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Abstract

Regular fluctuations and chaotic behaviour of wages and employment emerge as a robust finding from standard wages and labour quantities adjustment, in a simple neoclassical economy, when consumption and leisure are low substitutes. In particular when the unique walrasian equilibrium of this economy is destabilised, the economic variables (wage, demand for labor and supply of labor) are attracted in a stable attracting region within which their motion is chaotic. In this region we have shown that both average wage and average unemployment with respect to the total population are larger than those existing in the Walrasian Equilibrium, suggesting possible empirical implications in terms of a 'reminiscence' of a Phillips curve.

Introduction.

The relationship between wage inflation and unemployment has been extensively discussed since the early work of Phillips (1958) and Lipsey (1960).

Likewise a renewed interest to study fluctuations in aggregate economic variables has shed new light on both endogenously (Lorenz, 1993) and exogenously (Kydland-Prescott, 1982) determined business cycle.

In the business cycle literature there are several instances of models using the Phillips curve as a building block. An important example is represented for instance by Goodwin's model (Goodwin 1967) and its variants (Cugno and Montrucchio (1982), Sportelli (1995), Fanti-Manfredi (1998), Manfredi-Fanti (1999)). A question is then: of what types of dynamical phenomena is the Phillips relation responsible for? Rich dynamical behaviours in models incorporating the Phillips curve occur in many variants of the Goodwin's model. As known, these models are based on the Lotka-Volterra paradigm, which is one the most fertile generator of "rich" dynamics. This "richness" makes it often uneasy, unfortunately, to clarify the exact role played by the Phillips curve in determining the emergence of persistent cycles and complex dynamics. The dynamical behaviour of a system of equations based on the Phillips curve has been recently analysed in this Journal by Soliman (1996) and by Montoro, Paz and Roig (1998). The former author uses a very specific nonlinear formulation in discrete-time - justified by the author referring to a phenomenological rather than a theoretically microfounded approach - and illustrates the possibility of several dynamical phenomena. The latter authors extend Soliman's conclusions showing that the introduction of a stabilizing monetary policy may lead to cyclical and chaotic behaviors.

In this paper we consider a simple economic model describing the dynamics of the market of the unique input, the labour input. In this market, both demand and supply are micro-founded: the demand for labour is derived from a profit-maximiser firm with a usual diminishing return technology while the supply of labour is derived from an utility-maximiser worker-consumer with CES preferences. Moreover we postulate that both the labour demand and supply adjust only gradually to the respective optimal quantities, following a usual adjustment process in continuous-time. The final building block of the model is a Phillips-type curve based on the famous interpretation of Lipsey (1960): the wage rate is assumed to continuously adjust, following an adaptive rule, to the current excess demand for labour.

A paper which has recently investigated the dynamics of a single market similar to the present one is Chichilnisky-Heal-Lin (1995, CHL from now on). CHL have considered an economy with fixed costs, which in turn cause both increasing returns to scale and a discontinuous firm's labor demand. Moreover, as in our model, the supply of labour is a continuous function and the adjustment is the usual according to the laws of supply and demand. In CHL, due to the special choice of the production function, the demand and supply curve do not intersect. They could nonetheless prove the existence of a stable disequilibrium price. Further dynamical analysis proved the existence of complex behaviours as well. This is done by applying Keener's (1980) results on chaotic behaviour in piecewise continuous difference equations (the work by CHL is in a discrete-time framework governed by a discontinuous map). Although their result seems robust it relies on a highly special assumption, as the authors claim: "we may conclude that the fixed cost is responsible for the chaotic behaviour.."(CHL, p.284).

In this paper we obtain a robust chaotic behaviour in a continuous-time framework which seems mathematically more general and reasonable than the piecewise difference equation used by CHL (see for instance Gandolfo, 1996). Moreover we assumed a usual diminishing returns technology, and this latter is surely more general than the discontinuous technology assumed by CHL. As the only extra-ingredient with respect to the Lipsey's interpretation of the Phillips curve we postulated the existence of an adjustment process[1] also in the demanded and supplied quantities.

