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Labour supply, population dynamics, and persistent oscillations in a Goodwin-type growth cycle model

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ABSTRACT

A general Goodwin-type framework for the investigation of labour market dynamics is developed. The framework embeds: i)a standard "malthusian" wage dependent fertility schedule, ii)a participation schedule dependent on the state of the labour market, and especially iii)a simplified "mechanistic" representation of the age structure prevailing in the population.

The aforementioned framework is then used to address two main problems: a)to provide an alternative explanation to demoeconomic oscillations which is based on a model, Goodwin's one, that takes into account the overall macroeconomic structure, b) to compare different modelling strategies for the age structure mechanism.

With respect to the latter point, the (mainly numerical) results obtained from our general mechanistic model are also borne out by those provided by a simpler model, based on a simplified representation of age structure by means of a time-delay, which permits a more detailed mathematical analysis.

Two major facts emerge: a)the discovery of endogeous "labour-market induced" demoeconomic waves, which provide an alternative explanation to the traditional Easterlin effect; b)the feeling that even if the full treatment of age structure is complex, and its detailed effects may only be studied via numerical simulations, simplified models based on time-delays may be of great help in understanding the basic qualitative consequences of the introduction of age structure.

1 Introduction[1]

A major part of the recent efforts in the area of demo-economic interaction has been motivated by the need to provide sound mathematical foundations for the notion of "Easterlin cycle". Among these we recall the contributions by Lee (1974), Samuelson (1976), Frauenthal and Swick (1983), Feichtinger and Sorger (1989, 1990), Feichtinger and Doeckner (1990), and Chu and Lu (1995). The common target of these contributions is the investigation of persistent oscillations induced by the demo-economic interaction. In all these works the "economic side" is based either on a neoclassical framework, as in Feichtinger and Sorger (1990) and Feichtinger and Doeckner (1990), or is only implicit through some nonlinear demographic relationship, as in Lee, Samuelson, Frauenthal and Swick, Feichtinger and Sorger (1989), and Chu and Lu. All these last models are in fact purely "demographic": the demoeconomic interaction is taken into account by resorting to some clever modelling trick.

The recent work by Lee (1997) clarifies the nature of population fluctuations: "*Fluctuations in the population age distribution come about in three ways. In the simplest case fluctuations are imposed by some external force, such as the climate or the economy. It is also possible that the population renewal process itself creates damped waves (the so called "generational cycles"). The third possibility is that "Malthusian cycles" occur, due to the lags between the response of fertility to current labour market conditions and the time when the resulting births actually enter the labour force*" (Lee 1997, 1097). The Malthusian cycle, to which the notion of Easterlin cycle of course belongs, is the most important notion of demoeconomic cycle, and it calls for explanations which be an endogenous outcome of the interaction between economic and demographic forces. Despite the undoubted richness of his paper, Lee (1997) confines his analysis of Malthusian cycles solely to Easterlin's assumption, without any explicit consideration of the macroeconomic structure (along the lines of Lee (1974), Samuelson (1976) etc).

In this paper we try to offer a different perspective, by setting the demo-economic interaction within the framework of the classical Goodwin (1967) growth cycle model. This latter model represents, in our opinion, a rich and stimulating framework for the investigation of general (i.e.: not necessarily of the Easterlin type) demoeconomic relations, in that it takes into account for the overall macroeconomic structure. A second important motivation in favour of the Goodwin's framework is the fact that its "funding principle", the profit-squeeze mechanism, is ubiquitous in well formulated modern keynesian models, as sharply evidenced in recent research work by Chiarella and Flaschel (1999a,b). Chiarella and Flaschel have shown that Goodwin's principle is a highly robust dynamical principle that continues to be observed even when highly realistic generalisations of the model, including neoclassical substitution, are considered.

Among the many extensions of the classical Goodwin's model available in the macrodynamical

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literature there has not been, to our knowledge, a systematic investigation of the consequences of a full endogeneisation of population dynamics and age-structure. In this paper we start a systematic analysis of the effects of population dynamics, age structure and labour supply within Goodwin-type models (see also Manfredi and Fanti 1999a,b). This is done via a stage structured representation of the full age structure mechanism which appears a suitable and parsimonious tool. This representation permits to display the most relevant effects of the age structure on the supply of labour, by simply adding a third nonlinear equation (embedding several demographic parameters) to the basic two dimensional Goodwin's model. We stress that our framework is general and straightforwardly applicable to other types of growth models, such as the neoclassical growth model by Solow (1956), see Fanti and Manfredi (1999).

Our framework is then used to investigate the dynamical effects on the Goodwin's cycle of an "almost realistic" treatment of the labour supply process embedding, besides age structure: i)a standard "malthusian" wage dependent fertility schedule, ii)a participation schedule dependent on the state of the labour market. This set of ingredients appears to be taylored to the study of "malthusian cycles" following the previous definition by Lee.

Our analysis, besides some results on the steady state behaviour of the economy) shows the appearance of sustained "labour-market induced" demoeconomic oscillations via Hopf bifurcations. These oscillations seem to be a strict consequence of the mechanism of formation of the supply of labour: limit cycles appear essentially as the result of the balancing between the stabilising action due to the "participation" mechanism, and the demographic instability which is caused by the delayed entrance into the labour market due to the age structure mechanism. By choosing as a bifurcation parameter the age of entry into the labour force two main patterns appear from our simulations: strongly capital-intensive economies exhibit a quite large bifurcating age of entry into the labour force, whereas the opposite happens in weakly capital-intensive economies. The last fact seems to be in strong agreement with the empirical evidence contrasting developed with developing countries.

The central role played by the age of entry into the labour force as a critical bifurcation parameter (a role put in sharp evidence by the formulation used here) is suggestive in view of the possibility to use it in the planning of demo-economic policies aimed, for instance, at influencing family formation and fertility by acting on the process of transition to adulthood.

The results obtained on our general model have also been confirmed by those provided by a simpler model, based on a simplified representation of age structure by means of a time delay. As known from the literature on lags (Mac Donald (1978)) time-delays represent simplified but effective tools to capture the complex effects of age structure. The results provided by the simplified model are surprisingly close to those provided by the general one, with the advantage that its analysis is much simpler, so that we are in the position to analytically determine the whole bifurcation curve.

Two major facts thereby emerge from our analysis. First, the discovery of endogenous sustained "malthusian" demo-economic waves based on a dynamical mechanism not necessarily of the Easterlin type. Second, the feeling that even if the full treatment of age structure is complex, and its detailed effects may only be studied via numerical simulations, simplified models based on time-delays may be of great help in understanding its basic qualitative features.

The present paper is organised as follows. The second section is devoted to the derivation of our age structured demo-economic Goodwin-type framework. Section three is devoted to the basic properties of our general model. In the fourth section a complete local analysis of the alternative model based on a time-delay is provided. Section five is devoted to numerical simulations and to a comparison of the predictions provided by the two models considered. Conclusive remarks follow.

2 A general Goodwin-type demoeconomic framework embedding age structure and participation

2.1 The classical Goodwin's growth-cycle model

The well known Goodwin's model (1967) is the Lotka-Volterra theory of business cycle, derived by Goodwin to describe how the conflict between capitalists and workers on the labour market determines the distribution of the product and the employment level, and how these forces in turn affect the long term growth of the economy. The structure of the model is the following:

$$\begin{aligned} \frac{\dot{V}(t)}{V(t)} &= -(\alpha+\gamma)+\rho U \\ \frac{\dot{U}(t)}{U(t)} &= m(1-V)-(\alpha+n_s) \end{aligned} \tag{1}$$

where $U = U(t)$ is the employment rate at time t, defined as the ratio between the total labour force actually employed $L(t)$ and the supply of labour $N_s(t)$, while $V(t)$ is the distributive share of labour at time t, given by the ratio $w(t)L(t)/Q(t)$, where w is the real wage and Q the total product. V can be expressed also as: $V = w/A$ where A is the average productivity of labour. Moreover $\gamma > 0, \rho > 0$ represent characteristic parameters of the labour market, $m > 0$ is the output-capital ratio, and $\alpha > 0, n_s > 0$ respectively denote the rate of change of the productivity of labour and of the labour supply. The model (1) is obtained in two steps: first by expressing the relations $V = w/A$ and $U = Q/AN_s$ in terms of their rates of growth:

$$\frac{\dot{V}(t)}{V(t)} = \frac{\dot{w}(t)}{w(t)} - \frac{\dot{A}(t)}{A(t)} \quad \frac{\dot{U}(t)}{U(t)} = \frac{\dot{Q}(t)}{Q(t)} - \left(\frac{\dot{A}(t)}{A(t)} + \frac{\dot{N}_s(t)}{N_s(t)}\right) \tag{2}$$

and, second, by adding the following assumptions:

i)the labour market is driven by the linear Phillips relation:

$$\frac{\dot{w}(t)}{w(t)} = -\gamma + \rho U \quad (0 < \gamma < \rho) \tag{3}$$

ii)the accumulation rules are such that: a)the wage earners do not save, b)profits are entirely reinvested, c)the technology is Leontief-type, d)the capital output ratio $K/Q = m^{-1}$ is constant. These assumptions lead to the following equation for the rate of growth of the output:

$$\frac{\dot{Q}(t)}{Q(t)} = m(1 - V)m > 0 \tag{4}$$

iii)The supply of labour and the productivity of labour grow exogenously at the constant rates $n_s > 0$ and $\alpha > 0$.

By introducing (3) and (4) into (2) the formulation (1) quickly follows. The model (1) is a typical Lotka-Volterra system in which the labour share acts as the predator of the employment (the prey). In particular, when $m < \alpha + n_s$ the system has as a unique equilibrium the zero equilibrium $E_0 = (0,0)$ which is globally asymptotically stable[2] (GAS). Viceversa, provided $m > \alpha + n_s$, system (1) exhibits the traditional Lotka-Volterra conservative oscillations around the positive equilibrium E_1 of coordinates: $U_1 = (\alpha + \gamma)/\rho$; $V_1 = (m - \alpha - n_s)/m$ (notice that E_1 is economically meanigful provided $\rho > \alpha + \gamma$). The equilibrium values U_1, V_1 are the average values of (U, V) during the fluctuation period. As a consequence the average rate of growth of the output is $g = \alpha + n_s$.

The inequality $m > \alpha + n_s$ plays a critical role in the model. As discussed in more detail in Manfredi and Fanti (1999) it provides a threshold which at the same time governs whether: i)the E_0 equilibrium may be unstable, therefore providing the conditions for the economic "take off" which is a necessary "precondition" in a process of economic growth; ii)the positive equilibrium E_1 exists and is locally stable. In brief: the inequality $m > \alpha + n_s$ governs the stability switches between E_0 and E_1, therefore permitting the establishment of the conditions for a "structured" economic activity. This suggests, borrowing from the demographic dictionary, the following definition:

DEFINITION 1. *The ratio:*

$$R_E = \frac{m}{\alpha + n_s} \tag{5}$$

is the reproduction ratio of the economy.

