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A Separation Theorem in Alternative Theorems and Vector Optimization

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Sunto

Nei problemi di ottimizzazione vincolata, con vincoli espressi nella forma di disuguaglianza, le condizioni del primo ordine di Fritz-John sono stabilite ricorrendo al classico Teorema di separazione di Minkoswki applicato a due insiemi convessi dati, rispettivamente, l'interno dell'ortante di non negatività e un sotto-spazio lineare W costituito dall'insieme dei vettori aventi per componenti le derivate direzionali della funzione obiettivo e delle funzioni vincolari. Un tale teorema non permette però di avere informazioni riguardo alla stretta positività di uno o più moltiplicatori e per tale ragione, in presenza di vincoli di qualifica si ricorre ad un teorema di Alternativa (in genere il Teorema di Farkas) per ottenere le condizioni Kuhn-Tucker. Anche nella dimostrazione diretta dei più classici Teoremi di Alternativa l'uso del Teorema di Minkoswki è solo parziale, non permettendo come nel caso presentato precedentemente di stabilire il segno dei moltiplicatori.

Scopo di questo primo lavoro è di presentare una versione più raffinata del Teorema di Minkowski nello studio della separazione tra un sottospazio lineare e un cono poliedrico, che permetta di superare gli inconvenienti sopra descritti, ovvero di determinare in modo diretto sia le condizioni di Kuhn-Tucker, sia i classici teoremi di Farkas e di Motzkin.

In un successivo lavoro mostreremo come i più noti teoremi dell'alternativa quali ad esempio quelli recentemente illustrati nel lavoro di M.E. De Giuli, G.Giorgi, U. Magnani A General Linear Theorem of the Alternative: how to get its special cases quickly", P.U.M.A. vol. 8 (1997) n° 2,3,4, 215-232, possano essere ottenuti estendendo l'approccio suggerito in questo lavoro.

1. A separation theorem

Let $\Re^m_+ = \{x = (x_1, x_2, .. x_m) \in \Re^m \mid x_i \geq 0, i=1,..m\}$ be the Paretian cone of the m-dimensional space and let W be a linear subspace of $\Re^m$. It is well known that in such a case there exists a hyper plane $\Gamma = \{u \mid \langle \alpha \cdot u \rangle = 0\}$ separating W e $\Re^m_+$ with $\alpha \geq [0]$.

In many problems, for instance in optimality conditions, we are interested to know if some components of α are strictly positive. In order to deep this aspect, we will investigate the intersection between W and the boundary of $\Re^m_+$ instead of using Theorems of the Alternative, which will be obtained in a simple way by means of our approach.

Denote with $e^{(i)}$ i= 1,..,m, the unitary vector of $\Re^m_+$. Let $W^* = W - \Re^m_+$ the conic extension of W with respect to $\Re^m_+$.

The following properties holds (see [5],[6]).

P_1. W^* is a closed convex cone;

P_2. W^* is the intersection of all its supporting hyperplanes at the origin;

P_3. $W \cap \text{int}\, \Re^m_+ = \varnothing \Leftrightarrow W^* \cap \text{int}\, \Re^m_+ = \varnothing$;

P_4. $W \cap \text{Fr}\, \Re^m_+ = W° \Rightarrow W^* \cap \text{Fr}\, \Re^m_+ = F$, where F is the minimal face (with respect to the inclusion) of $\Re^m_+$ containing W°;

P_5. An hyperplane Γ separates W and $\Re^m_+$ $\Leftrightarrow$ Γ separates W^* and $\Re^m_+$ $\Leftrightarrow$ Γ is a supporting hyperplane at the origin to W^*.

P_6. $W \cap \text{int}\Re^m_+ = \varnothing \Leftrightarrow$ there exists an hyper plane $\Gamma = \{u \mid \langle \alpha \cdot u \rangle = 0\}$ separating W e $\Re^m_+$ with $\alpha \geq 0$ such that

$\langle \alpha \cdot w \rangle = 0, \forall w \in W,$ (1.1.a)

$\langle \alpha \cdot w \rangle \geq 0, \forall w \in \Re^m_+,$ (1.1.b)

$\langle \alpha \cdot w \rangle \leq 0, \forall w \in W^*.$ (1.1.c)

Corresponding to a face F of $\Re^m_+$, denote $I = \{i \mid e^{(i)} \in F\}$. The following separation theorem holds:

Theorem 1.1: Assume that $W \cap \text{int}\, \Re^m_+ = \varnothing$. Then $W^* \cap \Re^m_+ = F$ if and only if

i) for every hyperplane Γ, which separates W e $\Re^m_+$, it results $\alpha_i = 0 \ \forall \ i \in I$,

ii) there exists an hyperplane $\Gamma°$ which separates W e $\Re^m_+$, such that $\alpha_i > 0 \ \forall \ i \notin I$.