Our main finding is therefore that robust chaotic behaviour of prices and quantities in the labour market may occur even from a simple micro-founded economic model characterised by the simplest adjustment mechanism. In particular chaotic fluctuations occur when the substitution between consumption and leisure is sufficiently weak. Furthermore we observed the following non self-

[1] By passing, we recall that the use of a "disequilibrium" model does not imply a lack of optimizing behavior by economic agents nor that their behavior is irrational. This aspect is extensively discussed by Wymer (1992).

evident fact: the absence of flexibility on the side of the firm, as well as the absence of stickiness on the side of the supply of the worker, can destabilise the economy and lead it in a "trapping" chaotic region characterised by both an average wage and an average unemployment with respect to the total population larger than those existing in the Walrasian Equilibrium (WE from now on)[2] regime. This suggest as a policy implication, to preserve this asymmetry in flexibility between the firms and the workers.

The plan of the paper is as follows. In the second section we present the model. The analysis of the equilibria, their local stability and the existence of a Hopf bifurcation is presented in the third section. Section four illustrates through numerical simulation the analytical results of section three as well as the emergence of chaotic behaviour, whereas a less technical discussion of the working of the model is postponed to section five. Section six is devoted to some concluding remarks.

2. The model.

Our economy is composed by a single representative firm and a single representative worker-consumer. Labour is the only input, by means of which the firm produces a single consumption good. The model is "essentially" neo-classical in that the firm and the consumer are both maximiser of, respectively, profit and utility, subject to prescribed technology and preference constraints. These latter are set according to a Cobb-Douglas production function and a CES utility function. The technology is represented by the following production function:

$$Y = DL^{a} \qquad 0 < a < 1, D > 0 \qquad (1)$$

where the parameters a,D have the usual meaning. Let Π and w respectively define the total profit and the wage rate. By setting the price of the unit output equal to one, the profit function is defined as

[2] A social welfare comparison (in a statistical sense) between the stable WE point and the stable chaotic set is beyond the scope of this paper. We notice however that the stable WE point does

$$\Pi = DL^a - wL \qquad (2)$$

A standard maximization of (2) gives the optimal demand for labour:

$$L^D = f_1(w) = \left(\frac{w}{aD}\right)^{\frac{1}{a-1}} \qquad (3)$$

The worker-consumer maximises the following CES utility function, as in CHL:

$$U(C, L) = \left[C^b + (N - L)^b\right]^{1/b} \qquad b \in (-\infty, 1] \qquad (4)$$

where N>0 is the maximal labour supply and C>0 the worker's consumption level. By still following CHL, let us assume N=1. The standard utility maximization gives the optimal labor supply:

$$L^S = f_2(w) = \frac{1}{1 + w^{\frac{b}{b-1}}} \qquad (5)$$

As well-known the elasticity of substitution between consumption and leisure, (*e*), is: *e=1/(1-b)*. The following features hold. The amount of labour supplied is an increasing function of the wage for *b*>0 and a decreasing one for *b<0*. Moreover, for *b<0* consumption and leisure are "low" substitutes. In particular the larger b is in absolute value, the more consumption and leisure tend to be "consumed" in fixed proportions, such as in the case of any recreational and shopping activities. These facts are summarised in the following table.

$b=1 \Rightarrow e=\infty$	Infinite substitution
$0< b < 1 \Rightarrow e> 1$	High substitution
$b\to 0 \Rightarrow e \to 1$	Cobb-Douglas

not seem necessarily better in terms of welfare.