The previous definition[3] is suggested from the fact that, provided $\alpha + n_s > 0$, the growth of the economy is possible IFF $R_E > 1$. The interpretation of R_E is the following (let us assume

[2]Far from being a trivial fact, the stability of the "zero" equilibrium corresponds to a situation in which accumulation is too weak to permit the birth of a "structured" economic activity, as stated by a labour market plus a production structure.

[3]In demographic analysis the net reproduction ratio (NRR) R_0 is a commonly used index (having genetic and epidemiological counterparts) of the reproductive ability of a population, based on the ratio between the number of newborn individuals in two subsequent generations. Under constant conditions it predicts the growth of the population when $R_0 > 1$ and its decay in the opposite case.

for simplicity $\alpha = 0$): if the supply of labour is steadily growing at the rate n_s, in order to guarantee the growth of the economy, the accumulation conditions (m) must be able to provide more than one additional job place for every new worker entering the labour market at least in the optimal condition in which the entire product is distributed to profit. This definition appears quite natural in Goodwin-type economies in which all the profit is reinvested in new labour, and will be used in our subsequent investigation in this paper.

2.2 A general demoeconomic framework

We now introduce our general Goodwin-type demo-economic framework (see also Manfredi and Fanti (1999a,b)). This framework aims to be a first step toward realistic formulations of the labour market. Compared to standard formulations of Goodwin's model we have added a detailed, but parsimonious, representation of the process of formation of the supply of labour. The supply of labour at time t, $N_s(t)$, has to be defined as the product between the total number of individuals in the working age span $N(t)$, and their participation rate $s(t)$: $N_s(t) = s(t)N(t)$. The rate of change of the supply of labour then satisfies [4]:

$$n_s = \frac{\dot{N}_s(t)}{N_s(t)} = \frac{\dot{s}(t)}{s(t)} + \frac{\dot{N}(t)}{N(t)} = \frac{\dot{s}(t)}{s(t)} + n \tag{6}$$

where n denotes the rate of change of the population in the working age span. For what concerns the participation term a general form is the following:

$$\frac{\dot{s}(t)}{s(t)} = q(w, U) \tag{7}$$

with: i)$\partial q/\partial w > 0$ (this amounts to postulate that the relation between the labour supply and the wage is not backward-bending) and: ii)$\partial q/\partial U > 0$, which is based on the discouraged worker hypothesis (Mincer 1966). In this work we will consider only linear participation:

$$\frac{\dot{s}\ (t)}{s(t)} = -\delta + q_1 U + q_2 w \quad q_1 > 0, q_2 > 0 \tag{8}$$

2.2.1 Modelling the dynamics of the population in the working age span

The modellisation of the rate of change of the population in the working age span $n(t) = \dot{N}(t)/N(t)$ represents a distinct purpose of this paper. The whole population process has been modelled by means of a simplified three-stage structure aimed at representing the most relevant

[4]The relation (6) permits to fully evaluate the lack of realism of the assumption of a constant rate of change of the supply of labour, which is common not only to the Goodwin's literature but also to the standard (descriptive) neoclassical growth model (Solow (1956)). Such an assumption implies both: i)a constant rate of change of the population in the working age span, and ii)a constant participation rate.

stages of the individual life-cycle: pre-work ages ("young"), working age span ("adult"), and retirement. Let us call them stage 1,2,3. We assume that only adult individuals (stage 2) do reproduce. Table 1 reports the definitions for the involved demographic parameters .

Label	Description
μ_1, μ_2, μ_3	mortality rates in stages 1,2,3
v_1	rate of transition from stage 1 to stage 2
	(rate of transition to adulthood)
v_2	rate of transition from stage 2 to stage 3
	(rate of retirement)
b	rate of fertility of adults

Table 1 Demographic parameters employed in the model

Let us now derive here our main demographic relations, which are based on an age-stages formulation already used in mathematical biology (Li and Hallam (1988)). Under the assumption of constant coefficients, the general partial differential equation describing the dynamics of an age structured population is reducible to an ordinary differential equations (ODE's) formulation. This latter is able, in many cases, to capture in a sufficiently realistic way the main effects of age-structure whilst mantaining at the same time simplicity of treatment. Feichtinger and Sorger (1989) used the same ODE system but without explicit reference to the underlying PDE for age structure, and in any case they did not derive any equation resembling our equation (25). The present analysis clarifies in detail the limitations of the approach used by Feichtinger and Sorger (1989).

Let $N(a,t)$ be the age-time density of a given population, defining the density of individuals aged a at time t. The dynamics of a (one-sex) age-structured population is governed by its Von Foerster partial differential equation (Keyfitz 1985):

$$\left[\frac{\partial}{\partial a} + \frac{\partial}{\partial t}\right] N(a,t) = -\mu(a,t)N(a,t) \tag{9}$$

plus the usual boundary and initial conditions:

$$N(0,t) = B(t) \; ; \; N(a,0) = \varphi(a) \tag{10}$$

where $\mu(a,t)$ is the death rate at age a, $B(t)$ the birth function at time t, and $\varphi(a)$ is a prescribed function of age assigning the age density at time zero. To derive from (9)-(10) an ODE formulation let us now subdivide our population into the following three broad age classes: $(0, A_1)$=pre-work age (for instance $A_1 = 15$ years), (A_1, A_2)=working age span (for instance: $(15, 65)$), (A_2, ∞)=retirement ages, and let the functions $P_1(t), P_2(t), P_3(t)$ respectively denote the number of individuals (young, adult and retired) in the three groups at time t.[5] By perform-

[5]We could of course introduce an arbitrary number of stages (and this could of interest from other points of view) but this is the simplest reasonable choice.

ing separate integrations of the basic PDE over the three age groups, and by remembering that, by definition: $P(0,t) = B(t)$, $P(\infty,t) = 0$, we get the following relations:

$$\begin{aligned} \dot{P}_1(t) &= B(t) - D_1(t) - P(A_1,t) \\ \dot{P}_2(t) &= [P(A_1,t) - P(A_2,t)] - D_2(t) \\ \dot{P}_3(t) &= P(A_2,t) - D_3(t) \end{aligned} \tag{11}$$

where the $D_i(t)$ terms denote the number of deaths within each class at time t, while $P(0,t)$, $P(A_1,t), P(A_2,t)$ are the flows from one class to the next one due to the aging process. The interpretation of (11) is straightforward. For instance the dynamics of the number $P_2(t)$ of individuals in the working age span is the outcome of the balance between the entries from the pre-work class $(P(A_1,t))$ and the exits due to aging $(P(A_2,t))$ and mortality $(D_2(t))$. Notice that we may write:

$$D_i\,(t) = \bar{\mu}_i\,(t)\,P_i\,(t) \tag{12}$$

where $\bar{\mu}_i\ (t)$ is the average death rate in the class (which is usually time-varying unless the population is in a stable state). By assuming that i) the death rate is constant in each class ($\bar{\mu}_i\ (t) = \mu_i$) we get $D_i(t) = \mu_i P_i\,(t)$. Let us now consider the quantities $P(A_1,t), P\,(A_2,t)$ describing transitions from one age-class to the next one. Clearly:

$$P(A_1,t) = B\,(t - A_1)\,p\,(A_1) \quad P(A_2,t) = B\,(t - A_2)\,p\,(A_2) \tag{13}$$

where $p\,(a)$, is the survival probability up to age A_1. Let us now assume that:

$$B(t) = b\,(t)\,P_2(t) \tag{14}$$

i.e that all births take place in the adult population, where b is the fertility rate of the adult population, assumed age-independent. In this case:

$$P(A_1,t) = b\,(t - A_1)\,P_2(t - A_1)p\,(A_1) \tag{15}$$

Hence:

$$\dot{P}_2(t) = b\,(t - A_1)\,[P_2(t - A_1)p\,(A_1) - P_2(t - A_2)p\,(A_2))] - \mu_2 P_2(t) \tag{16}$$

The last relation provides the exact representation of the dynamics of the population in the working age span. In other words: even if we assume that fertility is age-independent the mathematical description of population dynamics within macroeconomic models needs differential-difference equations, the tractability of which, once embedded within a macro-economic model, may be quite limited.[6] A strong simplification comes about if we assume that: $P(A,t) = v_1 P_1(t)$, $P(B,t) = v_2 P_2(t)$. In this case (11) collapses into the ODE system:

$$\begin{aligned} \dot{P}_1(t) &= bP\,(t) - (\mu_1 + v_1)\,P_1(t) \\ \dot{P}_2(t) &= v_1 P_1 - (\mu_2 + v_2)\,P_2(t) \\ \dot{P}_3(t) &= v_2 P_2(t) - \mu_3 P_3(t) \end{aligned} \tag{17}$$

[6]The analysis of population frameworks as (16) within macro-economic models is a future step of our research agenda.

where $\mu_i (i = 1, 2, 3)$ are the death rates and $v_i (i = 1, 2)$ are the transition rates between the three classes. Both types of rates will be assumed as constant since now on.

Some remarks are useful. The system (17) provides, being derived under several assumptions, a necessarily simplified view of the demographic reproduction process. Fertility is age-independent. Moreover, by working on broad age stages its description can not give information on facts concerning smaller time scales, i.e. it disregards age structure inside $(0, A)$ and (A, B). On the other side, it has the advantage of providing a hopefully tractable demographic framework for most of the basic macro-dynamical models, which are expressed in terms of ODE. As stated by Feichtinger and Sorger (1989,279): *".. the approach may not be suited for quantitative calculations but it has avdantages if one wants to get insights into the qualitative population dynamics."*

Let us now go one step further and introduce explicitly (17) within the Goodwin's basic Goodwin's fomulation. This leads to the following 5-dimensional system in the (w, U) variables:

$$\begin{aligned} \dot{w}(t) &= w(t)\left(-\gamma + \rho U\right) \\ \dot{U}(t) &= U(t)\left(m(1-w) - \tfrac{\dot{s}(t)}{s(t)} - n\right) \\ \dot{P}_1(t) &= B(t) - (\mu_1 + v_1)\,P_1(t) \\ \dot{P}_2(t) &= v_1 P_1(t) - (\mu_2 + v_2)\,P_2(t) \\ \dot{P}_3(t) &= v_2 P_2(t) - \mu_3 P_3(t) \end{aligned} \tag{18}$$

In order to concentrate on the pure effects of the labour market, in (18) we set $\alpha = 0$. This implies that productivity is assumed to be constant over time at a prescribed value A_0 (for simplicity we have rescaled A_0 to one). This permits to work on the wage variable (w), rather than on the wage share (V) used in (1).