Proof: $\Rightarrow$ Since $W \cap \text{int } \Re^m_+ = \varnothing$ there exists an hyperplane which separates W e $\Re^m_+$ such that (1.1.a,b,c) hold. i) $F = W^* \cap \text{Fr } \Re^m_+$ implies $\forall i \in I$, $e^{(i)} \in W^* \cap \text{Fr } \Re^m_+$. From (1.1.b) and (1.1c) $<\alpha \cdot e^{(i)}> = \alpha_i = 0 \ \forall i \in I$. ii) Now we consider $e^{(i)} \notin W^*$. Property P_2 implies the existence of a supporting hyperplanes of W* at the origin which does not contain $e^{(i)}$. Taking into account property P_3 there exists $\forall j \notin I \ \alpha^{(J)}$ such that $<\alpha^{(J)} \cdot w> = 0 \ \forall \ w \in W^*$ and $<\alpha^{(J)} \cdot e^{(i)}> > 0$. It is now easy to verify that the vector $\alpha° = \sum \alpha^{(J)} j \notin I$ is such that $<\alpha°, e^{(i)}> = \alpha°_j > 0$.

$\Leftarrow$ Obviously i) and ii) imply $W \cap \text{int}\Re^m_+ = \varnothing$, $W^* \cap \text{int } \Re^m_+ = \varnothing$ and that (1.1.a,b,c) hold. Since $<\alpha \cdot e^{(i)}> = \alpha_i > 0 \ \forall i \notin I$ it follows from (1.1.b) $\forall i \notin I$ $e^{(i)} \notin W^*$. It remains to prove that $\forall i \in I$ $e^{(i)} \in F$ implies $e^{(i)} \in W^*$. Assume $e^{(k)} \notin W^*$ $k \in I$, ii) implies an hyperplane such that $<\alpha' \cdot w> = 0 \ \forall \ w \in W$ and $<\alpha' \cdot e^{(k)}> > 0$ and this contradicts i).

Now we will consider the case when W is a linear subspace of $\Re^m$ and $\Re^m$ is the Cartesian product of two or more Euclidean vector spaces, i.e. $\Re^m_+ = \Re^q_+ \times \Re^s_+ \times \Re^t_+$, m = q + s + t. Denote with $w=(w_q,w_p,w_t)$ an element of W and set $C = (\text{int } \Re^q_+ \times (\Re^s_+ \setminus \{0\}) \times \Re^t_+)$, the following separation theorem holds:

Theorem 1.2: $W \cap C = \varnothing$ if and only if there exists a hyper plane $\Gamma = \{(u,v,w) \mid <\alpha \cdot u> + <\beta \cdot v> + <\gamma \cdot w> = 0\}$ separating W and C with $\alpha \geq [0]$, $\beta \geqq [0]$, $\gamma \geqq [0]$ or with $\alpha = [0]$, $\beta > [0]$, $\gamma \geqq [0]$[1].

Proof: $\Rightarrow$ The assumption implies $W \cap \text{int } \Re^m_+ = \varnothing$, so there exists a hyper plane Γ with $(\alpha,\beta,\gamma) \geq [0]$, if $\alpha \geq [0]$, the thesis holds; if $\alpha = [0]$ for every hyper plane separating W and C we have to prove that $\beta > [0]$. Suppose one element of β is different to zero

[1] If $x \geqq z$ then $x_j \geqq z_j \ \forall j$, if $x > z$ then $x_j > z_j \ \forall j$, if $x \geq z$ then $x_j \geqq z_j \ \forall j$, $x \neq z$.

then for Theorem 1.1 this means that F contains all edges of $\Re^q_+$ and one of $\Re^s_+$. Since F is the minimal face containing W*, there exists an $x' = \sum \lambda_i e^{(i)}$ $\lambda_i > 0$ where $e^{(i)}$ are all edges of $\Re^q_+$ and one of $\Re^s_+$, so $x' \in C$. This is not possible since $W \cap C = \emptyset$.

