(C)$ and $f(x_2) \in f(x_1) - \text{Int}(C)$.
Since f is locally $(\text{Int}(C), C)$-quasiconcave and $f(x_1) \in f(x_2) + \text{Int}(C)$ then $\exists \lambda^* \in (0,1)$ such that $f(x_2 + \lambda(x_1 - x_2)) \in f(x_2) + C \ \forall \lambda \in (0, \lambda^*)$, that is to say that $f(x_0 + tw) \in f(x_2) + C \ \forall t \in (t_2, t_2 + \lambda^*(t_1 - t_2))$.
This is a contradiction since $f(x_0 + tw) \notin f(x_2) + C \ \forall t \in (t_2, 0)$ and the proof is then complete.

Theorem 13. *Function f is a $(C^*, \text{Int}(C))$-pseudoconcave function if and only if the following conditions hold:*

i) f is a locally (C^*, C^*)-quasiconcave function,

ii) $\forall x_0 \in A$ and $\forall d \in \Re^n$, $d \neq 0$, such that $J_f(x_0)d = 0$ either $d^T H_f(x_0)d \notin C$ or $d^T H_f(x_0)d \in \text{Fr}(C)$ and the function $\phi(\lambda) = f(x_0 + \lambda d)$, $\lambda \geq 0$, attains a local C^*-efficient point at $\lambda = 0$,

iii) $\forall x_0 \in A$ and $\forall d \in \Re^n$, $d \neq 0$, such that $J_f(x_0)d \in \text{Fr}(C)$, $J_f(x_0)d \neq 0$, the function $\phi(\lambda) = f(x_0 + \lambda d)$, $\lambda \geq 0$, attains a local C^*-efficient point at $\lambda = 0$.

Proof. $\Rightarrow$) The result follows directly from Theorem 3, Theorem 12 and the definition of $(C^*, \text{Int}(C))$-pseudoconcavity.
$\Leftarrow$) Suppose by contradiction that there exist $x, x_0 \in \Re^n$, $x \neq x_0$, such that $f(x) \in f(x_0) + C^*$ and $J_f(x_0)(x - x_0) \notin \text{Int}(C)$. By means of hypothesis i) it results $f(x_0 + \lambda(x - x_0)) \in f(x_0) + C^* \ \forall \lambda \in (0, \lambda^*)$; this implies that $\lambda = 0$ is not a local C^*-efficient point for $\phi(\lambda)$ and, by means of Theorem 2, that $J_f(x_0)(x - x_0) \in C$, which implies that $J_f(x_0)(x - x_0) \in \text{Fr}(C)$.
If $J_f(x_0)(x - x_0) = 0$ then by means of hypothesis i) and the previous Theorem 2 it is $(x - x_0)^T H_f(x_0)(x - x_0) \in C$; this implies for hypothesis ii) that $(x - x_0)^T H_f(x_0)(x - x_0) \in \text{Fr}(C)$ and that $\phi(\lambda)$ attains a local C^*-efficient point at $\lambda = 0$, which is a contradiction.
Suppose now $J_f(x_0)(x - x_0) \in \text{Fr}(C)$, $J_f(x_0)(x - x_0) \neq 0$; hypothesis iii) then implies again that $\phi(\lambda)$ attains a local C^*-efficient point at $\lambda = 0$, which is a contradiction. The proof is then complete.

Let us note that the previous Example 3 points out that no one among the three conditions of Theorem 13 is redundant.

7 Further Results

In this section we will point out that the pseudoconcavity of a scalar or a vector valued function can be characterized by means of the optimality of any point x_0 with respect of a constrained scalar/vector optimization problem P_{x_0}. This kind of results may be used in order to state some more second order characterizations of the "bordered Hessian" type.

In the scalar case it is possible to prove the following result.

Theorem 14. *Let us consider a differentiable scalar function f, defined on an open convex set A, and the following problem, depending on $x_0 \in A$:*

$$P_{x_0} = \begin{cases} Max\ f(x) \\ \nabla f(x_0)^T(x - x_0) \leq 0 \end{cases}$$

Then the three following conditions are equivalent:

i) f is [strictly] pseudoconcave,
ii) x_0 is a [strict] global maximum for P_{x_0}, $\forall x_0 \in A$,
iii) x_0 is a [strict] local maximum for P_{x_0}, $\forall x_0 \in A$.

