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Macro -economic models

The case of the Goodwin growth cycle

Piero Manfredi and Luciano Fanti

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Piero Manfredi* and Luciano Fanti**
*Dipartimento di Statistica e Matematica Applicata all'Economia
Via Ridolfi 10, 56124 Pisa - Italy e-mail: manfredi@ec.unipi.it
**Dipartimento di Scienze Economiche Via Ridolfi 10, 56124 Pisa - Italy
e-mail: fanti@ec.unipi.it

ABSTRACT

This paper aims to start a broader investigation of the role played by labour supply and population dynamics within macroeconomic models. Here we concentrate on the Goodwin's growth-cycle framework. First, the role played by the population term within the classical Goodwin's model is discussed. By defining the conditions of economic growth as the conditions for instability of the zero equilibrium (in which the population exists without a "structured economic activity") some useful threshold results are provided. Second we investigate a basic Goodwin-type model in which the rate of growth of the supply of labour is related to the current wage. Subsequently, motivated by the lack of realism of the previous formulation, we essay to go one step further by explicitly introducing age structure within the basic model. This is done by developing a framework, recognising distinct stages of life, which represents a building block for introducing age-structure within macroeconomic models. As a last step we demonstrate the inherently destabilising role of age structure within the classical Goodwin's mode, in presence of a standard malthusian relation between wage and fertility. This suggest several research direction to be investigated in future work.

1 Introduction

A recognised merit of the classical Goodwin's growth cycle model (Goodwin 1967) is the fact that it proposes an articulated explanation of growth and cycle by keeping into account the overall macroeconomic structure. Despite the several extensions proposed in the subsequent literature in order to remove the oversimplicity of its basic assumptions, the attention paid to the problem of the formation of the supply of labour has been very scarce: up to now the Goodwin's model has not been of interest for demo-economic scholars. A major part of the recent efforts in the area of demo-economic interaction has in fact been motivated by the need to provide sound mathematical foundations for the notion of "Easterlin cycle" (Lee (1974,1997), Samuelson (1976), Frauenthal and Swick (1983), Feichtinger and Sorger (1989, 1990), Feichtinger and Doeckner (1990), and Chu and Lu (1995)). The common target of these contributions is the investigation for persistent oscillations induced by the demoeconomic interaction. In all these works the "economic side" is based either on a neoclassical framework, as in Feichtinger and Sorger (1990) and Feichtinger and Doeckner (1990), or is only implicit through some nonlinear demographic relationship, as in Lee, Samuelson, Frauenthal and Swick, Feichtinger and Sorger (1989), and Chu and Lu. No one of these efforts has been concerned with the Goodwinian framework.

The aim of this paper is to start fill this gap by defining a general framework for more refined investigations of demo-economic extensions of Goodwin's model. This is done through the following steps. First we discuss the role played by the supply of labour term within the original Goodwin's model and some simplified variants of it in which we assume that the rate of change of the supply of labour depends on the current wage prevailing in the economy (a standard assumption in the demoeconomic literature, see Day et al. (1989)). Second we introduce a framework aimed at a general treatment of the process of formation of the supply of labour within Goodwin's model which will be investigated in depth in a series of companion papers (Manfredi and Fanti 1999a,b,c). We stress here the fact that the present approach is not ad hoc for Goodwin's model but, rather, it is highly general and applicable to all the economic growth frameworks, such as the neoclassical descriptive model by Solow (1956), depending on the supply of labour only through its rate of change. A forthcoming paper is devoted to the embedding of the present framework within Solow's model (Fanti and Manfredi 1999).

The present paper is organised as follows. In the second section the role played by the supply of labour term within the classical Goodwin's model is discussed with respect to both its static and dynamic implications. The third section investigates the effects of a wage-related rate of change of the supply of labour. In the fourth section our general demo-econonomic framework is introduced, and two distinct alternative formulations of a general-purpose Goodwin-type macroeconomic framework are provided. Several remarks on the problem of embedding age structure and, more in general, population dynamics within macro-economic models, are also added. In the fifth section some remarks are given with reference to the basic case of constant (wage independent) fertility. In the sixth section the inherently destabilising role played, within the Goodwin's model, by age structure in presence of a "classical" (Day et al. 1989) relation between fertility and the real wage is illustrated. The last section sketches out some possible variants of our basic Goodwin-type demo-economic framework which are able to correct such deficiency and which are investigated in subsequent works (Manfredi and Fanti 1999a,b,c).

2 The Goodwin's growth-cycle model: a closer look to the role of the population term

The well known Goodwin's model (1967) is the Lotka-Volterra theory of business cycle, derived in an elegant way by Richard Goodwin in order to describe how the conflict between capitalist and workers on the labour market affects the distribution of the product and the employment, and, so, in the end, the growth capabilities of the economy. The basic relations of the model are a "Phillips" curve, relating the rate of growth of wages at time t ($w(t)$) and the rate of employment ($U(t) = L(t)/N_s(t)$ where L is total labour force actually employed and N_s the supply of labour):

$$\frac{\dot{w}(t)}{w(t)} = -\gamma + \rho U(t) \quad (0 < \gamma < \rho) \tag{1}$$

and an accumulation rule, relating the dynamics of the output ($Q(t)$) to investments ($I(t)$) and profits ($P(t)$):

$$\frac{\dot{Q}(t)}{Q(t)} = m(1 - V(t)) > 0 \tag{2}$$

In equation (2) $Q = aL$ is the national product per unit time ($a(t)$ is the average productivity of labour) and $V = w/a = wL/Q$ is the distributive share of labour. The definition (2) follows from the classical assumption that all profits are reinvested[1] and from the assumption of a fixed technology, i.e. of a fixed capital/output ratio:

$$K/Q = 1/m$$

We can so write:

$$\frac{\dot{Q}(t)}{Q(t)} = \frac{\dot{K}(t)}{K(t)} = \frac{I(t)}{K(t)} = \frac{P(t)}{K(t)} = \frac{P(t)}{Q(t)} \cdot \frac{Q(t)}{K(t)} = m(1 - V(t)) \quad m > 0 \tag{3}$$

Using the dynamical identities:

$$\frac{\dot{V}(t)}{V(t)} = \frac{\dot{w}(t)}{w(t)} - \frac{\dot{a}(t)}{a(t)} \quad \frac{\dot{U}(t)}{U(t)} = \frac{\dot{Q}(t)}{Q(t)} - \left(\frac{\dot{a}(t)}{a(t)} + \frac{\dot{N}_s(t)}{N_s(t)}\right) \tag{4}$$

and by adding the assumptions of a constant growth rate of labour supply ($n_s > 0$) and productivity ($\alpha > 0$), we are finally driven to the standard formulation of Goodwin's model:

$$\begin{aligned} \frac{\dot{V}(t)}{V(t)} &= -(\alpha + \gamma) + \rho U(t) \\ \frac{\dot{U}(t)}{U(t)} &= m(1 - V(t)) - (\alpha + n_s) \end{aligned} \tag{5}$$

[1] Actually this hold even if only a constant proportion s of the profits is reinvested.