$-\infty < b < 0 \Rightarrow e < 1$	Low substitution
$b \to -\infty \Rightarrow e \to 0$	Leontiev (fixed proportions)

Let us now investigate the Walrasian dynamics of this economy. The excess demand for labour (z) can be expressed as

$$z(w) = L - S \qquad (6)$$

where L,S are the current demand and supply of labour. If we assume that the optimal demand and supply of labour instantaneously adjust to their current counterparts, then it would also holds

$$L = L^D; \quad S = L^S$$

The price of labour (the wage rate) is not assumed to immediately adjust to labour market disequilibrium situations. Rather we postulate that the wage continuously adjust to the current excess demand for labour, according to an adaptive rule with speed of adjustment l>0:

$$\dot{w} = lz(w) = l(L - S) \qquad (7)$$

Equation (7), which is nothing else than the Phillips equation in the famous interpretation of Lipsey (1960), is the bare bone of the Walrasian dynamic theory. More relevant "dynamical" flesh can be added by considering the possibility that the current demand for labour (L) does not immediately adjust to the optimal demand (L^D), as well as the possibility that the current supply of labour (S) does not immediately adjust to its optimal counterpart (L^S). Both these assumptions seem to be quite natural. For instance in the case of the firm, let us assume that a wage change occurs. The firm reacts to this change by computing a new value of the optimal demand for labour but this new value is unlikely to be effective until a certain time has elapsed (because of, namely, negotiations with the union or other legal procedures). Or in the case of the worker (assumed with a new-born baby) the new optimal supply (assumed increased) due to a wage change cannot be effectively expressed until the nursery-school is closed for summer holidays. By assuming that both the current demand and supply adjust to their optimal counterparts following an adaptive adjustment, our model finally takes the form:[3]

$$\dot{L} = g\left(L^D - L\right) = g\left(\left(\frac{w}{aD}\right)^{1/(a-1)} - L\right) \qquad g > 0$$

$$\dot{S} = d(L^S - L) = d\left(\frac{1}{1 + w^{b/(b-1)}} - S\right) \qquad d > 0 \qquad (8)$$

$$\dot{w} = l\left(L - S\right) \qquad \lambda > 0$$

So far, it has been usual to analyse the dynamics of traditional wage adjustments, such as that implicit in (7), by considering an exogenous cyclical fluctuation either of demand, for instance of the type $L^D(t) = \cos(t)$ (Hansen, 1970)[4], or both of demand and supply (Bowden, 1980). We have built a different mechanism, based on a subsystem of demand and supply adjustments, which can also display some endogenous cyclical features which in turn interact with the Phillips-Lipsey mechanism.

3. Equilibria and their local stability

1. Existence of equilibria

The equilibria of the system are defined as the solutions of the equation:

$$\mathrm{kw}^{-\frac{1}{1-a}} = \frac{1}{1 + \mathrm{w}^{\frac{b}{b-1}}} \qquad \text{where } \mathrm{k} = \left(\frac{1}{\mathrm{aD}}\right)^{\frac{1}{a-1}} > 0 \qquad (9)$$

for $0<a<1, b<1$. This corresponds to the intersections between the two curves $f_1(w)$ and $f_2(w)$ denoting optimal demand and supply.

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[3] Due to the fact that the maximum of labour supply is set to one, it should also be assumed that the upper bound for L is necessarily one. But we do not impose explicitly this bound in order to avoid the further non-linearity due to the existence of a "barrier" in the dynamics (in fact in this event also locally unstable *linear systems* could behave chaotically (Simonovits, 1982)). Our main goal is in fact to investigate the dynamical behaviour of our economy under the "minimal" number of " nonlinear" ingredients, i.e. just those due to the neo-classical assumptions. Nevertheless we can assume that, temporarily, during the cycle a situation of overemployment, due to, for instance, overtime work and so on, could exist. Note that this problem arise also in the Goodwin's model (1967), in which the rate of employment is not bounded, and for which similar temporary overemployment situations can be postulated, as in Flaschel-Groth (1995)).

[4] "Instead of building up a complete model for labour demand, we shall impose an exogenous, cyclical fluctuation of demand for labour upon this system." (Hansen (p.19)).