Proof. i) $\Rightarrow$ ii) Suppose by contradiction that $\exists x_0 \in A$ such that x_0 is not a [strict] global maximum for P_{x_0}, so that $\exists y \in A$, such that $\nabla f(x_0)^T(y - x_0) \leq 0$ and $f(y) > f(x_0)$ $[f(y) \geq f(x_0)]$; then by means of i) it results $\nabla f(x_0)^T(y - x_0) > 0$ which is a contradiction.
ii) $\Rightarrow$ iii) Trivial.
iii) $\Rightarrow$ i) Suppose by contradiction that f is not [strictly] pseudoconcave, that is to say that $\exists y, x_0 \in A$, $y \neq x_0$, with $f(y) > f(x_0)$ $[f(y) \geq f(x_0)]$ and $\nabla f(x_0)^T(y - x_0) \leq 0$; note that the whole line segment $[x_0, y]$ comes out to be feasible for P_{x_0}. Being the function $\phi(\lambda) = f(x_0 + \lambda(y - x_0))$ continuous over $[0, 1]$, there exists the minimum value m_v for it, so that we can define the following $\overline{\lambda} \in [0, 1]$:

$$\overline{\lambda} = sup\{\lambda \in [0, 1] \text{ such that } f(x_0 + \lambda(y - x_0)) = m_v\};$$

being $f(y) > f(x_0)$ $[f(y) \geq f(x_0)]$, being x_0 a [strict] local maximum for P_{x_0} and being the segment $[x_0, y]$ feasible for P_{x_0} it results $0 < \overline{\lambda} < 1$; by means of the continuity of f we can easily prove also that:

$$\begin{gathered} f(x_0 + \overline{\lambda}(y - x_0)) = m_v \text{ and that} \\ f(x_0 + \overline{\lambda}(y - x_0)) < f(x_0 + \lambda(y - x_0)) \quad \forall \lambda \in (\overline{\lambda}, 1]; \end{gathered} \tag{7}$$

it then results $\frac{d\phi}{d\lambda}(\overline{\lambda}) = 0 = \nabla f(\overline{x})^T(y - x_0)$ where $\overline{x} = x_0 + \overline{\lambda}(y - x_0)$.
Let us consider now the problem $P_{\overline{x}}$; being:

$$\nabla f(\overline{x})^T(y - \overline{x}) = (1 - \overline{\lambda})\nabla f(\overline{x})^T(y - x_0) = 0$$

it comes out that the line segment $[\overline{x}, y] \subset [x_0, y]$ is feasible also for $P_{\overline{x}}$ so that, by means of the hypothesis, $\overline{x}$ is a [strict] local maximum for f over $[\overline{x}, y]$ and this contradicts (7).

Note that the second order optimality conditions applied to the previous problem P_{x_0} allow us to find the well known characterization of pseudoconcave scalar functions by means of the bordered hessian.
In the vector case the following analogous result holds.

Theorem 15. *Let us consider function f and the following problem, depending on the point x_0:*

$$P_{x_0} = \begin{cases} C^*\text{-}max\, f(x) \\ J_f(x_0)(x - x_0) \notin \text{Int}(C) \end{cases}$$

Then the three following conditions are equivalent:

i) f *is* $(C^*, \text{Int}(C))$*-pseudoconcave,*
ii) x_0 *is a global* C^**-efficient point for* P_{x_0}, $\forall x_0 \in A$,
iii) f *is locally* (C^*, C^*)*-quasiconcave and* x_0 *is a local* C^**-efficient point for* P_{x_0}, $\forall x_0 \in A$.

Proof. i) ⇒ ii) Suppose by contradiction that $\exists x_0 \in A$ such that x_0 is not a global C^*-efficient point for P_{x_0}, so that $\exists y \in A$, such that $J_f(x_0)(y - x_0) \notin \text{Int}(C)$ and $f(y) \in f(x_0) + C^*$; then by means of i) it results $J_f(x_0)(y - x_0) \in \text{Int}(C)$ which is a contradiction.
ii) ⇒ i) Suppose by contradiction that $\exists x_0, y \in A$, $y \neq x_0$, such that $f(y) \in f(x_0) + C^*$ and $J_f(x_0)(y - x_0) \notin \text{Int}(C)$; y comes out to be feasible for P_{x_0} and x_0 results not to be a global C^*-efficient point for P_{x_0}, which is a contradiction.
ii) ⇒ iii) The thesis follows since a global C^*-efficient point is also local and since ii) implies i) and any $(C^*, \text{Int}(C))$-pseudoconcave function is also locally (C^*, C^*)-quasiconcave for Theorem 3.
iii) ⇒ ii) Suppose by contradiction that $\exists x_0 \in A$ such that x_0 is not a global C^*-efficient point for P_{x_0}, so that $\exists y \in A$, such that $J_f(x_0)(y - x_0) \notin \text{Int}(C)$ and $f(y) \in f(x_0) + C^*$; by means of the local (C^*, C^*)-quasiconcavity of f it is $f(x_0 + \lambda(y - x_0)) \in f(x_0) + C^*$ $\forall \lambda \in (0, \lambda^*)$, $\lambda^* \in (0, 1)$, and this contradicts the local C^*-efficiency of x_0 for P_{x_0}.

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