The system (5) is a typical Lotka-Volterra system in which the labour share acts as the predator of the employment (the prey). Provided $m-\alpha-n_s$ system (5) is characterised by a Lotka-Volterra conservative dynamics of (U, V) around the positive equilibrium E_1 of coordinates:

$$U^* = (\alpha+\gamma)/\rho; \quad V^* = (m-\alpha-n_s)/m$$

E_1 is of course assumed economically meanigful ($0 \leq U^* \leq 1, 0 \leq V^* \leq 1$). The equilibrium values U^*, V^* can be interpreted as the average values of (U, V) during the fluctuation period. This in turn allows to define the average rate of growth of the product as $g = a + n_s$.

2.1 Role of the supply of labour: static and dynamical considerations

As a preliminary step, let us discuss more deeply the effects of the supply of labour, via the constant term n_s, within the basic Goodwin model. As n_s denotes the rate of change of the labour supply, a constant n_s may be justified by assuming: i)a fully exogenous population dynamics; ii)unchanging conditions in the participation to the labour market (we will make this statement more rigorous later on).

The system (5) always has the zero equilibrium $E_0 = (0, 0)$, whatever be n_s. For what concerns the existence of the positive equilibrium, provided the condition for having an economically meaningful equilibrium for employment ($\rho > \alpha + \gamma$) is met, we have to distinguish:

i)$n_s > 0$; in this case, the equilibrium value of the labour share: $V^* = (m - \alpha - n_s)/m$ will be meaningful as long as:

$$m - \alpha > n_s$$

i.e. provided the rate of growth of population does not exceed the difference $m - \alpha$.

b)$n_s < 0$; in this case, the equilibrium value of the labour share V^* will loose economic significance when $(\alpha + n_s)<0$, taking a value greater than one. This will always happen if we explicitly assume as constant ($\alpha = 0$) the average productivity of labour. This fact seems to indicate that the standard Goodwin's formulation is not appropriate for the study of decaying populations, unless in those cases in which the growth of productivity is fast enough to keep $V^* < 1$.

Before continuing our discussion it is of help to make some clarification for what concerns the role of the equilibria within Goodwin models. In traditional economic analysis of Goodwin-type models we are used to consider the E_0 equilibrium as the "trivial" one. In fact in the model-building process we usually seek conditions aiming to guarantee the existence of a positive ("non-trivial") equilibrium, and very often dynamical analysis concentrate on the properties of this positive equilibrium (stability, existence of persistent periodic behaviours and so on). Viceversa, if we aim to a global view of the process of economic growth we must recognize that a crucial role is played by both equilibria, and in particular (assuming for simplicity that no other equilibria exist), by the stability switches between E_0 and E_1.

In fact in presence of a positive rate of growth of the supply of labour, the E_0 equilibrium has a clear meaning: it defines the equilibrium state of a community surviving without a "structured" economic activity, as stated by a common labour market plus a production structure. This state (notice that the present considerations are totally independent from the present Goodwin-type environment) may be even "stable". We expect that this happens for instance when the accumulation rule is weaker compared to population growth, thereby preventing the take-off of structured economic activity. Vice-versa we expect that when accumulation is sufficiently high, E_0 may even become unstable and perhaps a new equilibrium E_1, characterized by a "structured" economic activity, appears. We expect that this new equilibrium will then be stable.

The previous considerations, which are motivated by our study of Goodwin's model but are in fact much more general, suggest how to seek conditions for the overall "take-off" of the economy, corresponding to the establishment of structured economic activity. Among the possible candidates there are:

- the conditions of (local) instability of the E_0 equilibrium
- the conditions for the appearance of a strictly positive equilibrium E_1 and for its (local) stability.

As we will see later on, these conditions collapse in a unique condition in the basic Goodwin's model (but our work on Goodwin-type models suggests that this fact is much more common, so to deserve more interest).

Let us tehrefore consider the jacobian at E_0, given by:

$$J(E_0) = \begin{pmatrix} -(\alpha+\gamma) & 0 \\ 0 & m-\alpha-n_s \end{pmatrix}$$

It follows that:

i) when $n_s > 0$, then E_0 is unstable as long as n_s is not "too" large to prevent E_1 to be positive, i.e as long as $m - \alpha - n_s > 0$. When n_s is too large ($n_s > m - \alpha$) it prevents the economy from establishing a positive equilibrium in wage and employment: in this case E_0 is of course stable (LAS and GAS).

ii) when $n_s < 0$ (in this case we may have or not an economically meaningful positive equilibrium) E_0 is unstable unless α is so large to more than counterbalance the positive term $m - n_s > m$. In particular, if $\alpha = 0$, then E_0 is always unstable. This has the following explanation: if the supply of labour is decaying in presence of a positive accumulation ($m > 0$), then employment necessarily increases, therefore pushing upward the wages: this is an unstable mechanism with respect to E_0.

Let us collect our results. Let us consider first the simplified case $\alpha = 0$, which permits to enlight the pure-state balance between accumulation and population growth. To fix the ideas let us call the E_0 situation the case of a "subsistence economy", in which the rule is stationarity (this is a typical ancient world (AW) case), and E_1 the case of a developing economy (MW case), in which growth is the rule. In this way we may interpret the tendency of the E_0 equilibrium to loose its stability in favour of E_1, as the natural footprint of the historical process of economic development.