The system (18) can be somewhat simplified. Notice first that the equation of the retired population does not provide any input to the system and can be neglected, thereby reducing one dimension. Moreover, the rate of change of the adult population $n = \dot{P}_2/P_2$ satisfies:

$$\frac{\dot{P}_2}{P_2} = v_1 \frac{P_1}{P_2} - (\mu_2 + v_2) \tag{19}$$

The last equation shows that the rate of change of the labour force essentially depends on the ratio between young and adults prevailing in the population. It is possible to reduce the system one further dimension by introducing the new variable $Z = P_1/P_2$. As: $\dot{Z}= Z\left(\frac{\dot{P}_1}{P_1} - \frac{\dot{P}_2}{P_2}\right)$ we have:

$$\dot{Z} = Z\left[\frac{b}{Z} - (\mu_1 + v_1) - v_1 Z + (\mu_2 + v_2)\right]$$

leading to:

$$\dot{Z}= b - HZ - v_1 Z^2 \tag{20}$$

where:

$$H = (\mu_1 + v_1) - (\mu_2 + v_2) \gtreqless 0 \tag{21}$$

Hence (18) has been reduced to the 3-dimensional system:

$$\begin{aligned} \dot{w} &= w\left(-\gamma+\rho U\right) \\ \dot{U} &= U\left[m(1-w)-\tfrac{\dot{s}(t)}{s(t)}-\left(v_1 Z-\left(\mu_2+v_2\right)\right)\right] \\ \dot{Z} &= b-HZ-v_1 Z^2 \end{aligned} \tag{22}$$

The system (22) is the minimal modelling structure for the investigation of demoeconomic interactions within Goodwin's model. The demographic subsystem appears through the ratio $Z=Z(t)$ between the number of young and adult individuals in the population.

An alternative formulation is obtained by working on the rate of change $n(t)=\dot{P}_2/P_2$ of the population in the working age class (instead of Z). As:

$$n(t)=\frac{\dot{P}_2}{P_2}=v_1 Z(t)-(\mu_2+v_2) \tag{23}$$

then:

$$v_1 Z(t)=(\mu_2+v_2)+n(t)\ \ ;\ \ \dot{n}(t)=v_1\,\dot{Z}\,(t) \tag{24}$$

We therefore get:

$$\dot{n}(t)=(-1)\left[n^2+(H+2(\mu_2+v_2))\,n-\left(v_1 b-H(\mu_2+v_2)-(\mu_2+v_2)^2\right)\right]$$

This leads to the dynamical equation:

$$\dot{n}(t)=(-1)\left[n^2+Pn-B\right] \tag{25}$$

where:

$$\begin{aligned} P &= (\mu_1+v_1)+(\mu_2+v_2)>0 \\ B &= v_1 b-Q=Q\,(R_0-1) \\ Q &= (\mu_2+v_2)(\mu_1+v_1) \end{aligned} \tag{26}$$

In particular the quantity:

$$R_0=\frac{v_1 b}{Q}=\frac{v_1 b}{(\mu_1+v_1)\,(\mu_2+v_2)} \tag{27}$$

is the net reproduction ratio of the population in our three stages demographic system (Manfredi and Fanti 1999a).

By adopting the formulation (25)-(26), and by using (8) we arrive to the following Goodwin-type demo-economic model:

$$\begin{aligned} \dot{w} &= w\left[-\gamma+\rho U\right] \\ \dot{U} &= U\left[m(1-w)-(-\delta+q_1 U+q_2 w)-n\right] \\ \dot{n} &= (-1)\left[n^2+Pn-B(w)\right] \end{aligned} \tag{28}$$

In (28) the economic subsystem is influenced by the demographic one via the labour supply term : $n_s = \frac{\dot{s}}{s} + n$, which is affected by i) the changes in the population in the working age span: $n = \dot{P}_2/P_2$, and by ii) the changes in the participation rate $s(t)$. Viceversa there can be several inputs from the economic system to the demographic subsystem: virtually all the demographic parameters are influenced by the living conditions and other economic factors. For instance, following the malthusian tradition, both the fertility and the mortality rates should be modelled as functions of the living conditions: $b = b(w), \mu_i = \mu_i(w)$. Similarly the rates of transition to adulthood (v_1) and retirement (v_2) are probably influenced by the state of labour market and so on.

We will call (28) a "mechanistic" model, in that it embodies age structure without intentional approximations. The model (28) represents a flexible framework for the investigation of the role of population dynamics and participation in demoeconomic interactions. Here we will use it in order to investigate the joint effects of population dynamics and participation under a malthusian fertility function: $b = b(w)$. Of course the potential applications of the demographic framework introduced here go well beyond Goodwin-type economies. For instance we already applied it to the study of demoeconomic dynamics in the Solow's neoclassical growth model.

3 Analysis of the mechanistic model

The model (28) is mathematically well-posed and admits a unique solution for every initial condition. Moreover the variables U, w preserve positivity. To avoid the risk of being overburdened with non-essential details, let us simplify the formulation of the participation function, by writing simply $\dot{s}/s = qU$, $q > 0$. This is motivated by the fact that the general formulation (8): $q(w,U) = -\delta + q_1 U + q_2 w$ does not add, in the present case, further substantive dynamical information.[7] This leads to the model:

$$\begin{aligned} \dot{w} &= w\left(-\gamma + \rho U\right) = w f_1(U) \\ \dot{U} &= U\left[m(1-w) - qU - n\right] = U f_2(w, U, n) \\ \dot{n} &= (-1)\left[n^2 + Pn - B(w)\right] \end{aligned} \tag{29}$$

where as usual:

$$B = v_1 b(w) - Q \ ; \ Q = (\mu_2 + v_2)(\mu_1 + v_1) \tag{30}$$

It is useful to define:

$$B = Q\left(R_0(w) - 1\right) \tag{31}$$

where:

$$R_0(w) = \frac{v_1 b(w)}{(\mu_1 + v_1)(\mu_2 + v_2)} \tag{32}$$

[7]The equation $\dot{s}/s = qU$ has the drawback that when the employment is in equilibrium the participation rate should increase exponentially and so reach its upper bound. Our formulation is to be regarded just as a starting point.

is the wage-related net reproduction ratio (NRR) of the population. $R_0(w)$ associates a value of the reproduction ratio of the population to each level of the wage observable in the economy. In particular: $R_0(0) = v_1 b(0)/(\mu_1 + v_1)(\mu_2 + v_2)$ is the NRR at zero wage, which assesses the ability of the population to grow in absence of a "structured" economic activity. As already stated we consider only classical "standard" fertility: the fertility rate is a monotonically increasing and possibly saturating function of the wage: $b = b(w)$, $b(0) \geq 0, b'(w) \geq 0$. We will sometimes write $b(w)$ in the form: $b(w) = b_0 + b_1(w)$, $b_0 \geq 0, b_1(0) = 0$. We notice that all the results of this section are preserved under the simplifying assumption of linear fertility: $b(w) = b_0 + b_1 w \;\; b_0 \geq 0, b_1 > 0$.

Manfredi and Fanti (1999a) have studied (??) under the sole effects of wage dependent fertility (in absence of participation effects) and have shown that the original Goodwin's model is always destabilised[8] when fertility has a "classical" standard form, whereas more complex facts occur when fertility exhibits "postclassical" features such as downturns at high wage levels. Manfredi and Fanti (1999b) have considered the effects of heterogeneity in fertility behaviour of employed and unemployed individuals in presence of a general Phillips curve embedding "insider-outsider" behaviour. In this last case they also found chaotic oscillations.

3.1 Equilibria and their local stability analysis

The discussion on existence and meaningfulness of equilibria is quite involved, and we relegate the details to appendix 1. As $\dot{w} = \dot{U} = 0$ for $U = w = 0$, the system (29) always has an equilibrium (let us denote it as E_0) with zero wage and employment and a nonzero rate of change of the working population. This latter is a solution of the equation:

$$n^2 + Pn - B(0) = 0 \tag{33}$$

The last equation has only one nontrivial solution (i.e. suitable to represent a rate of change), the positive one namely, given by[9]:

$$n_0 = \frac{1}{2}\left(-P + \sqrt{P^2 + 4B(0)}\right) \tag{34}$$

Hence $n_0 \gtrless 0$ depending on whether $B(0) \gtrless 0$. This implies: $R_0(0) \gtrless 1$ where $R_0(0)$ is the aforementioned NRR at zero wage. The jacobian evaluated at E_0 is:

$$J(E_0) = \begin{pmatrix} -\gamma & 0 & 0 \\ 0 & m - n_0 & 0 \\ b'(0) & 0 & -(2n_0 + P) \end{pmatrix}$$

[8] This fact may be appreciated by observing that when a "trivial" population dynamics mechanism, based on a relation between the current fertility and the current wage, is considered, then Goodwin's conservative oscillations are preserved.

[9] The condition: $\Delta = P^2 + 4B(0) > 0$ needed to ensure that the involved root be real is always satisfied. In fact:

$$\Delta = P^2 + 4B(0) = (P^2 - 4Q) + 4v_1 b(0) = ((\mu_1 + v_1) - (\mu_2 + v_2))^2 + 4v_1 b(0) > 0$$

with eigenvalues: $\lambda_1 = -\gamma$; $\lambda_2 = m - n_0$; $\lambda_3 = -(2n_0 + P)$. Notice that: $-(2n_0 + P) = -\sqrt{P^2 + 4B(0)} < 0$. Hence, when $R_0(0) < 1$, as $n_0 < 0$, then E_0 is always locally unstable (a saddle point). In the more interesting case $n_0 > 0$, i.e. $R_0(0) > 1$ then E_0 is locally asymptotically stable (LAS) as long as $m < n_0$, and it becomes locally unstable when $m > n_0$.

REMARK 1. *The last result may be put in the form introduced in section 2. By defining the reproduction ratio of the economy at zero wage as the quantity:*

$$R_E(0) = \frac{m}{n_0} \tag{35}$$

we see that E_0 becomes locally unstable when $R_E(0) > 1$.

In words: economic development, as synthesized by the instability of E_0, becomes possible when accumulation is sufficiently fast and/or the rate of growth of population is not so large to overtake accumulation. This generalizes Goodwin's basic threshold result.