It is possible to show that the system always has a unique equilibrium which is economically meaningful. The optimal demand curve has the traditional strictly decreasing form over the set of positive values of the wage for all the possible values of its characteristic parameters *a,D*. In particular when $w \to 0^+$, $f_1(w) \to +\infty$. Vice-versa, for what concerns the supply curve there are two broadly distinct cases: i) 0<b<1, and ii) b<0. In the first case the ratio *b/(b-1)* is negative. Let us denote $b/(b-1)=-\beta$, with $0<\beta<\infty$. As:

$$f_2(w) = \frac{1}{1+w^{\frac{b}{b-1}}} = \frac{1}{1+w^{-\beta}} = \frac{w^\beta}{1+w^\beta}$$

in this case the optimal supply is strictly increasing and saturating to its unit upper bound. Moreover $f_2(0)=0$. Hence a unique economically meaningful equilibrium always exists (fig. 2).

Let us now consider the case b<0. By writing: b=-c we have: $b/(b-1)=c/c+1=\delta$, with $0<\delta<1$. In this case:

$$f_2(w) = \frac{1}{1+w^{\frac{b}{b-1}}} = \frac{1}{1+w^\delta}$$

Hence, by defining: $1/(1-a)=\gamma$, $\gamma>1$, the detection of equilibria leads to the equation:

$$k\left(1+w^\delta\right) = w^\gamma$$

to be studied for w>0.

The right-member curve has a zero-intercept and is strictly increasing. As $\gamma>1$ it increases more faster than the 45° line. On the other side the left-hand curve has a strictly positive intercept (given by k>0) and is strictly increasing as well, but, as $0<\delta<1$, it increases more slowly than the 45° line. Hence also in this case a unique economically meaningful equilibrium exists (fig. 2).

The following proposition summarises our steady state analysis:

Proposition 1: the model (8) always admits a unique equilibrium point $E_1=(L^*,S^*,w^*)$. E_1 is always economically meaningful.

Notice therefore that the previous proposition holds both when consumption and leisure are substitutes as well as when they are complements.

Fig. 1. Equilibrium analysis: optimal demand and supply curve: the case b>0 (a=0.15, b=0.5)

Fig. 2. Equilibrium analysis: optimal demand and supply curve: the case b<0 (a=0.15,b=-10)

2. Local stability analysis

To investigate stability let us write our model in the form:

$$\begin{aligned} \dot{L} &= g(f_1(w) - L) \\ \dot{S} &= d(f_2(w) - S) \qquad g, d, l > 0 \qquad (10) \\ \dot{w} &= l(L - S) \end{aligned}$$

For simplicity let us suppress (*) and denote by (L,S,w) the equilibrium values of the state variables. The jacobian evaluated at E_1 is:

$$J(L,S,w) = J(E_1) = \begin{pmatrix} -g & 0 & gf_1'(w) \\ 0 & -d & df_2'(w) \\ l & -l & 0 \end{pmatrix} \qquad (11)$$

The corresponding characteristic equation is:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where the coefficients a_i (i=1,2,3) are defined as:

$$\begin{aligned} a_1 &= g + d \\ a_2 &= gd - l(gf_1'(w) - df_2'(w)) \qquad (12) \\ a_3 &= ldg(f_2'(w) - f_1'(w)) \end{aligned}$$

We note that a_1 is always positive. Moreover a_3 is always positive when $b>0$, as in this case the optimal supply curve is strictly increasing (Vice-versa when $b<0$, a_3 is not necessarily positive). Let us concentrate first on the "traditional" case b>0. In this case, according to the Routh-Hurwicz test, E_1 is locally stable when the further inequality $a_1a_2-a_3>0$ holds. The latter condition leads to the inequality:

$$(g+d)(gd-l(gf_1'(w)-df_2'(w)))-ldg(f_2'(w)-f_1'(w))>0$$

or:

$$(g+d)gd+l(d^2f_2'(w)-g^2f_1'(w))>0 \qquad (13)$$

By recalling that $f_1'(w)<0$ the last inequality is always true, showing that when $b>0$ the E_1 equilibrium is always locally asymptotically stable LAS.
These fact are summarised in the following:

Proposition 2: if consumption and leisure are "strong" substitutes ($0<b<1$, corresponding to a monotonically increasing supply function) the unique equilibrium E_1 is LAS independently from the values of the adjustment parameters.