PROPOSITION 1. *In presence of a positive rate of growth of the labour supply ($n_s>0$) then the economy may experience a transition from the AW to the MW (this happens mathematically via the instability of E_0) only if the accumulation forces exceed the rate of growth of the population ($m > n_s$). Moreover, if $m > n_s$ holds, then accumulation permits the establishment of a positive equilibrium state, to which it corresponds the classical conservative LVG dynamics.*

Viceversa, when the rate of growth of the population is so high, or the rate of accumulation is so low, in such a manner that $m < n_s$, then no positive equilibrium is possible in the economy. In this case the E_0 equilibrium can never become unstable, and the economy can not make transitions from the AW state to growth.

REMARK 1. *The crucial threshold parameter for the reproduction of the economy appears to be the ratio:*

$$R_E = \frac{m}{n_s} \tag{6}$$

Provided $n_s > 0$ the growth of the economy is possible IFF $R_E > 1$.

Borrowing from the demographic dictionary we will call the ratio R_E the "reproduction ratio of the economy"[2]. Its interpretation is the following: if the supply of labour is steadily growing at the rate n_s, to guarantee the growth of the economy, the accumulation conditions (m) must be able to provide more than one additional job place for every new worker entering the labour market at least in the optimal condition in which the entire product is distributed to the profit. In Goodwin-type economies this definition appears quite natural as all the profit is reinvested in new labour.

The case $n_s < 0$ is less interesting. From the U equation (remembering that $0 < V < 1$), the employment rate is forced to grow systematically, and will reach, in the long term its upper bound (unity for simplicity). At that time the asymptotic dynamics of the labour share would be described by the equation:

$$\frac{\dot{V}(t)}{V(t)} = -(\alpha + \gamma) + \rho \tag{7}$$

leading to long term exponential growth of V (provided the basic assumption $\rho > \alpha + \gamma$ holds), which in turn will stop the accumulation process, leading to stationarity again. It is anyway clear

[2] In demographic analysis the net reproduction ratio R_0 (see Keyfitz (1968)) is a commonly used index (having a genetic counterpart) of the reproductive ability of a population, based on the ratio between the number of newborn individuals in two subsequent generations. In stationary conditions it predicts the growth of the population when $R_0 > 1$ and its decay in the opposite case.

that the model is no more in condition to explain what's going on in the economy: the stationary point $(1,1)$ is not a stationary point of the model itself. These remarks seem to definitively exclude the possibility to use Goodwin's model as a model for economies with constantly decaying populations, unless, as already pointed out, the growth of productivity is so high to more than counterbalance the decay of the supply of labour.

3 Effects of wage-related labour supply dynamics

3.1 Fertility depending on the current wage

At the very least the supply of labour at time t, $N_s(t)$, has to be defined as the product between the total number of individuals in the working age span $N(t)$, and the participation rate $s(t)$: $N_s(t) = s(t)N(t)$. In dynamical terms we have[3]:

$$\frac{\dot{N}_s(t)}{N_s(t)} = \frac{\dot{s}(t)}{s(t)} + \frac{\dot{N}(t)}{N(t)} \tag{8}$$

Let us write $\dot{N}(t)/N(t) = n(t)$, defining the rate of change of the population in the working age span. We take $n(t) = b(t) - \mu(t)$ where $b(t), \mu(t)$ are respectively the enrollment rate and the exit rate prevailing in the working age-span. As a first, very rough, approximation, we may assume that $b(t)$ is the birth rate of the population and μ is the death rate.[4] The fact to estimate the enrollment rate with the birth rate is of course quite coarse: it amounts to assume that the recruitment into the labour force occurs without time-delays (i.e.: at birth), and hence it is to be intended as a purely preliminary step toward more general investigations. Following the "classical" tradition (Lee (1974, 1998), Day et al. (1989)) let us assume that the fertility rate depends on the living conditions, summarised by the real wage prevailing in the economy: $b = b(w)$, where b has the standard malthusian form (monotonically increasing and possibly saturating). By temporarily neglecting the effects of the participation term (i.e. by assuming $s(t) = const$), and by working for simplicity on the (w, U) environment[5] and using the relation $w = aV$ we obtain the following variant of Goodwin's model (we suppress the time variable):

$$\begin{aligned} \dot{w} &= w\left(-\gamma + \rho U\right) \\ \dot{U} &= U\left(m(1-w) - (b(w) - \mu)\right) \end{aligned} \tag{9}$$

[3] The relation (8) permits to evaluate the lack of realism of the assumption of a constant rate of change of the supply of labour, which is common not only to the Goodwin's literature but also to the standard (descriptive) neoclassical growth model (Solow (1956)). Such an assumption implies both: i)a constant rate of change of the population in the working age span, and ii)a constant participation rate.

[4] We are therefore assuming that our population is closed to migrations. There are no difficulties in considering an open population.

[5] This is done by assuming $\alpha = 0$. This amouns to study the impact of population growth in a world in which it does not exist technical progress.

For simplicity the productivity index a has been rescaled to one: this permits to preserve the interpretation of rate of profit for the term $(1-w)$. Hence during the dynamics the wage w (which is equivalent in this case to the wage share) is expected to remain nonnegative and upperly bounded by one, i.e. by the level of the productivity.

We notice that when $n(w)$ is linear the model (9) preserves the conservative Goodwin's oscillations but this is true in many other situations. We discuss here some of the basic properties of (9).

The system (9) always has the zero equilibrium $E_0 = (0,0)$. Moreover, the employment has the positive equilibrium value $U_1 = \gamma/\rho$ which is meaningful provided that $\rho > \gamma$. Viceversa no positive equilibrium values of the wage exists if the growth rate of the population $n(w) = b\,(w) - \mu$ is negative for all the admissible values of the wage. For the subsequent discussions the following definitions are useful:

DEFINITION 1. *(Wage-related net reproduction ratio of the population) It is the ratio:*

$$R_0(w) = \frac{b(w)}{\mu} \tag{10}$$

Obviously, for a prescribed w, the population would tend to increase for $R_0(w) > 1$, and to decrease for $R_0(w) < 1$. Notice that if b were constant, then also R_0 would be constant, and defined by the ratio b/μ, which is the traditional definition of net reproduction ratio for aggregate populations.