Moreover there may be an equilibrium with zero wage and positive employment $E_2 = (0, U_2, n_2)$ and a nonzero equilibrium $E_1 = (w_1, U_1, n_1)$. The E_2 equilibrium is found by putting $w = 0$ in the second and third equation leading to the system:

$$m - qU - n = 0 \qquad n^2 + Pn - B(0) = 0 \tag{36}$$

The rate of growth of the population is hence determined from the second equation (36) and it is obviously given by $n_2 = n_0$. Moreover:

$$U_2 = \frac{m - n_0}{q} \tag{37}$$

Hence when $n_0 > 0$ then U_2 is positive for: $m - n_0 > 0$, i.e. $R_E(0) > 1$. The condition $m - n_0 < q$ ensures economic meaningfulness. The jacobian evaluated at E_2 gives:

$$J(E_2) = \begin{pmatrix} -\gamma + \rho U_2 & 0 & 0 \\ -mU_2 & -qU_2 & -U_2 \\ B'(0) & 0 & -(2n_2 + P) \end{pmatrix}$$

with eigenvalues:

$$\lambda_1 = -\gamma \ + \rho U_2; \ \lambda_2 = -qU_2 \ ; \ \ \lambda_3 = -(2n_2 + P) \tag{38}$$

Therefore E_2 is a saddle point or a stable node depending on whether:

$$-\gamma + \rho U_2 \gtrless 0 \ \rightarrow \ \rho\,(U_2 - U_1) \gtrless 0 \tag{39}$$

where $U_1 = \gamma/\rho$ is the employment value at the E_1 equilibrium. Hence, if E_2 coexists with the nonzero equilibrium E_1, then E_2 is unstable, while it is LAS when it exists without E_1.

Finally, for what concerns the nonzero equilibrium $E_1 = (w_1, U_1, n_1)$, from the wage equation we find the equilibrium value of employment: $U_1 = \gamma/\rho$. We hence get the following sub-system in the two variables (w, n):

$$\begin{aligned} &m(1 - w) - qU_1 - n = 0 \\ &n^2 + Pn - B(w) = 0 \end{aligned} \tag{40}$$

the solutions of which are the intersections of the lines: $n_A(w) = m - qU_1 - mw$ and $n_B(w) = \frac{1}{2}\left(-P + \sqrt{P^2 + 4B(w)}\right)$. The existence of E_1 again depends on the mutual position of the intercepts $n_A(0)$, $n_B(0)$. It is easy to see that if $n_B(0) > n_A(0)$ the system can never admit nonzero equilibria. In the opposite case, $n_B(0) < n_A(0)$, the system always admits a unique nonzero equilibrium E_1 (details on the full meaningfulness of E_1 are given in appendix 1). This result can be expressed again in terms of suitable reproduction rates of the economy. Provided $qU_1 + n(0) > 0$ the correct threshold will be: $R_E(w = 0; U = U_1) = m/qU_1 + n(0)$.

REMARK 2. *The equilibrium value of the rate of growth of the population at E_1 may be positive or negative. It will be certainly positive when $n_B(0) > 0$ ($R_0(0) > 1$), and in the event $n_B(0) < 0$ it will be positive as long as: $n_B(\frac{m-qU_1}{m}) > 0$. It will be negative otherwise.*

The local stability analysis of E_1 leads to the jacobian:

$$J(E_1) = \begin{pmatrix} 0 & \rho w_1 & 0 \\ -mU_1 & -qU_1 & -U_1 \\ B'(w_1) & 0 & -(2n_1 + P) \end{pmatrix}$$

The characteristic polynomial $K^3 + AK^2 + BK + C = 0$ has the coefficients:

$$\begin{aligned} A &= X + qU \\ B &= U(qX + m\rho w) \\ C &= U(mX + B'(w))\rho w \end{aligned}$$

where we defined: $2n + P = X$, and we suppressed the suffix, by writing w instead of w_1 and so on. As all the coefficients are strictly positive, the Routh-Hurwicz stability test gives:

$$qX^2 + q^2UX + qm\rho Uw - B'(w)\rho w > 0 \tag{41}$$

Unfortunately, a full analysis of the stability condition (41) is quite difficult, both in the general case of a general fertility function $b(w)$, where equilibria can not be determined explicitly, but also in the case of linear fertility. The stability properties of the model (29) will therefore be investigated numerically (see section five).

4 An alternative formulation based on time-lags: fertility depending on the lagged wage

It is of interest to compare the results of the model developed in the previous sections with a simplified formulation in which the age structure process is embedded via a time delay in the rate of change of the supply of labour. As well recognised in the specialised literature, time delays represent a powerful strategy in order to provide simplified, and hence more tractable, representations of age structure processes (McDonald 1978). In rough terms we may say that the rate of change of the population in the working age span n is a function of past fertility, which in turn is a function of the past levels of the wage (see also Fanti and Manfredi (1998a)). We can therefore reformulate our demo-economic model in the following way:

$$\begin{array}{l} \frac{\dot{w}(t)}{w(t)} = -\gamma + \rho U \\ \frac{\dot{U}(t)}{U(t)} = m(1-w) - qU - \left(\int_{-\infty}^{t} b(w(\tau))G(t-\tau)d\tau - \mu\right) \end{array} \tag{42}$$

where $b(w(\tau))$ denotes past levels of the wage-related fertility rate, G is the delaying kernel, and μ denotes the total exit rate in the adult ages (it may be thought as the sum of the mortality rate plus the retirement rate of the general model). The previous system of nonlinear integro-differential equations (IDE's) is difficult to manage analytically but useful information can be obtained by resorting to the case of linear fertility. This leads to:

$$\begin{array}{l} \dot{w}(t) = w(t)\left(-\gamma + \rho U\right) \\ \dot{U}(t) = U(t)\left[m(1-w) - qU - \left(b_0 + b_1 \int_{-\infty}^{t} w(\tau)G(t-\tau)d\tau - \mu\right)\right] \end{array} \tag{43}$$

In what follows we adopt the assumption of an exponentially fading memory, i.e. we assume that the delaying kernel G is of the exponential type:

$$G(x) = ae^{-ax} \quad x > 0, a > 0 \tag{44}$$

This assumption is in some sense coherent with the general model, where transition to the adult state was regulated by constant transition rates, i.e. by an exponential survival mechanism. Moreover it permits us to reduce the IDE system (43) to a 3-dimensional dynamical system. This is done by introducing the new variable:

$$S(t) = \int_{-\infty}^{t} w(\tau)G(t-\tau)d\tau \tag{45}$$

(which defines the "average wage over the past") and formally applying the linear chain trick (McDonald 1978). This leads to the system[10]:

$$\begin{array}{l} \dot{w} = w\left(-\gamma + \rho U\right) \\ \dot{U} = U\left[m(1-w) - qU - (b_0 + b_1 S - \mu)\right] \\ \dot{S} = a(w - S) \end{array} \tag{46}$$

where the quantity $qU + (b_0 + b_1 S - \mu)$ denotes the total rate of change of the supply of labour.

[10] Some further conditions are needed to ensure the equivalency between the initial conditions of the ODE system (46) and the initial "functions" of the IDE system (43).

4.1 Equilibria

The equilibrium properties of the model (46) (details in appendix 3) are analogous to those of the general mechanistic model. The model (46) always has the zero equilibrium $E_0 = (0,0,0)$. Moreover, for $w = 0$ ($S = 0$), provided $m - n_0 > 0$, we have an axis equilibrium $E_2 = (0, U_2, 0)$ with:

$$U_2 = \frac{m + \mu - b_0}{q} = \frac{m - n_0}{q} \tag{47}$$

which is economically meaningful provided that: $0 < m - n_0 < q$. Finally, remembering that $0 \leq w \leq 1$, positive equilibria of the wage may exist only if the total rate of change of the supply of labour at equilibrium is strictly positive at least for some w. By denoting $U_1 = \gamma/\rho$, positive equilibrium values of the wage are solutions of the equation: $m - n_0 - qU_1 - (m + b_1)\,w = 0$. Hence, provided that $m - n_0 - qU_1 > 0$, then a positive equilibrium (E_1) with coordinates:

$$w_1 = \tfrac{m-n_0-qU_1}{m+b_1} \quad U_1 = \gamma/\rho \quad S_1 = w_1 \tag{48}$$

exists. The condition $m - n_0 - qU_1 > 0$ indicates that even if accumulation is sufficiently high so as to "absorb" population growth ($m - n_0 > 0$), the existence of an equilibrium with a positive wage can be prevented by a too strong participation effect and/or a too high employment level. By writing the last condition in terms of q we have: $q < (m - n_0)/U_1 = q_1$, meaning that a too large participation rate would, coeteris paribus, decrease the rate of growth of the employment, thereby making the employment itself unable to effectively sustain the wage. Notice finally that E_1 will be economically meaningful provided:

$$\frac{m - n_0 - qU_1}{m + b_1} < 1$$

This is always true (provided $m - n_0 - qU_1 > 0$ still holds) if $n_0 > 0$. In the opposite case the further condition:

$$b_1 + b_0 - \mu > -qU_1 \;\rightarrow\; n_1 + qU_1 > 0 \tag{49}$$

has to be imposed. The condition (49) states that, as previously pointed out, in order to have a fully economically meaningful equilibrium value of the wage, the total rate of change $qU + n$ of the supply of labour must be strictly positive at least when the wages are set up at their maximal values.

4.2 Local stability analysis of the equilibria

At the zero equilibrium we have the jacobian:

$$J(E_0) = \begin{bmatrix} -\gamma & 0 & 0 \\ 0 & m - n_0 & 0 \\ a & 0 & -a \end{bmatrix}$$

with eigenvalues $(-\gamma, m - n_0, -a)$. Hence, as already pointed out the zero equilibrium is locally unstable only when $m - n_0 > 0$, i.e. provided that $R_E(0) > 1$. For what concerns the axis equilibrium we have:

$$J(E_2) = \begin{bmatrix} -\gamma + \rho U_2 & 0 & 0 \\ -mU_2 & -qU_2 & -b_1 U_2 \\ a & 0 & -a \end{bmatrix}$$

The characteristic polynomial has the eigenvalues: $(-\gamma + \rho U_2, -qU_2, -a)$. Hence E_2 will be LAS or not depending on whether $-\gamma + \rho U_2$ is negative or not. But:

$$-\gamma + \rho U_2 = \rho (U_2 - U_1) = \frac{\rho}{q} (m - n_0 - qU_1)$$

Hence: $-\gamma + \rho U_2 < 0$ for $m - n_0 - qU_1 < 0$: this shows that E_2 will be LAS only in absence of E_1. Otherwise, when E_1 exists, E_2 will always be unstable (independently on the stability properties of E_1 itself).[11]

Finally, the local stability analysis around E_1 gives the jacobian (writing for simplicity U, w instead of U_1, w_1):

$$J(E_1) = \begin{bmatrix} 0 & \rho w & 0 \\ -mU & -qU & -b_1 U \\ a & 0 & -a \end{bmatrix} \tag{50}$$

The corresponding characteristic polynomial $P(K) = K^3 + A_1 K^2 + A_2 K + A_3 C = 0$ has the coefficients:

$$A_1 = a + qU \; ; \; A_2 = aqU + m\rho U w \; ; \; A_3 = a\rho U w \, (m + b_1)$$

As all the coefficients of the characteristic equation are strictly positive, the Routh-Hurwicz test for local stability analysis only needs to check the sign of the quantity: $\Delta_2 = A_1 A_2 - A_3 > 0$. As our focus is on the role of the extra parameters a, q introduced via our formulation embedding population dynamics and participation, let us write such a condition as:

$$f_q(a) = A(q)a^2 + B(q)a + C(q) > 0 \tag{51}$$

where:

$$A(q) = q \;\; ; \; B(q) = q^2 U - \rho w b_1 \;\; ; \;\; C(q) = qm\rho U w$$

The set of parameter constellations satisfying:

$$f_q(a) = A(q)a^2 + B(q)a + C(q) = 0 \tag{52}$$

[11]Despite the richness of economic interpretation of the overall process of birth and death of equilibria in the model, we will not pay anymore attention to the the equilibria E_0, E_2 since the main object of this work is the investigation of the existence of sustained oscillations via Hopf bifurcations. From this point of view the only relevant candidate for subsequent analysis is the positive equilibrium E_1. In fact, since the system preserves positivity no meaningful oscillations are possible around E_0, E_2.

defines the stability boundary of E_1, which separates (locally) stable from unstable behaviours. As we will see later on, on the set $f_q(a) = 0$ a Hopf bifurcation occurs.