$\Leftarrow$ If there exists an hyperplane $\Gamma = \{(u,v,w) \mid <\alpha \cdot u> + <\beta \cdot v> + <\gamma \cdot w> = 0 \ \forall (u,v,w) \in W^*\}$ with $\alpha \geq [0]$, $\beta \geqq [0]$, $\gamma \geqq [0]$ or with $\alpha = [0]$, $\beta > [0]$, $\gamma \geqq [0]$ then $W \cap C = \emptyset$. Suppose there exists a $w' = (w'_q, w'_p, w_t') \in W \cap C$, i.e. such that $w'_q > [0]$, $w'_p \geq [0]$, $w_t' \geqq [0]$ then $<\alpha, w'_q> + <\beta, w'_p> + <\gamma, w_t'> > 0$ and for (1.1.a) this contradicts $w' \in \Gamma$.

Taking into account the previous result we have the following corollaries.

Let $C = (\Re^s_+ \setminus \{0\} \times \Re^t_+)$, we have:

Corollary 1.1: $W \cap C = \emptyset$ if and only if there exists an hyper plane $\Gamma = \{(u,v) \mid <\alpha \cdot w_s> + <\beta \cdot w_t> = [0], \forall (w_s, w_t) \in W^*\}$ with $\alpha > [0]$, $\beta \geqq [0]$.

Let $C = (\text{int}\ \Re^s_+ \times \Re^t_+)$ we have

Corollary 1.2: $W \cap C = \emptyset$ if and only if there exists a hyper plane $\Gamma = \{(u, v) \mid <\alpha \cdot w_s> + <\beta \cdot w_t> = 0, \forall (w_s, w_t) \in W^*\}$ with $\alpha_t \geq [0]$, $\beta \geqq [0]$.

Let $C = (\text{int}\ \Re^s_+ \times \Re^t_+ \setminus \{0\})$ we have

Corollary 1.3: $W \cap C = \emptyset$ if and only if there exists a hyper plane $\Gamma = \{(u,v) \mid <\alpha \cdot w_s> + <\beta \cdot w_t> = 0, \forall (w_s, w_t) \in W^*\}$ with $\alpha \geq 0$, $\beta \geqq 0$ or with $\alpha = [0]$, $\beta > [0]$.

2. Applications

In this section we will utilize the Separation Theorem presented in the previous section in order to show how it is possible to prove, in a simply way, Kuhn-Tucker's conditions [7] and the classic theorems of the alternative of Farkas and Motzkin [3].

With this aim we state, preliminary, some classic definitions and results. A multi objective programming problem is defined as [6]:

$$P: \max F(x) = (f_1(x), \ldots, f_p(x)),$$

$$x \in S = \{x \in \Re^n \mid G(x) = (g_1(x), \ldots, g_m(x)) \geqq [0]\} \subseteq \Re^n$$

where f_i and g_i are assumed to be continuously differentiable on a open set containing S.

A point $x_0 \in S$ is said to be:

- a **Pareto optimal solution** for problem P if there is no $x \in S$ such that $F(x) \geq F(x_0)$;
- a **weak Pareto optimal solution** for problem P if there is no $x \in S$ such that $F(x) > F(x_0)$;
- a **properly Pareto optimal solution** if it is Pareto optimal solution and if there is no $d \in \Re^n$ such that

$$< \nabla F(x_0), d > \geq [0] \quad (2.1.a)$$

$$< \nabla G(x_0), d > \geqq [0], \quad (2.1.b)$$

where $\nabla F(x_0) = (\nabla f_i(x)^T, i = 1,..,p)$, $\nabla G(x) = (\nabla g_j(x)^T, j \in I(x_0))$ with $I(x_0) = \{ j \mid g_j(x_0)=0, i=1,..,m\}$, $s = \# I(x_0)$.

If x_0 is a weak Pareto optimal solution for problem P, then it is well known that there is no $d \in T(S, x_0)$ such that $<\nabla f_i(x)^T d>>0$, $<\nabla g_i(x)^T d> \geqq 0$, where $T(S, x_0)$ is the Bouligand tangent cone to the feasible set of S at x_0.

Problem P is said to satisfy the Abadie constraint qualification at $x_0 \in S$, if $T(S, x_0)$ is the linearizing cone to S at x_0. This implies that: there is no d such that:

$$< \nabla F(x_0), d > > [0], \quad (2.2.a)$$

$$< \nabla G(x_0), d > \geqq [0] \quad (2.2.b)$$

Consider the linear subspace: $W = \{z \in \Re^{p+s} \mid z = \begin{bmatrix} \nabla F(x_0) \\ \nabla G(x_0) \end{bmatrix} d, d \in \Re^n\}$. Obviously, it results that:

- System (2.1) is equivalent to the condition

$$W \cap (\Re^p_+ \setminus \{0\} \times \Re^s_+) = \varnothing, \quad (2.3.a)$$

- System (2.2) is equivalent to the condition

$W \cap (\text{int } \Re^p_+ \times \Re^s_+) = \varnothing$ (2.3.b).