DEFINITION 2. *(Wage-related net reproduction ratio of the economy)It is the ratio:*

$$R_E(w) = \frac{m}{n(w)} \tag{11}$$

3.1.1 The case of linearly increasing fertility

If $b(w) = b_0 + b_1 w$ $(b_0 \geq 0;\ b_1 > 0)$ the U equation becomes:

$$\dot{U} = U\left(m(1-w) - (b_0 + b_1 w - \mu)\right)$$

or:

$$\dot{U} = U\left(m(1-w) - (n_0 + b_1 w)\right) \tag{12}$$

where $n_0 = b_0 - \mu$ is the growth rate at zero wage, and $m + b_1 > 0$. Let us moreover define as $n(1) = n_0 + b_1$ the maximal growth rate of the population, corresponding to the maximal wage $w = 1$. Hence, when $n_0 > 0$ (i.e.: $R_0(0) > 1$) we have two cases: i)if $m < n_0$ ($R_E(0) < 1$) no positive equilibrium of the wage exists (the growth of the population at low wages is too fast to be supported by the available technology); ii)$0 < n_0 < m$ (i.e.: $R_E(0) > 1$) a unique equilibrium exists (with a positive growth rate of the population).

Moreover, when $n_0 < 0$: iii)if $-b_1 < n_0 < 0$ then a positive equilibrium of the wage still exists, with a positive growth rate of the population. Notice that: $-b_1 < n_0$ or: $n_0 + b_1 > 0$ means: $n(1) > 0$ or $R_0(1) > 1$, i.e. that the demographic environment is capable of generating a positive rate of growth of the population at least at high levels of the wage (although it can not at low wages); iv)if $n_0 < -b_1 < 0$ then $n(1) < 0$, and no positive equilibrium of the wage exists. Hence, in the cases ii) and iii) the system (9) has a positive equilibrium $E_1 = (w_1, U_1)$, where: $w_1 = (m - n_0) / (m + b_1)$, $U_1 = \gamma/\rho$, which is always meaningful from the economic point of view.

Dynamically, as long as E_1 exists, the system (9) is a LVG-type system describing a predator-prey interaction and hence preserving the traditional LVG conservative oscillations.

Let us summarise the main results of this section in the following:

PROPOSITION 3. *If the fertility rate is a linear function of the wage, then an economically meaningful positive equilibrium of the economy exists provided $-b_1 < n_0 < m$. In this event Goodwin's conservative oscillations are preserved and generate a strictly positive rate of growth of the population.*[6]

The case of more general increasing fertility functions The assumption of a fertility rate linearly related to the real wage is unrealistic. It is possible to show tha all the results of the previous subsection are preserved if we more realistically assume that fertility is an increasing (possibly saturating) function of the wage. For ease of comparison with the previous section let us write:

$$b(w) = b_0 + b_1(w)$$

where $b_0 \geq 0$ and $b_1(0) = 0, b_1'(w) > 0, b_1(\infty) = b_\infty < +\infty$. The employment equation becomes:

$$\dot{U} = U\left((m - n_0) - (mw + b_1(w))\right)$$

The same discussion on equilibria of the previous subsection holds (we just have to replace b_1 with $b_1(1)$). Hence, provided a positive equilibrium of the wage still exists, LVG conservative oscillations are still the rule. This is just a standard result on generalised LVG models of the form:

$$\dot{w} = wf(U) \;\; ; \;\; \dot{U} = Ug(w)$$

[6] Positive equilibria of the wage would coexist with decaying populations only with wages exceeding their physical upper bound of productivity.

4 Toward more realistic formulations: an ODE formulation as a first step toward realistic age structured demoeconomic models

In this section we remove the simplistic assumptions used in the previous sections to model population dynamics by developing a general framework for the study of the role played by population dynamics and age-structure within demoeconomic models[7]. Such a framework has the advantage of capture in a sufficiently realistic way the effects of age-structure whilst mantaining at the same time a reasonable degree of tractability. By adopting an assumption of age-independent demographic coefficients we will able to simplify the general PDE for age structure into a more economical ODE formulation (hence mantaining compatibily with the traditional formulations of economic growth models) describing the population process by means of a sequence of irreversible age-stages. An approach of this type was used for instance by Hallam and Zhien (1988) in the mathematical biology literature. Feichtinger and Sorger (1990) used the same equations in a demoeconomical problem but without explicit reference to the underlying age structure framework and in any case their assumptions lead to a different formulation. We want to stress here that the present approach is not ad hoc for the Goodwin's model but rather it is highly general and applicable to all the economic growth frameworks, such as the neoclassical descriptive model by Solow (1956), depending on the supply of labour only through its rate of change.

Let us suppose that $N(a,t)$ be the age-time density of a given population, driven by the usual McKendrick-Von Foerster PDE:

$$\left[\frac{\partial}{\partial}+\frac{\partial}{\partial}\right]N(a,t) = -\mu(a,t)N(a,t) \tag{13}$$

plus the usual boundary and initial conditions:

$$N(0,t) = B(t)\ ;\ N(a,0) = \varphi(a)$$

where $\mu(a,N,t)$ is the age-dependent mortality rate, $B(t)$ the birth function at time t and $\varphi(a)$ a prescribed function of age assigning the age density at time zero. To derive from (13) an ODE formulation let us subdivide our population into three broad age classes: $(0,A)$=pre-work age (or "young", for simplicity $(0, 15years)$), (A,B)=adult or working age span (for instance: $(15,65)$), (B,∞)=retirement, and let the functions $P_1(t), P_2(t), P_3(t)$ denote the number of individuals in the three stage at time t.[8] By integrating the basic PDE over each one of the three age classes

[7]The framework introduced here is fully general. Therefore, although the present applications are essentially devoted to the investigation of Goodwin-type economies, it may be fruitfully applied to whatever demoeconomic framework. For instance we applied it to the study of demoeconomic dynamics in the Solow's neoclassical model.