The stability parabola $f_q(a)$ defined by (51) is convex and has a positive intercept. When the abscissa of its vertex is negative $f_q(a)$ will be strictly positive for every positive value of a and no loss of stability may occur. This happens when: $B(q) = q^2 U - \rho w b_1 > 0$, i.e., by using (48), when:

$$\Lambda(q) = (m + b_1)\, U q^2 + \rho b_1 U q - \rho b_1 (m - n_0) > 0 \tag{53}$$

Remembering that $m - n_0 > 0$ (necessary for the existence of E_1), the parabola $\Lambda(q)$ is convex, with a negative vertex abscissa and a negative intercept. This implies that $B(q) > 0$ for $q > q^*$[12], where q^* is the unique positive solution of the equation: $\Lambda(q) = 0$. Hence the following holds:

PROPOSITION 1. *The positive equilibrium E_1 can never be destabilised when the participation rate exceeds the threshold value q^*.*

The last proposition is in agreement with the fact that a very large q causes a very strong stabilizing pressure, therefore preventing possible instability patterns[13].

In the opposite case $B(q) < 0$, i.e. for $0 < q < q^*$, the parabola $f_q(a)$ has a positive vertex abscissa, and therefore it may become negative for positive values of a. If its discriminant:

$$\Delta = q^4 U^2 - 2q^2 U \rho w b_1 + \rho^2 w^2 b_1^2 - 4q^2 m \rho U w \tag{54}$$

is negative bifurcations are necessarily ruled out. Viceversa if $\Delta \geq 0$ bifurcations may occur. Let us study the discriminant Δ as an explicit function of q, $\Delta(q)$, where:

$$\Delta(q) = U^2 q^4 - 2\gamma\,(b_1 + 2m)\, w q^2 + \rho^2 b_1^2 w^2 \tag{55}$$

By substituting in (55) the equilibrium value of the wage: $w = w_1 = (m - n_0 - qU_1)\,/\,(m + b_1)$ we find:

$$\Delta(q) = U^2 q^4 - 2\gamma\,(b_1 + 2m)\,(C - Dq)\, q^2 + \rho^2 b_1^2\,(C - Dq)^2 = 0$$

By expanding the last expression we arrive to the following quartic equation in q:

$$A_0 q^4 + A_1 q^3 + A_2 q^2 + A_3 q + A_4 = 0 \tag{56}$$

[12]The threshold q^* satisfies: $q^* < q_1$. In fact:

$$q^* = \frac{-b_1\gamma + \sqrt{b_1^2\gamma^2 + 4\,(m + b_1)\, b_1 \gamma (m - n_0)}}{2\,(m + b_1)\, U_1}$$

A direct comparison between q^* and q_1 quickly proves the statement.

[13]In addition, when $B(q) = 0$, i.e. $q = q^*$, we have:

$$f_q(a) = qa^2 + qm\rho U w$$

which is always positive. This shows that no bifurcation may occur for $q \geq q^*$.

where:

$$\begin{aligned} A_0 &= U^2 > 0\ ;\ A_1 = 2D\gamma\,(b_1+2m) > 0\ ; \\ A_2 &= D^2\rho^2 b_1^2 - 2C\gamma\,(b_1+2m)\ \ ;\ A_3 = -2CD\rho^2 b_1^2 < 0 \\ A_4 &= \rho^2 b_1^2 C^2 > 0 \\ C &= \frac{m-n_0}{m+b_1}\quad ;\quad D = \frac{U_1}{m+b_1} \end{aligned}$$

The quartic (56) has to be studied for $0 < q < q^*$. The only coefficient whose sign is umbiguous is A_2.[14] The following lemma holds:

LEMMA 2. *There exists a unique q^{**} strictly smaller than q^* such that the discriminant $\Delta(q)$ is strictly positive for $0 < q < q^{**}$ and strictly negative for $q^{**} < q < q^*$.*

Proof. For the equation (56) the two following sequences of signs of coefficients are possible: $(++--+)$ and $(+++-+)$, both admitting two sign changes. Therefore, from Descartes rule of coefficients the following facts holds: i) there are at most two real positive roots, ii)there are either two or zero real positive roots. Let us now observe that, as $B(q^*)=0$, it follows:

$$\Delta(q^*) = B^2(q^*) - 4A(q^*)C(q^*) = -4A(q^*)C(q^*) < 0$$

Hence $\Delta(q)$ is strictly positive at $q=0$, while it is strictly negative at $q=q^*$. This implies that $\Delta(q)$ has at least one real positive root in the interval $(0,q^*)$ which will be denoted as q^{**}. From Descartes rule, as $\Delta(q)$ has at least one real positive root, it must necessarily have two real positive roots. Let us denote as q_s the second of these roots. Clearly, due to the fact that $\Delta(q)$ will eventually be positive again $(\Delta(+\infty)=+\infty)$, q_s necessarily lies in the set $(q^*,+\infty)$. Definitively, $\Delta(q)$ is strictly positive for $(0,q^{**})$, it changes sign at q^{**} and remains negative in the whole interval (q^{**},q^*) (end proof).

The previous lemma implies that the parabola $f_q(a)$ becomes negative (i.e. Hopf bifurcations are possible), only in the interval $(0,q^{**})$.

In particular, for $0 < q < q^{**}$, the stability parabola has two real solution, both of which are admissible, i.e. we have two admissible bifurcating values of the a parameter given by:

$$a_{1,2} = \frac{1}{2q}\left(-q^2U + \rho w b_1 \mp \sqrt{\Delta}\right) \tag{57}$$

We are now in the position to state our main result concerning the dynamical properties of the system (29):

[14] As:

$$A_2 = \frac{\gamma}{(m+b_1)^2}\left[\gamma b_1^2 - 2\,(m-n_0)\,(m+b_1)\,(2m+b_1)\right]$$

from the bracketed term we see that A_2 will be negative for small values of γ, but may well become positive for large values of γ.

PROPOSITION 3. *When the participation rate q exceeds a prescribed threshold ($q > q^{**}$) the positive equilibrium E_1 of the system (46) is LAS independently on the delay. When $0 < q < q^{**}$ the system (46) continues to converge to the locally stable equilibrium E_1 only when the delay parameter a is sufficiently large or sufficiently small, i.e. for $a > a_2$ and $a < a_1$. In the whole window $a_1 < a < a_2$ the equilibrium E_1 is locally unstable. At the points $a = a_1, a = a_2$ Hopf bifurcations occur.*

Proof: the part of the proof of our main proposition concerning the local stability/instability of E_1 is evident from our previous discussion. In particular, to formally prove that at the points $a = a_1, a = a_2$, where stability is lost, a Hopf bifurcation occurs, we need to show that: i)purely imaginay eigenvalues exist for the linearised system at $a = a_1, a = a_2$ due to a "continuous" movement of a pair of complex eigenvalues; ii)the crossing of the imaginary axis by the involved complex pair occurs with nonzero speed. The proof of i) is evident, see for instance Liu (1994) or Fanti and Manfredi (1998b). To show that the crossing of the imaginary axis by the bifurcating eigenvalues occurs with nonzero speed, we have to consider (Liu 1994) the sign of the derivative of the higher order determinant of the Routh-Hurwicz theorem with respect to the chosen bifurcation parameter (a in the present case), evaluated at the bifurcation point. We have:

$$\frac{d\Delta_2}{da} = \frac{d}{da}(A_1A_2 - A_3) = \frac{d}{da}f_q(a) = 2A(q)a + B(q)$$

Remembering that the bifurcating values are:

$$a_{1,2} = \frac{1}{2A(q)}\left(-B(q) \pm \sqrt{\Delta}\right)$$

we immediately find:

$$\left(\frac{df_q(a)}{da}\right)_{a_2} = 2A(q)\frac{-B(q) + \sqrt{\Delta}}{2A(q)} + B(q) = \sqrt{B^2(q) - 4A(q)C(q)} > 0$$

and similarly:

$$\left(\frac{df_q(a)}{da}\right)_{a_1} = -\sqrt{B^2(q) - 4A(q)C(q)} < 0$$

This completes the proof, by confirming the existence of a Hopf bifurcation at both the points a_1, a_2.

The smooth functions of the structural parameters: $a_1 = a_1(q, ..)$; $a_2 = a_2(q, ..)$ where: $a_1(q, ..) \leq a_2(q, ..)$, define the appropriate bifurcation surfaces in the parameter space. The special structure of the bifurcation curve in the (a, q) plane is illustrated in fig. 1[15]. It is defined by the union of the two curves $a_1 = a_1(q, ..)$, $a_2 = a_2(q, ..)$ which collapse in the same point for $q = q^*$.

[15]The bifurcation curve of fig. 1 is calculated by using the same values of the main economic parameter used in the simulations of the next section. For instance, for: $m = 0.33$; $\gamma = 1$; $\rho = 2$; $b_1 = 0.10$; $n_0 = 0$; we find: $q_1 \cong 0.66$; $q^* \cong 0.368$; $q^{**} \cong 0.125$.

Fig. 1. Structure of the bifurcation curve

of the E_1 equilibrium

The main finding of the previous bifurcation analysis is therefore the existence, of a twofold Hopf bifurcation: for every value of the participation coefficient in the window $(0, q^{**})$ (other parameters being equal), there are two bifurcating values of the delay parameter a: one occurring for very short average delays, the second one occurring for relatively large delays. The substantive aspects of this result will be discussed in the next section.