As a consequence we have the following necessary Kuhn-Tuckers optimality conditions:

Theorem 2.1 : Let $x_0 \in S$ be a properly Pareto optimal solution for problem P. Then there exists $\alpha \in \Re^p$, $\beta \in \Re^s$ such that:

$$<\alpha, \nabla F(x_0)> + <\beta, \nabla G(x_0)> = 0$$

$$\alpha > [0], \beta \geqq [0].$$

Proof: The assumption of the theorem implies (2.3.a), the thesis follows for Corollary (1.1).

Theorem 2.2: Let $x_0 \in S$ be a properly Pareto optimal solution for problem P assume that Abadie constraint qualification holds at $x_0 \in S$. Then there exists $\alpha \in \Re^p$, $\beta \in \Re^s$ such that:

$$<\alpha, \nabla F(x_0)> + <\beta, \nabla G(x_0)> = 0$$

$$\alpha \geq [0], \beta \geqq [0].$$

Proof: In the same way of Theorem 2.1, taking into account of Corollary 1.2.

Now we will give a proof of Farkas' theorem and Motzkin's theorem, utilizing the previous results. Let a real matrix A of order (m;n) and the column-vector $b \in \Re^m$ and $x \in \Re^n$, we have

Farkas'theorem: $S = \begin{cases} yA \geqq [0] \\ yb < [0] \end{cases}$ is impossible $\Leftrightarrow$ $S' = \begin{cases} Ax = b \\ x \geqq [0] \end{cases}$ has solution.

Proof: $\Rightarrow$ Set $W = \{ y \begin{bmatrix} A \\ -b \end{bmatrix} \in \Re^{n+1} \mid y \in \Re^m \}$ and $C = \Re^m_+ \times \text{int } \Re$. S is impossible $\Leftrightarrow$ $W \cap C = \varnothing$. From Corollary 1.1 there exists a hyperplane separating W and C such that $< k \cdot \alpha > + < h \cdot \beta > = 0$, $\forall (k,h) \in W$ $\alpha \geqq [0]$ and $\beta \geq [0]$. Set $k = y^T [A]$ and $h = y^T$

[-b] we have: y^T [A α -b β] = 0 with α ≧ [0], β > [0]. In this way, set x = α/β we obtain a solution of system S'.

⇐ If S' has solution then there exists a x' $\in \Re^n$ such that Ax' = b, x' ≧ [0]. We consider y^T [A x' -b] = 0 then the hyper plane < k· x' > + <h ·1 > = 0, ∀(k,h)∈W separates W e C. For Corollary 1.1 W ∩ C = ∅ and S has no solutions.

Motzkin's Theorem: The system

$$S: \begin{cases} Vk = [0] \\ Wk \leqq [0], \\ Zk < 0 \end{cases}$$

has no solution if and only if system

$$S': \begin{cases} vV + wW + zZ = [0] \\ v \text{ sign unrestrected}, \\ \quad w \geqq 0, z \geq 0 \end{cases} \quad \text{has solution.}$$

Proof: ⇒ Set W = { y^T [V:-V:W:Z] | y∈$\Re^m$} = { y^T [K:H] | y∈$\Re^m$} where K = [V:-V:W] and H = [Z] and C = $\Re^s_+ \times \text{int}\, \Re^t_+$. S is impossible if and only if W ∩ C = ∅. From Corollary 1.1 there exists a hyperplane separating W and C such that <α· k> + <β· h > = 0 ∀ (h,k)∈W α ≧ [0] and β ≥ [0]. Set k = y^T [V: -V:W] and h = y^T [Z]

y^T [$(\alpha_2 - \alpha_1)V + \alpha_3 W + \beta Z$] = 0 (*), $v = (\alpha_2 - \alpha_1)$, $w = \alpha_3 \geqq [0]$, $z = \beta \geq [0]$ we have a solution of system S'.

⇐ If S' has solution then there exists a $v = (\alpha_2 - \alpha_1)$, $w = \alpha_3 \geqq [0]$, $z = \beta \geq [0]$that verifies system S' and condition (*). So for Corollary (1.1) W ∩ C = ∅ and S is impossible.

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