[8]We could of course introduce an arbitrary number of stages (and this could of interest from some points of view) but this is the simplest reasonable choice.

and by using the assumptions: $P(0,t) = B(t),\ P(\infty,t) = 0$, we get the following relations:

$$\begin{array}{l} \dot{P}_1(t) = B(t) - D_1(t) - P(A,t) \\ \dot{P}_2(t) = [P(A,t) - P(B,t)] - D_2(t) \\ \dot{P}_3(t) = P(B,t) - D_3(t) \end{array} \tag{14}$$

where the D_i terms denote the number of deaths within each class at time t, while $P(A,t)$ is the number of transitions from class one to class two at time t due to the aging process, and similarly $P(B,t)$ is the number of transitions from class two to class three. Obviously age-class three is an absorbing state. The interpretation of (14) is straightforward. For instance the dynamics of the number of individuals in the working age span $P_2(t)$ is the outcome of the balance between the entries from the pre-work class ($P(A,t)$) and the exits due to aging ($P(B,t)$) and mortality ($D_2(t)$). Let us now add the following assumptions: i) both the number of deaths in each class and the number of transitions from each class to the next-one at any time t are linearly related with the population in that class, i.e.:

$$D_i(t) = \mu_i\,(t)\,P_i\,(t) \quad ; \quad P\,(A,t) = v_1\,(t)\,P_1\,(t) \quad ; \quad P\,(B,t) = v_2\,(t)\,P_2\,(t)$$

ii)both the death and the transition rates μ_i, v_i are constant, iii)the birth function is defined as:

$$B(t) = b\,(t)\,P_2 \tag{15}$$

meaning that all the births take place in adult population, where $b\,(t)$ is the fertility rate of the population in the working age span. The assumptions i),ii),iii) lead finally to the ODE system:

$$\begin{array}{l} \dot{P}_1(t) = b\,(t)\,P_2 - (\mu_1 + v_1)\,P_1(t) \\ \dot{P}_2(t) = v_1 P_1 - (\mu_2 + v_2)\,P_2(t) \\ \dot{P}_3(t) = v_2 P_2(t) - \mu_3 P_3(t) \end{array} \tag{16}$$

Let us now combine the demographic "subsystem" (16) with the basic Goodwin's "subsystem" (9). We obtain the following 5-dimensional demo-economic model:

$$\begin{array}{l} \dot{w}(t) = w(t)\,(-\gamma + \rho U\,(t)) \\ \dot{U}(t) = U(t)\,(m(1 - w\,(t)) - n_s) \\ \dot{P}_1(t) = b(t) P_2\,(t) - (\mu_1 + v_1)\,P_1(t) \\ \dot{P}_2(t) = v_1 P_1(t) - (\mu_2 + v_2)\,P_2(t) \\ \dot{P}_3(t) = v_2 P_2(t) - \mu_3 P_3(t) \end{array} \tag{17}$$

In the last formulation the economic subsystem is influenced by the demographic one via the rate of change of the labour supply n_s:

$$n_s = \frac{\dot{N}_s}{N_s} = \frac{\dot{s}}{s} + \frac{\dot{N}}{N} \tag{18}$$

which in turn is affected by i) the changes in the total population in the adult (or working) age span:

$$\frac{\dot{N}}{N} = \frac{\dot{P}_2}{P_2} \tag{19}$$

and by ii) the changes in the participation rate $s(t)$. Viceversa there can be several inputs from the economic system to the demographic subsystem: the simplest is certainly represented by the influence the wage level may play on fertility.

The system (17) can be somewhat simplified. Notice first that the equation of the elderly population does not provide any input to the system and can be neglected, thereby reducing one dimension. Moreover, by temporarily neglecting the effect of the activity rates (i.e. by putting $\dot{s}/s = 0$), it holds:

$$n_s \equiv n = \frac{\dot{N}}{N} = \frac{\dot{P}_2}{P_2}$$

where:

$$\frac{\dot{P}_2}{P_2} = v_1 \frac{P_1}{P_2} - (\mu_2 + v_2) \tag{20}$$

The system (17) therefore simplifies to:

$$\begin{aligned} \dot{w} &= w(t)\left(-\gamma + \rho U\right) \\ \dot{U} &= U\left[m(1-w) - \left(v_1 \tfrac{P_1}{P_2} - (\mu_2 + v_2)\right)\right] \\ \dot{P}_1 &= b\left(t\right) P_2 - \left(\mu_1 + v_1\right) P_1 \\ \dot{P}_2 &= v_1 P_1 - \left(\mu_2 + v_2\right) P_2 \end{aligned} \tag{21}$$

From (21) we notice that, if the demographic system does not receive any input from the economic side, i.e. all the coefficients of the demographic subsystem are constant (=independent on the overall set of economic variables), i.e. if in particular the fertility rate is taken as constant, then Goodwin's model is recovered asymptotically. In fact the demographic subsystem, described by the third and fourth equations in (21), will follow in the long term a stable exponential path in which both P_1 and P_2 evolve exponentially at the same rate. This means that the ratio P_1/P_2 between the young and the adult population will be asymptotically constant. This provides a constant demographic input to the employment equation, which is then formally equivalent to the traditional simplified Goodwin's formulation, provided that:

$$n_s = v_1 \left(\frac{P_1}{P_2}\right)_\infty - (\mu_2 + v_2) > 0$$

In this case the formulation (21) thereby extends the original Goodwin's formulation in a straightforward, alhough more realistic, manner. The main new is represented by the presence of a "transient" phase during which the stage composition of the population reaches its equilibrium level.

As the economic system depends on the demographic input only via the ratio P_1/P_2, it is convenient to reduce the system one further dimension by introducing the new variable P_1/P_2, which follows the ODE:

$$\dot{\left(\frac{P_1}{P_2}\right)} = \left(\frac{P_1}{P_2}\right)\left[\left(\frac{\dot{P_1}}{P_1}\right) - \left(\frac{\dot{P_2}}{P_2}\right)\right]$$

By posing $P_1/P_2 = Z$ we have:

$$\dot{Z} = Z\left[\frac{bP_2 - (\mu_1 + v_1)P_1}{P_1} - \frac{v_1 P_1 - (\mu_2 + v_2)P_2}{P_2}\right]$$

leading to the equation:

$$\dot{Z} = b - HZ - v_1 Z^2 \tag{22}$$

where:

$$H = (\mu_1 + v_1) - (\mu_2 + v_2) \gtreqless 0 \tag{23}$$

We have therefore obtained the three-dimensional system:

$$\begin{aligned} \dot{w} &= w\,(-\gamma + \rho U) \\ \dot{U} &= U\left[m(1-w) - (v_1 Z - (\mu_2 + v_2))\right] \\ \dot{Z} &= b - HZ - v_1 Z^2 \end{aligned} \tag{24}$$

The system (24) defines a flexible framework for the study of Goodwin-type dynamics in presence of an "almost realistic" population description. It is the minimal modelling structure for the investigation of demoeconomic interactions within Goodwin's model (and also other models of demoeconomical interactions, such as the Solows' model). The demographic subsystem appears through the ratio young/adult $Z = Z(t)$.