5 Simulation evidence and substantive facts: comparing the "mechanistic" with the delayed system

Surprisingly, and this is a major result from our simulation runs, the general mechanistic model exhibits an almost complete similarity, both for what concerns qualitative and quantitative predictions, with its simplified delayed version.

To organise the discussion let us begin from the delayed model, which, in view of its sharper analytical properties, seems the appropriate starting point. We must first of all note that the pure existence of a Hopf bifurcation says nothing about the stability properties of the involved periodic orbits, i.e. it does not say whether the emerging periodic orbit is locally stable (supercritical bifurcation) or unstable (subcritical bifurcation). Unfortunately the investigation of the stability properties of periodic orbits appeared via Hopf bifurcation at dimensions greater than the second is a hard task (see Marsden and MacCracken 1976). Moreover the predictions of the Hopf theorem are local in nature: they nothing say about global behaviour.

We therefore have resorted to numerical simulation to clarify the stability nature of the Hopf bifurcations occurring at the points $a = a_1, a = a_2$, and more generally to investigate the global properties of our model, in particular its more substantive demo-economic properties.

The first remarkable simulation evidence is that both the points $a = a_1, a = a_2$ generate supercritical bifurcations (i.e. locally stable limit cycles). In particular the whole window $a_1 < a < a_2$ is a region of stable oscillations. Moreover, all the properties of the model seem[16] to hold globally: when the E_1 equilibrium is locally stable, then its stability seems to be global rather than only local; when E_1 switches its stability with the stable limit cycles emerging at the bifurcation points, then the involved periodic orbit seems not only locally stable but also globally stable.

We now summarise our main findings, by illustrating the working of the model in terms

[16]We could not produce a formal proof of this fact.

of the two pivotal parameters (a, q). In the region of strong "participation effect" (defined by $q^{**} < q < q_1$) the model is stable independently on the delay (let us call this situation "strong stability") and the economy converges to the long term (seemingly globally stable) steady state E_1. This makes sense: the stabilizing action of participation is so strong to always prevail against the destabilizing action of the delay.

Viceversa, when the participation effect is weak ($0 < q < q^{**}$) the E_1 equilibrium may be destabilised by the action of the delay. More in detail, as long as a is very large (in relative terms), i.e. for $a > a_2$, which corresponds to "very small" values of the mean delay $T = 1/a$, the E_1 equilibrium preserves its stability. But as a is decreased (this happens for increasing mean delays) stability may be lost. This happens for $a = a_2$ where a first Hopf bifurcation occurs and E_1 exchanges its stability with a stable limit cycle. The whole window $a_1 < a < a_2$ (characterised by intermediate values of the mean delay) is characterised by stable oscillations. Finally, by furtherly decreasing a, a further bifurcation occurs at $a = a_1$ where the stability of the E_1 equilibrium is restored: hence for very large mean delays the local (global?) stability of the economy is recovered. This last point gives an important economic example in which the role of the delay is not purely destabilizing.

The process of switching and reswitching of stability between the E_1 equilibrium and the limit cycles emerging via Hopf bifurcation appears of particular interest. At the points $a = a_1, a = a_2$ distinct Hopf bifurcations occur leading to periodic behaviours in suitable neighborhoods of these points. A prediction of the Hopf theorem is that the radius of the emerging periodic orbit depends (approximately) linearly on the distance between the actual value of the bifurcation parameter and its value at the bifurcation point. In simple words: consider a dynamical system depending on a bifurcation parameter μ and undergoing a Hopf bifurcation at μ_0. Let us suppose that the bifurcation is supercritical in a right neighborhood $(\mu_0, \mu_0 + \sigma)$. This means that for every $\mu \in (\mu_0, \mu_0 + \sigma)$ a stable limit cycle exists with radius proportional to $\|\mu - \mu_0\|$. Now, as simulations show, in our systems these neighborhoods are of the type $(a_1, a_1 + \sigma), (a_2, a_2 - \rho)$. But simulations also show that the radius of the periodic orbits emerging at $a = a_2$ is strictly increasing as a decreases from a_2 to a threshold value a^* and then decreasing as a is further decreased from a^* to a_2 where the fluctuations are reabsorbed (and the radius converges to zero). This seems to denote that the process of switching between the two regimes of bifurcation is a smooth one. This agrees with the fact that, although $a = a_1, a = a_2$ are distinct Hopf bifurcation point, the whole bifurcation process is due to the "activity" of a unique complex pair of eigenvalues which has negative real parts for large a, it crosses (with nonzero speed) the imaginary axis a first time at $a = a_2$, it keeps positive real part as long as $a_1 < a < a_2$, and crosses anew (always with nonzero speed) the imaginary axis at $a = a_1$.

From the substantive point of view the first bifurcation (appearing at a_2, i.e. for short delays) may be interpreted as a sustained oscillation induced by a delayed (but relatively "quick") adjustment of wage-dependent participation (we will analyse it further it in a subsequent paper). Viceversa, the second one (appearing at a_1, i.e. for large delays), appears as a typical population induced oscillation.[17]

[17] We must admit that the "smallest delay" oscillation was not at all expected. Its appearance reveals that in

As previously pointed out, all the findings relative to the simplified delayed model are fully confirmed, by numerical simulations, for the general mechanistic model. In fact not only the mechanistic model exhibits sustained oscillations as well, but, more important, the structure of its bifurcation process seems to be completely analogous to the one exhibited by the delayed model. Indeed, by choosing the rate of transition to adulthood v_1 as the relevant bifurcation parameter for the mechanistic model, simulation indicates that, exactly as for the delayed model, when losses of stability occur, then there are always two bifurcations values, one occurring for very small values of the bifurcation parameter, i.e. for relatively large delays, and the other one for very small delays. In particular the structure of the bifurcation curve in the (v_1, q) plane is qualitatively identical to that found in the (a, q) plane for the delayed-model. Though we will not deep here the relation beween the mechanistic and the delay model we remark that the main difference is represented by the quadratic term in the n equation in the mechanistic model, which has no counterpart in the delay model. Now, though the quantitative impact of this term could be substantial, its qualitative impact will in general be limited.[18]

Here we report a few numerical results, the aim of which is purely illustrative; more detailed investigations are postponed to future work. For ease of comparison we simulated both models under the same conditions: $\gamma = 1, \rho = 2$ (identical labour market conditions), $b_0 = 0.02, b_1 = 0.03$ (the fertility rate is assumed to be linearly related to the wage). For what concerns the mortality rates and the rates of transition to adulthood the comparison needs more care, as the simplified delayed model contains just an overall exit parameter from the adult state (μ). The value of μ in the delayed model was therefore fixed to the level $\mu_2 + v_2$ taken by the total exit rate from the adult class in the mechanistic model. Finally, we fixed the participation coefficient to $q = 0.015$ in both models.

Table 2 reports the "smaller" (i.e. those corresponding to the "demographic" bifurcation) bifurcating values of a (for the delayed model) and of v_1 (for the general mechanistic model), for two broadly different situations, the first one typical of highly capital-intensive economies ($\sigma = m^{-1} = 5$), and the second one typical of weekly capital-intensive economies ($\sigma = 2$). As the table shows, sustained oscillations appear for quite realistic values of the demo-economic parameters involved.

	D model: bif. value of a	M model: bif. value of v_1
$m = 0.2$ ($\sigma = 5$)	$0.05/year$ ($a^{-1} \cong 20\,ys$)	$0.055/year$ ($v_1^{-1} \cong 18\,years$)
$m = 0.5$ ($\sigma = 2$)	0.135 ($a^{-1} \cong 8\,years$)	$0.144/year$ ($v_1^{-1} \cong 7\,years$)

Tab. 2. Bifurcation values of the parameters a and v_1 in the

two models (legenda: D=delay,M=mechanistic)

some cases the existence of delayed wage-dependent participation may be a further source of dynamic complexity.

[18] A very heuristic argument is the following: at zero wage the n equation in the mechanistic model looks as: $\dot{n} = -n^2 - Pn + Q$. Provided it admits only one positive equilibrium, the previous equation will not differ much, in qualitative terms, from the equation: $\dot{n} = -Pn + Q$.

The figures 2-3 report a phase space dynamics of the "D" and "M" models for values of a and v_1 quite close to their bifurcating values, while the fig. 4-5 report the time paths of the corresponding long term cyclical behaviors. Both models exhibit a quite appealing "viability". Notice that, from (23):

$$n(t) = v_1 Z(t) - (\mu_2 + v_2)$$

where $Z(t)$ is the ratio between the numbers of young and adults at time t, the fluctuations in the rate of growth of the adult population are directly mirrored in fluctuations in the age distribution of the overall population.

Moreover the following main facts emerge:

i)the similarity of dynamical behaviour in the two models is "complete". This appears to be of first importance as it suggests that up to a certain degree even highly simplified (i.e. poorly parametrised) delay models can be of great help in understanding the basic qualitative features of more complex age-structured models.

ii)For what concerns purely quantitative predictions, the results from both models remain quite close, with the exception of the dynamics of the rate of growth of the population which seems quite overestimated by the delayed model.

iii)the transition to adulthood appears to be the main demographic mechanism capable of generating dynamic instability (and therefore lead to oscillations). With the exception of the birth rate at age zero (b_0) which has also a slight destabilising effect, all the remaining demographic factors, as summarised by the demographic parameters (μ_i, v_2), have essentially a stabilising role on E_1.

Fig. 2 Projection onto the (s,w) plane of a limit cycle in the delayed model

Fig. 3 Projection onto the (n, w) plane of a limit cycle in the mechanistic model

Fig. 4 Long term cyclical oscillations in the delayed model

Fig. 5 Long term cyclical oscillations in the mechanistic model

6 Conclusions

In this paper an attempt is made to model the demoeconomic interaction by introducing a mechanistic treatment of age structure within an economic framework, that of the classical Goodwin's (1967) growth cycle model, which takes into account the overall macroeconomic structure. In particular a detailed (although perhaps still very simplistic) representation of the process of determination of labour supply is considered, which in our opinion suits very much the idea of "malthusian mechanism", as clearly defined by Lee (1997).

Two main conclusions emerge. In the first place persistent endogenous demoeconomic oscillations may be the natural outcome of the working of the labour market, i.e. the outcome of the dynamical balancing between the destabilising effect played by the delay of transition to the adult state, and the stabilizing force implicit in standard participation schedules (based as in this paper on highly traditional assumptions, such as the discouraged worker hypothesis). This idea is alternative to the notion of fertility feedback implicit in the traditional Easterlin effect. Second, the feeling that even if the full treatment of age structure is complex, and its detailed effects may only be studied via numerical simulations, simplified models based on time-delays may be of great help to understand the basic qualitative consequences of the introduction of age structure.