An intermediate case is represented by b=0 which causes a rigid optimal labour supply. In this case we have the system:

$$\begin{aligned}\dot{L}&=g(f_1(w)-L)\\ \dot{S}&=d(1-S) \qquad g,d,l>0 \qquad (14)\\ \dot{w}&=l(L-S)\end{aligned}$$

In this case the equilibrium E_1 is explicit: $E_1=(1,1,k^{1/\gamma})$, and it is always LAS. In fact the coefficients of the characteristic polynomial are simply:

$$a_1=g+d \ ; \quad a_2=gd-\lg f_1'(w) \quad ; \quad a_3=-ldgf_1'(w) \qquad (15)$$

In this case the Routh-Hurwicz test quickly shows that E_1 is always LAS:

$$a_1a_2 - a_3 = (g+d)(gd - \lg f_1'(w)) + ldgf_1'(w) = (g+d)gd - (g+d)\lg f_1'(w) > 0$$

The case b=0 is simple and we will not deep it. Notice that it leads to the system:

$$S(t) = S_0 + (1-S_0)e^{-dt}$$
$$\dot{L} = g(f_1(w) - L)$$
$$\dot{w} = l(L - S_0 + (1-S_0)e^{-dt})$$

which is non autonomous but asymptotically autonomous.

Let us now consider more in depth the case b<0. In this case both the derivatives of the optimal labour supply and demand are negative. Let us put for simplicity:

$$f_1'(w) = -A \quad ; \quad f_2'(w) = -B$$

Hence, as:

$$a_3 = f_2'(w) - f_1'(w) = A - B$$

for A-B<0 (A<B) (corresponding to the case of a supply curve steeper than the demand curve at equilibrium) the equilibrium is always locally unstable (a saddle point).[5] Vice-versa, for A>B the coefficient a_3 remains positive and more interesting dynamical effects may appear. In this case the condition $a_1a_2-a_3>0$ becomes:

$$(g+d)gd + l(Ag^2 - Bd^2) > 0$$

or:

[5] The equality A=B causes a saddle-node bifurcation. In this paper we have not considered the whole spectrum of possible bifurcation patterns of the equilibrium as we are mainly interested in the mechanisms leading to complex behaviour. A full picture of the bifurcation patterms is obtained via equilibria continuation methods (Kutznetzov 1995) in a different paper (Fanti and Manfredi (1999)).

$$g^2(d+Al)+d^2(g-Bl)>0 \qquad (16)$$

We immediately notice that for d=g we get:

$$a_1a_2-a_3=2d^3+ld^2(A-B)>0$$

Hence, for d=g the system is always locally stable in a neighborhood of the E_1 equilibrium. This shows that, a fortiori, the system remain LAS for all combinations (g,d) satisfying g>d (i.e. when the speed of adjustment of the demand sector of the economy is larger of the corresponding quantity of the supply side of the economy). On the contrary, when g<d, the system may definitively loose its stability. Instability arises when, provided that g<d, it holds:

$$(g+d)gd+l(Ag^2-Bd^2)<0 \qquad (17)$$

We claim that a Hopf bifurcation arises when the equality:

$$(g+d)gd+l(Ag^2-Bd^2)=0 \qquad (18)$$

holds. The previous equality defines the stability boundary of the system. To simplify the analysis in this study we study the bifurcation process using as a bifurcation parameters the speed of adjustment of the demand for labor to its optimal level. To do this we write the bifurcation locus as:

$$H(g,d)=(d+lA)g^2+d^2g-lBd^2=0$$

For d=0 we have g=0 showing that the bifurcation locus passes through the origin. The full structure of the curve is found by solving the previous equation. with respect to g; we get:

$$g_{H,1}=\frac{-d^2-\sqrt{d^4+4lB(d+lA)d^2}}{2(d+lA)} \quad ; \quad g_{H,2}=\frac{-d^2+\sqrt{d^4+4lB(d+lA)d^2}}{2(d+lA)} \qquad (19)$$