4.1 An alternative formulation

Here we derive an alternative formulation using as third dynamical variable the rate of change $n(t)$ of the population in the working age class (instead of Z):

$$n = \frac{\dot{N}}{N} = \frac{\dot{P_2}}{P_2}$$

(still assuming $\dot{s} = 0$). Since:

$$n(t) = \frac{\dot{P_2}}{P_2} = \frac{v_1 P_1 - (\mu_2 + v_2)P_2}{P_2} = v_1 Z(t) - (\mu_2 + v_2) \tag{25}$$

it holds:

$$\dot{n}(t) = v_1 \dot{Z}(t) \tag{26}$$

As:

$$v_1 Z(t) = (\mu_2 + v_2) + n(t)$$

using the previous developments we get:

$$\begin{aligned}\dot{n}(t) &= v_1 b - H v_1 Z - (v_1 Z)^2 = \\ &= (-1)\left[n^2 + (H + 2(\mu_2 + v_2))\, n - \left(v_1 b - H(\mu_2 + v_2) - (\mu_2 + v_2)^2\right)\right]\end{aligned}$$

Definitively:

$$\dot{n}(t) = (-1)\left[n^2 + Pn - B\right] \tag{27}$$

where:

$$P = H + 2(\mu_2 + v_2) = (\mu_1 + v_1) + (\mu_2 + v_2) > 0 \tag{28}$$

and:

$$\begin{aligned}B &= v_1 b - H(\mu_2 + v_2) - (\mu_2 + v_2)^2 = v_1 b - (\mu_2 + v_2)(\mu_1 + v_1) = \\ &= (\mu_2 + v_2)(\mu_1 + v_1)\left(\frac{v_1 b}{(\mu_2 + v_2)(\mu_1 + v_1)} - 1\right)\end{aligned}$$

It is then useful to define:

$$B = (\mu_2 + v_2)(\mu_1 + v_1)\,(R_0 - 1) \tag{29}$$

where:

$$R_0 = \frac{v_1 b}{(\mu_2 + v_2)(\mu_1 + v_1)} \tag{30}$$

is the net reproduction ratio (NRR) for our age-structured population. In fact it is the product of the quantity $b/(\mu_2 + v_2)$ representing the average number of offspring produced by an average individual during his sejour in the adult state, times the probability $v_1/(\mu_1 + v_1)$ to reach the adult state. Let us write: $B = Q(R_0 - 1)$, which is positive or negative depending on whether the NRR is greater or smaller than one. In sum we get the following alternative formulation of our basic Goodwin-type model:

$$\begin{array}{l}\dot{w}(t) = w(t)\,(-\gamma + \rho U(t)) \\ \dot{U}(t) = U(t)\,[m(1 - w(t)) - n(t)] \\ \dot{n} = (-1)\,[n^2 + Pn - B]\end{array} \tag{31}$$

where:

$$\begin{array}{l}P = (\mu_1 + v_1) + (\mu_2 + v_2) > 0 \\ B = Q\,(R_0 - 1) \\ Q = (\mu_2 + v_2)(\mu_1 + v_1)\end{array} \tag{32}$$

The formulation (31) is equivalent to (24).

5 A preliminary: the Goodwin's model with age structure and constant fertility

As pointed out in the previous section, under the assumption of constant fertility, Goodwin's model is recovered in the long term. In fact the evolution of the population is independent on economic conditions and the population itself reaches in the long term a stable state in which the Z ratio is constant over time at some value Z^*. This implies that the demographic subsystem provides a constant (long term) input to the U equation given by:

$$n^* = v_1 Z^* - (\mu_2 + v_2)$$

The condition: $n^* > 0$ is therefore necessary in order to guarantee an economically meaningful long term Goodwin-type dynamics with an economically meaningful equilibrium value of the wage. Notice that n^* is the rate of growth of the population in the working age span (and of course of all the age classesi.e. of one of the age classes). Hence the condition $n^* > 0$ coincides with the growth condition of the overall population, namely: $R_0 > 1$. To prove this let us consider, from the third equation (24) the unique long term equilibrium value Z^* of Z:

$$Z^* = \frac{-H \pm \sqrt{H^2 + 4v_1 b}}{2v_1}$$

The condition $n^* > 0$ therefore leads to:

$$-H \pm \sqrt{H^2 + 4v_1 b} > 2(\mu_2 + v_2)$$

i.e:

$$H^2 + 4v_1 b > H^2 + 4C^2 + 4HC$$

where $C = \mu_2 + v_2$). This leads finally to the condition: $v_1 b > C(C + H)$, i.e.:

$$\frac{v_1 b}{C(C+H)} = \frac{b}{\mu_2 + v_2} \frac{v_1}{\mu_1 + v_1} = R_0 > 1$$

which is the desired condition. When the population has reached its long term stable state the dynamics of the economy will be driven by the following asymptotic 2-dimensional Goodwin-type system:

$$\begin{aligned} \dot{w} &= w(-\gamma + \rho U) \\ \dot{U} &= U\left[m(1-w) - (v_1 Z^* - (\mu_2 + v_2))\right] \end{aligned} \qquad (33)$$

In particular the long term rate of growth of output at equilibrium is given by:

$$\left(\frac{\dot{Q}}{Q}\right)_E = \frac{\dot{U}}{U} + \frac{\dot{N}}{N} = \left(\frac{\dot{N}}{N}\right)_E = v_1 Z^* - (\mu_2 + v_2)$$

This relationship makes very sharp an aspect which was not "transparent" from the basic aggregate Goodwin's formulation:

PROPOSITION 1. *The rate of growth of the output in the long-term equilibrium is a strictly increasing function of the ratio Z^* between young and adults prevailing in the population. More precisely, it is a strictly increasing function of the "relative" flow $v_1 Z^*$ of young people becoming adult each unit of time.*

6 The age structured Goodwin-type model with wage-dependent fertility

Let us now move to the more realistic case of wage-related fertility: $b = b(w)$. This leads to the model (by working on the variables (w, U, n):

$$\begin{array}{l} \dot{w} = w\left(-\gamma + \rho U\right) = w f_1(U) \\ \dot{U} = U\left[m(1-w) - n\right] = U f_2(w,n) \\ \dot{n} = (-1)\left[n^2 + Pn - B(w)\right] = f_3(w,n) \end{array} \tag{34}$$

where:

$$\begin{array}{l} P = (\mu_1 + v_1) + (\mu_2 + v_2) > 0 \\ B = Q\left(R_0(w) - 1\right) = v_1 b(w) - Q \\ Q = (\mu_2 + v_2)(\mu_1 + v_1) \end{array} \tag{35}$$

In this paper we consider only the "standard case" in which the fertility rate is a monotonically increasing and saturating function of the wage: $b = b(w)$, $b(0) \geq 0, b'(w) \geq 0$ (as before we will sometimes write: $b(w) = b_0 + b_1(w)$, $b_0 \geq 0, b_1(0) = 0$). The "postclassical" case of downturning fertility, motivated both on the theoretical and empirical side by the demo-economic literature (Day et al. 1989) will be considered elsewhere (Manfredi and Fanti 1999c).

6.1 Equilibria and their local stability

As $\dot{w} = \dot{U} = 0$ for $U = w = 0$, the system (34-35) always has an equilibrium with zero wage and employment and a nonzero rate of change of population. The equilibrium of the rate of change of the (adult) population is the solution of:

$$n^2 + Pn - B(0) = 0 \tag{36}$$

The last equation has only one nontrivial (i.e. suitable to represent a rate of change) solution, the positive one namely, given by[9]:

$$n_0 = \frac{1}{2}\left(-P + \sqrt{P^2 + 4B(0)}\right) \tag{37}$$

Hence $n_0 \gtrless 0$ depending on whether $B(0) \gtrless 0$. This implies:

$$R_0(0) \gtrless 1 \tag{38}$$

[9]The condition: $\Delta = P^2 + 4B(0) > 0$ needed to ensure that the involved root be real is always satisfied. In fact:

$$\begin{aligned} \Delta &= P^2 + 4B(0) = (P^2 - 4Q) + 4v_1 b(0) = \\ &= \left((\mu_1 + v_1) - (\mu_2 + v_2)\right)^2 + 4v_1 b(0) > 0 \end{aligned}$$

where $R_0(0)$ is the NRR prevailing at zero wage. In the particular case $b(0) = 0$ we find:

$$B = -Q = -(\mu_2 + v_2)(\mu_1 + v_1)$$

leading to: $n_0 = -(\mu_2 + v_2) < 0$ implying: $Z_0 = 0$.

Local stability of E_0 The general jacobian of the system is:

$$J(w, U, n) = \begin{pmatrix} -\gamma + \rho U & \rho w & 0 \\ -mU & m(1-w) - n & -U \\ B'(w) & 0 & -(2n + P) \end{pmatrix} \tag{39}$$

Hence:

$$J(E_0) = \begin{pmatrix} -\gamma & 0 & 0 \\ 0 & m - n_0 & 0 \\ B'(0) & 0 & -(2n_0 + P) \end{pmatrix}$$

with eigenvalues:

$$\lambda_{1,2,3} = (-\gamma, m - n_0, -(2n_0 + P)) \tag{40}$$

Notice that:

$$-(2n_0 + P) = -\sqrt{P^2 + 4B(0)} < 0$$

Hence, when $R_0(0) < 1$, as $n_0 < 0$, then E_0 is always locally unstable (a saddle point). In the more interesting case $n_0 > 0$, i.e. $R_0(0) > 1$ then E_0 is locally stable as long as $m < n_0$, and it becomes locally unstable when $m > n_0$, i.e. when $R_E(0) > 1$.

In words: economic development, as synthesized by the instability of E_0, becomes possible when accumulation is sufficiently fast and/or the rate of growth of population is not so large to overtake accumulation. The condition $m > n_0$ ($R_E(0) > 1$) may be expressed also as:

$$m > n_0 \quad \rightarrow \quad m > \frac{1}{2}\left(-P + \sqrt{P^2 + 4B(0)}\right)$$

from which:

$$1 < R_0(0) < 1 + \frac{m(m+P)}{Q} \tag{41}$$

The condition (41) defines the window of values of $R_0(0)$ in which population induced economic growth is possible.

Other equilibria The system does not admit axis equilibria, and admits up to one positive equilibrium, denoted in the sequel by E_1. From the first equation we have the usual Goodwin's equilibrium value of employment: $U^* = \gamma/\rho$. The equilibrium values for w and n follow from the subsystem:

$$\begin{aligned} m(1-w) - n &= 0 \\ n^2 + Pn - B(w) &= 0 \end{aligned} \tag{42}$$

Let us study the intersections between the two curves: $n_A(w) = m(1-w)$ and $n_B(w) = \frac{1}{2}\left(-P+\sqrt{P^2+4B(w)}\right)$. As the curve $B(w)$ is monotonically increasing, $n_2(w)$ is monotonically increasing as well. Therefore the existence of intersections depends essentially on the mutual location of the intercepts $n_A(0) = m$, $n_B(0) = n_0$ defined in (37). We have two basic situations:

i)when $n_B(0) = n_0 > 0$, i.e.when $R_0(0) > 1$, then we will have a positive equilibrium provided $m > n_B(0) = n_0$ i.e. $R_E(0) > 1$, and none if $R_E(0) < 1$. Notice that this is the condition that guarantees the instability of E_0.

ii)when $n_B(0) = n_0 < 0$ i.e. when $R_0(0) < 1$ we will certainly have a positive equilibrium provided:

$$n_B(w=1) > 0 \quad \rightarrow \quad R_0(1) > 1$$

In fact in this case $n_B(1) = n(1) < m$, or: $R_E(1) > 1$ In brief:

- when the net reproduction rate of the population at zero wage ($R_0(0)$) is greater than one (therefore generating a positive rate of growth of the population at zero wage, n_0), then a positive equilibrium will certainly exist, provided the corresponding reproduction rate of the economy $R_E(0)$ is greater than one. It is intuitive in fact that if the population is capable to grow even for very low wages, we may expect that its growth may become desruptive for accumulation when wages are high (remember that fertility is positively related to the wage by assumption).
- Nonetheless the condition $R_0(0) > 1$, i.e. $n_0 > 0$, is not necessary for a positive equilibrium. Even when $n_0 < 0$ population induced economic growth may emerge provided that the population be able to generate a strictly positive rate of growth at least in correspondence of the situation in which the whole product is distributed to wages, i.e. provided that:

  $$n(w=1) > 0 \leftrightarrow \quad R_0(1) > 1$$

  We notice that in this last case the E_0 equilibrium is always unstable. Notice moreover that we do do not have the triviality of the case $n < 0$ in the original Goodwin's model, where the meaningfulness of the positive equilibrium was lost.

6.1.1 Local stability of the positive equilibrium

The relevant jacobian matrix is:

$$J(E_1) = \begin{pmatrix} 0 & \rho w & 0 \\ -mU & 0 & -U \\ B' & 0 & -(2n+P) \end{pmatrix}$$

leading to the characteristic polynomial:

$$P(X) = X^3 + (2n + P)X^2 + m\rho UwX + \rho Uw\left[m(2n + P) + B'(w)\right]$$

the coefficients of which are all strictly positive. By a straightforward application of the Routh-Hurwicz criterion we get:

$$\begin{aligned} a_1a_2 - a_3 &= (2n + P)m\rho Uw - \rho Uw\left[m(2n + P) + B'(w)\right] = \\ &= -\rho wUB'(w) < 0 \end{aligned}$$

implying that E_1 is always (locally) unstable. Hence the following important result follows:

PROPOSITION 1. *as long as the fertility of individuals is an increasing function of the wage the positive equilibrium is always unstable.*

7 Final remarks: general demoeconomic framework and research directions

The analysis of the previous section shows that the introduction of a realistic demographic mechanism which delays the entrance of newborn people into the labour market always destabilizes the positive equilibrium of Goodwin's model. This result opens a broad series of research questions which will be attacked in a series of forthcoming papers. A main question is the following: are there possible supply-side mechanisms which are able to restore stability, and possibly to lead to sustained oscillations? Other demoeconomic questions are: which are the effects of fertility downturns at high wages? Which are effects of heterogeneity in fertility? In order to answer such questions we now formulate a broader Goodwin-type demoeconomic framework embedding also stabilising mechanisms which will be extensively investigated in subsequent work (Manfredi and Fanti (1999a,1999b, 1999c)). With respect to the standard basic Goodwin model with age structure, given in (24) or (31), we will also consider:

- participation effects. By recalling that $N_s(t) = s(t)N(t)$, a more complete description of the formation of the labour supply needs specific assumptions on participation as well. In Manfredi and Fanti (1999a) the classical discouraged worker hypothesis (Mincer 1966) is used:

$$\frac{\dot{s}(t)}{s(t)} = q(w, U) \tag{43}$$

  where $\partial q/\partial w > 0, \partial q/\partial U > 0$. More specifically we will consider linear participation:

$$\frac{\dot{s}(t)}{s(t)} = -\delta + q_1U + q_2w \quad q_1, q_2 > 0. \tag{44}$$

  Another interesting possibility, still giving rise to stabilising effects, is to assume that: $v_1 = v_1(U)$ i.e. that the rate of transition to adulthood be a function of the state of the labour market.

- a general fertility function allowing heterogeneity in fertility rates between employed and unemployed individuals. The form of the fertility rate is:

$$b = b(w, U) = b_1(w)U + b_2(1 - U) \tag{45}$$

The last relation says that the overall fertility rate is the weighted mean of the fertility of employed and unemployed (the weights being given by the rates of employment and unemployment). The use of a relation as (45) is allowed by a peculiar feature of the classical Goodwin's model: the existence of structural unemployment. The role of heterogeneity is well ackowledged in demographic theory (see for instance the synthesis in Keyfitz (1985)). The formulation (45) reduces to: $b = b_1(w)U$ in the special case the fertility rate of unemployed is assumed to be zero.

The interest for the explicit introduction of relationships as the present one is suggested by other work of the authors on the role of fertility within the Goodwin's model: the model is (locally) stabilised when the fertility of the employed population is higher than the fertility of the unemployed, but is destabilised when the converse is true (Manfredi and Fanti 1999c).

- a general Phillips relation given by:

$$\dot{w} = w\left[-\gamma + RU + \vartheta\,\dot{U}\right] \tag{46}$$

This formulation, which is the original Phillips' one (Phillips 1958, Lipsey 1960) has also been used by Sportelli (1995). It permits to take into account some further important facts of the labour market, namely: a)the influence of the particular phase of the business-cycle on the economy, b)the observed hysteresis (the state of a variable depends on its own past history) of the natural rate of employment in many Western countries (Blanchard-Summers (1990)).

8 Conclusions

In this paper a general framework has been developed in order to study the impact of population dynamics on the process of formation of the supply of labour within the classical Goodwin's model. This framework seems to be quite promising in that it allows a good balance between reality of description and mathematical tractability. In particular we have considered the behaviour of a Goodwin-type model with age structure (the main effect of which is that of delaying the entrance of individuals into the labour market) in presence of a classical malthusian wage related fertility schedule. The main result is that the demographic mechanism, always destabilizes the positive equilibrium of Goodwin's model (whereas Goodwin's conservative oscillations are preserved in a simplistic variant with wage related fertility but without age structure). This result opens a broad series of research questions, the main of which is: which dynamical outcomes are expected when we consider further realistic mechanisms which can be capable to restore stability? Other related demoeconomic questions are: which are the effects of fertility downturns? Which are effects of heterogeneity in fertility? In order to answer such questions we have also proposed several stabilising "supply side" mechanisms whose effects on our Goodwin-type demo-economic will be investigated in subsequent works (Manfredi and Fanti 1999a, 1999b, 1999c).

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