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8 Appendix 1: the non-zero equilibrium in the mechanistic model

The nonzero equilibrium E_1 of model (29) is the solution of the following subsystem in the variables (w, n):

$$\begin{array}{l} m(1-w) - qU_1 - n = 0 \\ n^2 + Pn - B(w) = 0 \end{array} \tag{58}$$

Hence, its solutions are the intersections of the lines: $n_A(w) = m - qU_1 - mw$ and $n_B(w) = \frac{1}{2}\left(-P + \sqrt{P^2 + 4B(w)}\right)$. The existence of E_1 depends on the mutual position of the intercepts $n_A(0)$,$n_B(0)$. By recalling that an equilibrium pair (w_1, n_1) is economically meaningful provided $0 \leq w_1 \leq 1$, we have the following cases:

A)When $n_B(0) > 0$ and $n_A(0) < 0$ no nonzero equilibrium may exist.

B)When $n_B(0) > 0$, $n_A(0) > 0$, we have the two possibilities: i)$n_B(0) > n_A(0)$ in which no nonzero equilibrium may exists; ii)$0 < n_B(0) < n_A(0)$ (or: $R_E(0) > 1$) in which E_1 exists, it is meaningful and provides a positive rate of growth n_1 of the population.

C)When $n_B(0) < 0$, $n_A(0) > 0$, then:

i)if $n_B(\frac{m-qU_1}{m}) > 0$ then E_1 exists (and it is meaningful) with $n_1 > 0$;

ii)if $n_B(\frac{m-qU_1}{m}) < 0$ but $-qU_1 < n_B(1) < 0$ then E_1 exists and is meaningful with $n_1 < 0$; iii)if $n_B(1) < -qU_1$ then E_1 exists but it will not be meaningful.

D)When $n_B(0) < 0$, $n_A(0) < 0$, then:

i)$0 > n_B(0) > n_A(0)$: no nonzero equilibrium exists; ii)$0 > n_A(0) > n_B(0)$: E_1 exists and it is meaningful as long as $n_B(1) > -qU_1$.

9 Appendix 2: equilibria in the delay-model

The equilibria of the delayed model (46) are the solutions of the system:

$$\begin{aligned} &w\left(-\gamma+\rho U\right)=0 \\ &U\left(m(1-w)-qU-\left(b_0+b_1 w-\mu\right)\right)=0 \end{aligned}$$

where the quantity: $qU + (b_0 + b_1 w - \mu)$defines the total rate of change of the supply of labour. The nullclines of the wage equation are $w = 0$ and $U = U_1 = \gamma/\rho$ $(\rho > \gamma)$. The nullclines of the U equation are: $U = 0$ and the line:

$$w = \frac{m - n_0}{m + b_1} - \frac{q}{m + b_1}U \tag{59}$$

Hence the zero equilibrium E_0 always exists. Notice moreover that meaningful equilibria of the system are given by those intersection of the nullclines which are located within the admissible "box" $(0 \leq w \leq 1, 0 \leq U \leq 1)$.

Let us first consider the case $n_0 > 0$ (notice that in this case the existence of a positive equilibrium of the wage necessarily implies a strictly positive rate of growth of the total population). In this case the intercept of the line (59) satisfies $(m - n_0)/(m + b_1) < 1$. Then, if $m - n_0 < 0$, neither positive equilibria nor axis equilibria exist. Vice-versa, $m - n_0 > 0$ then an axis equilibrium E_2 exists, with $U_2 = (m - n_0)/q$, and, provided that $U_2 > U_1$, a positive equilibrium E_1 also exists. In particular:

i)if E_1 does not exist, i.e. $U_2 < U_1$, then E_2 is economically meaningful, as: $U_1 = \gamma/\rho < 1$ by assumption.

ii)if E_1 does exist, i.e. $U_2 > U_1$, then it is necessarily economically meaningful. This is not necessarily the case for E_2; E_2 will be meaningful provided that:

$$U_2 = \frac{m - n_0}{q} < 1 \quad \rightarrow \quad m - n_0 < q \tag{60}$$

Let us now consider the case $n_0 < 0$. In this case the intercept $(m - n_0)/(m + b_1)$ may exceed one. We have to distinguish:

i)$(m - n_0)/(m + b_1) < 1$. In this case the intercept nonetheless remains in the admissible box. This corresponds to $n_0 + b_1 > 0$, i.e. $n(1) > 0$. In this case E_1 still exist and it is economically meaningful, while E_2 is not necessarily meaningful and the condition (60) is still needed.

ii)$(m - n_0)/(m + b_1) > 1$. In this case the intercept does not belong to the box. This corresponds to $n_0 + b_1 < 0$, i.e. $n(1) < 0$. In this case also E_1 is not necessarily meaningful. It will be meaningful provided that the condition:

$$w_1 = \frac{m - n_0 - qU_1}{m + b_1} < 1$$

i.e.:

$$n_0 + qU_1 + b_1 > 0 \quad \rightarrow \quad n(1) + qU_1 > 0 \tag{61}$$

holds. The meaning of the last condition is that, to have an economically meaningful positive equilibrium of the wage, it is necessary that the equilibrium growth rate of the total supply of labour, given by the sum of the equilibrium growth rate of the population and of the equilibrium rate of change of the particpation rate, be strictly positive.