Notice that the discriminant $\Delta = d^4 + 4lB(d+lA)d^2$ is always positive. So that the solutions (19) are always real. It is easy to check that only the solution $g_{H,2}$ being positive is adequate to represent the desired bifurcation process. This shows that a bifurcation value always exists. The structure of the bifurcation curve of the E_1 equilibrium is given by

$$g_H = \frac{-d^2 + \sqrt{d^4 + 4lB(d+lA)}}{2(d+lA)} \qquad (20)$$

It is of interest to study the shape of the curve (20) as a relation of the type $g = g_H(d)$ where g,d are the speeds of adjustment of the current demand and supply curves. It is not difficult to see that, as expected, the curve $g_H(d)$ lies always in the region $g < d$. Moreover $g_H(d)$ is a strictly increasing function of d. In fact we have, after some algebra:

$$g'_H(d) = \frac{1}{2}\frac{\left\{(d+lA)\left(-2d + \left(2\sqrt{\Delta}\right)^{-1}\left(4d^3 + 12lBd^2 + 8l^2ABd\right)\right) - \left(\sqrt{\Delta} - d^2\right)\right\}}{(d+lA)^2} =$$
$$= \frac{1}{2(d+lA)^2}\left(4d^6l^2B^2 + 24d^5l^3B^2A + 52d^4l^4A^2B^2 + 48d^3l^5A^3B^2 + 16l^6A^4B^2d^2\right) > 0$$

Finally it is possible to show that $g_H(d)$ is a convex curve. To complete the proof of the appearance of a Hopf bifurcation of the E_1 equilibrium, let us now show that the pair of bifurcating eigenvalues cross the imaginary axis with nonzero speed. This is equivalent to show that (Liu, 1994):

$$\left(\frac{d}{dg}(a_1a_2 - a_3)\right)_{g=g_H} \neq 0$$

We quickly have:

$$\left(\frac{d}{dg}(a_1 a_2 - a_3)\right)_{g=g_H} = \left(2(d+lA)g + d^2\right)_{g=g_H} = \sqrt{d^4 + 4lB(d+lA)}$$

which is always positive, thereby completing the proof.

The fig. 2 illustrates the bifurcation curve and the stability features of our basic system in the case A>B in the (d,g) positive quadrant. The straigh line g=d splits the plane into two regions. In the region above the line (g>d) the system is always LAS, and this happens on all the points of the line as well. Viceversa in the region below the line the system is stable in the region above the bifurcation locus.

Fig. 3. The structure of the bifurcation curve (a=0.15, D=1, b=-10, λ=4)

Let us summarise our dynamic finding in the following:

Proposition 3: if consumption and leisure are weak substitutes (b<0) (the case of a monotonically decreasing supply function), and provided some other condition is satisfied, there exists a bifurcation value of the adjustment parameter of the current demand $g_H = g_H(a,D,b,d,l)$ such that for $0<g<g_H$, the unique equilibrium E_1 is locally unstable. Moreover, a suitable left neighborhood of g_H exists in which a stable limit cycle (at least one) exists.

The inspection of the structure of the bifurcation curve $g_H(d)$ shows that the stability of the Walrasian equilibrium prevails for combination of values of the speeds of adjustment of the decisions of both the firm and the households which lie above a critical line, given by the bifurcation curve. Therefore the following remark holds

Remark: stickiness in the realization of the employment decisions of the households as well as flexibility in those of the firms tend to favour the stability of the Walrasian equilibrium.

Moreover, our simulation analysis of the next section will show that when g is furtherly decreased, chaotic behaviour can emerge.