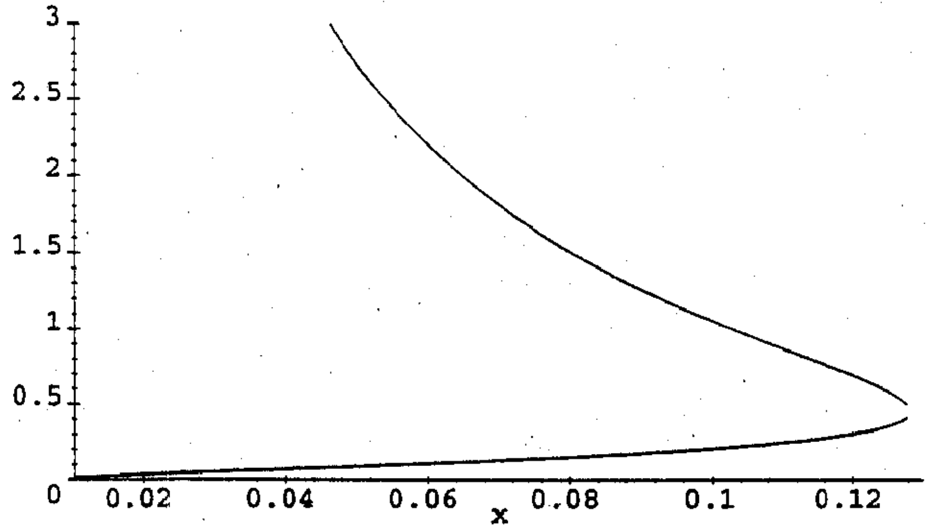

Fig. 1. Structure of the bifurcation curve of the E_1 equilibrium

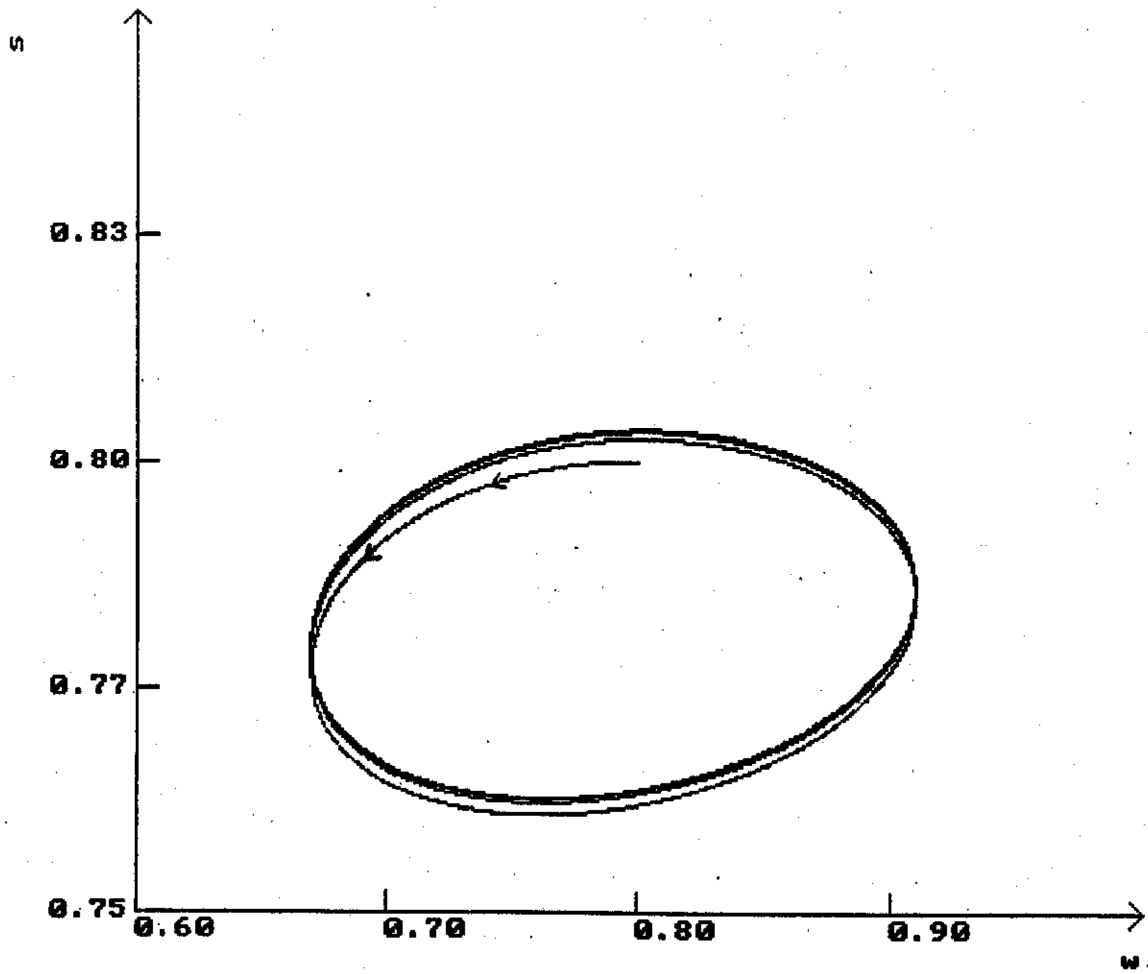

Fig. 2

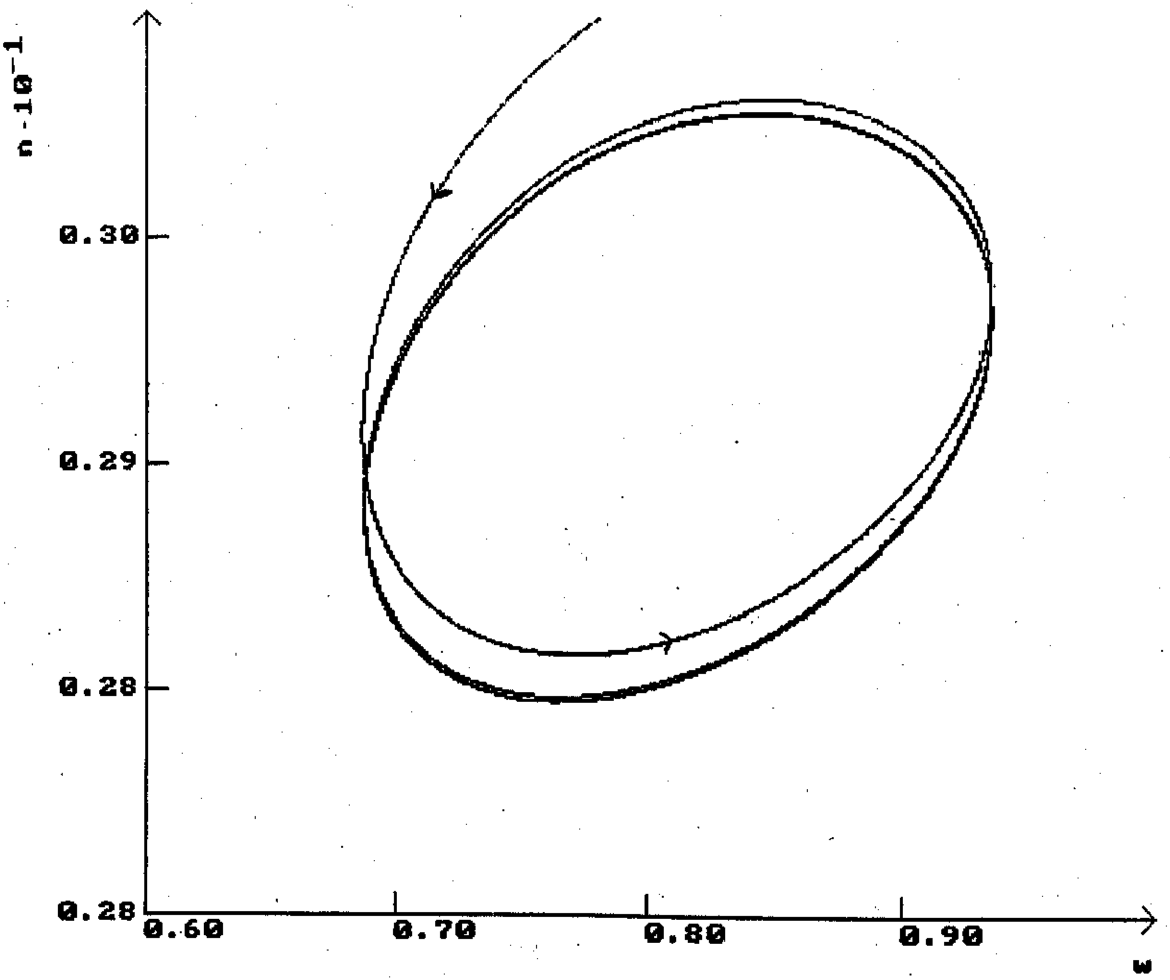

Fig. 3

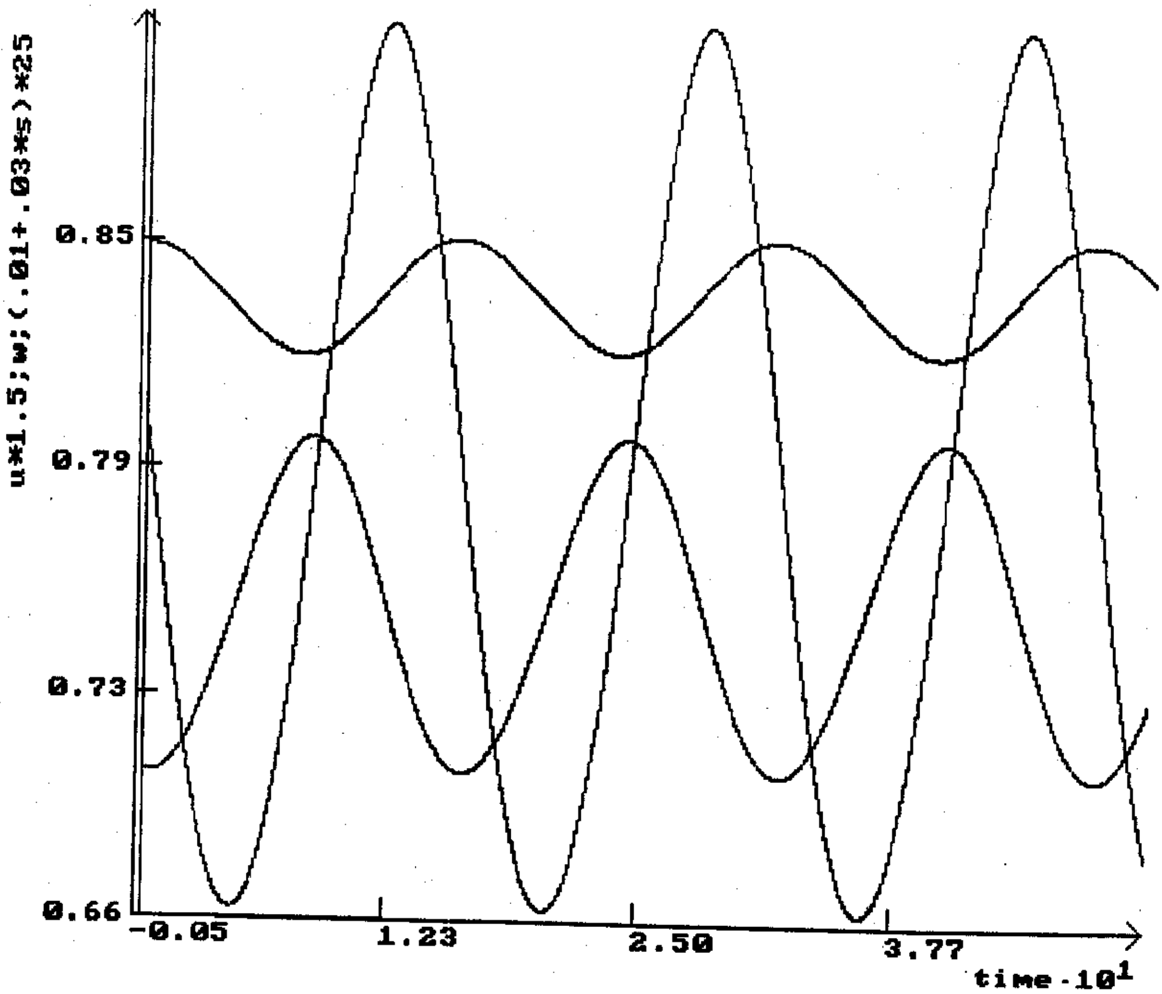

Fig. 4

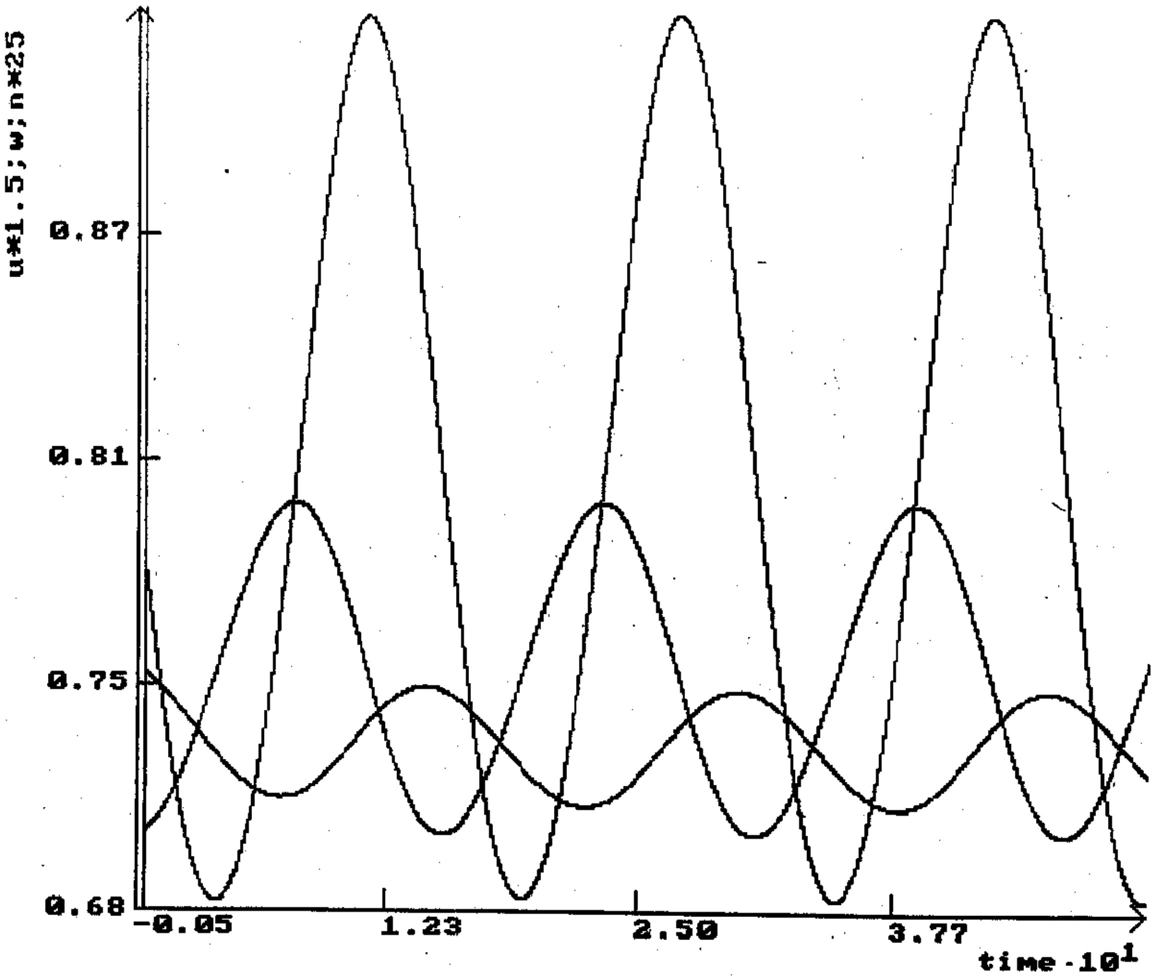

Fig. 5