4. Numerical simulations

Even for simple models as the one considered here it is difficult to obtain a global view of the dynamics. Nonetheless a good feeling in the global behaviour of the system may be obtained by simple strategies, i.e. by choosing a bifurcation (or "control") parameter and by analysing how the structure of the attractors evolve as the control parameter is varied while all the other parameters are kept fixed. In this work we have chosen as the bifurcation parameter the parameter *g*, which measures the degree of flexibility of the employment decisions of the firms. In what follows we report an example of the main simulative evidence from our simulation runs.

The sequence of windows of distinct dynamical behaviour reported below was observed for a very wide set of values of initial conditions and parameters.

The simulations reported in the figures 4-13 as illustration of the dynamics of the system (8), are based on the following specific parameter constellation: D=1, a=0.15, b= -10, d=4, λ=4. The corresponding equilibrium values are $L^*=S^*=0.832$ and $w^*=0.182$. All the reported simulations are performed with initial conditions very close to the equilibrium E_1 ($L^\circ= 0.825$, $S^\circ= 0.835$, $w^\circ= 0.18$).

Dynamical simulations are then performed focusing on how the equilibrium point changes its qualitative properties. For quite large values of *g* ($g>1.3$) E_1 is a stable node (see fig. 4 drawn for g=12). When g is decreased below 1.3 the equilibrium point becomes a stable focus (fig. 5, drawn for g=1.25): trajectories converge to the equilibrium with damped oscillations. As predicted by the analysis of the previous section, a Hopf bifurcation occurs when the speed of adjustment of the demand for labour falls below the critical threshold value defined by equation (20). This threshold is given by g_H=1.068 in our parameter constellation. The predictions of section 3 are confirmed by numerical simulation: trajectories starting sufficiently close to the steady state initially diverge, and subsequently converge to a stable limit cycle. Fig. 6 (drawn for g=1.066) reports a phase plane view of the dynamics of model (8) in the (S,L) plane. The involved cycle, which seems to be unique[6] from simulation, exhibits small oscillations between 0.82 and 0.84 both for demand and supply.

When we further decrease the adjustment parameter g, complex behaviours arise. Generally speaking the emergence of a strange or chaotic attractor may be detected through several measures[7]: 1) by "eye", i.e. by the visual inspection of highly irregular dynamical patterns both in the phase space and in the time paths; 2) through bifurcation diagrams; 3) through many numerical and statistical tests[8].

The direct visual inspection reveals that the trajectories of the system wander erratically in a bounded region of the phase plans L,S and w, S (see figg. 7-8). Furthermore a typical feature of the chaotic behaviour (which distinguishes it from the quasi-periodic behaviour), namely the SDIC (sensitive dependence on initial conditions), is neatly indicated by the comparison between any two time paths starting from very close initial conditions. Fig. 9a plots the trajectory of the wage in the time domain for the initial condition of L°=0.825, S°= 0.835 and w°=0.18, while fig. 9b reports the same plot for the same initial conditions for demand and supply but w°=0.181: the two paths, both of which display a highly irregular pattern, differ neatly, both in the amplitude and frequency of the cycles.

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[6] The Hopf theorem does not predict the uniqueness of the involved periodic orbit.

[7] The use of techniques for the global analysis of the system (8) in order to obtain an analytical or geometrical detection of "chaos" is beyond the scope of this paper (see Wiggins (1990)).

[8] Among these, we remark the computation of 1) the construction of a Poincarè map by numerical-graphical techniques which in the case of a simple bi-dimensional surface of section, permits to identify different types of dynamic behaviour, as limit cycle, subharmonic oscillations, quasiperiodic oscillations and the presence of a strange attractor; 2) the dominant first-order Lyapunov exponent for a reconstructed attractor which whether is positive gives a sign of existence of SDIC (Wolf at al, 1985); 3) the correlation dimension of the (reconstructed) attractor which whether is a non integer number indicates a fractal structure of the attractor (Grassberger – Procaccia, 1983). Such computations (for sake of brevity not reported here) have confirmed the presence of deterministic chaos in the system (8).

Furthermore, we can see that different initial conditions generate a different shape of the chaotic attractor: compare the figures 8 and 10.

The analysis of the bifurcation diagram (fig. 11) clearly indicates the onset of chaos: when still with reference to our initial parameter constellation, the g parameter is reduced below the threshold value g_C=0.62 the stable limit cycle, appeared via the Hopf bifurcation, bifurcates in its turn, and the system exhibits an evident route to chaos of a quasi-periodic type (still see fig. 11). Finally when a further reduction of g occurs, a so-called "catastrophic" crisis appears implying sudden increases in the size of the chaotic attractor, until the final exploding crisis (for g_E<0.53), which leads the system to global instability.

The figures 12-13 illustrate a remarkable fact: the chaotic fluctuations of the demand for and supply of labor are on average below their equilibrium value whereas they are systematically exceeding the equilibrium value for the wage.

FIG. 4– View of the monotonic convergence of the trajectories to the equilibrium in the phase space *L, S.* (g=12).

FIG. 5 – View of the oscillatory convergence of the trajectories to the equilibrium in the phase space *L, S.* (g=1.25).

FIG. 6 – View of the convergence of the trajectories to the stable limit cycle in the phase space *L, S.* (g=1.066).

FIG. 7 – Chaotic behaviour in the phase space *S, L.*

FIG. 8 – Chaotic behaviour in the phase space *w, S* (I.C.: L°=0.825, S°=0.835, w°=0.18).

FIG. 9 (a,b) – Two enlarged windows of the plot of the *time path* of the variable *w* in the case of a *very small* difference in one initial condition (L°= 0.825, S°= 0.835; time plotted 50000-61000): a) w°=0.18; b) w°= 0.181.

FIG. 10 – Chaotic behaviour in the phase space *w, S* (I.C.: L°=1, S°=0.4, w°=0.2).

FIG. 11 – Bifurcation diagram for the parameter g (between 0.535 and 0.62) and the labour supply *S*.

FIG. 12 - *Time path* of the variables *L, S* compared with the equilibrium value $L^*=S^*= 0.832$ (g= 0.55).

FIG. 13 - Plot of the *time path* of the variables *w* compared with the equilibrium value $w^*= 0.182$ (g= 0.55).

5. Working of the model

In this section we try to provide an heuristic view of the working of the model (8). As we have seen, in presence of a sufficiently limited flexibility of the employment decisions of the firm (as measured though the g parameter), i.e. for $g<g_H$, the stable Walrasian equilibrium is destabilised: an initial excess demand for labour will not be reabsorbed. Two broad dynamical situations may then occur: i) provided that g is in a suitable intermediate window ($g_C<g<g_H$), an initial excess demand for labour gives rise to an increasing wage which eventually leads to a cyclical balance between demand and supply; Vice-versa, ii) when $g<g_C$, the increase of the wage does not end in a balanced (though cyclical) regime, but in chaotic fluctuations.

In substantive terms, the increase in the wage tends to discourage both the optimal demand and the optimal supply (this is due to the fact that we are in a case of low substitution between consumption and leisure) though in different measures depending on their parameters (i.e. technological returns to scale and elasticity of substitution between of consumption-leisure). The balancing effect of the reduction in both demand and supply occurs in a region which is far from the WE, and lies prevalently below the WE value: after the initial reduction in both desired demand and supply , their different "speeds " permit the occurrence of chaotic fluctuations inside a "trapping region", but the wage is never able to decrease sufficiently to come back next to (and beyond) its (unstable) WE value. For example, in the fig. 12, we see that the maximum peak of the oscillation of the labour supply lies always below its WE value. In other words the system, once that the WE is abandoned, behaves either regularly or chaotically, *fluctuating in a range of values of the employment and of the wages that are <u>on average</u> respectively lower and higher than the WE values.*

CHL find that[9] in the chaotic region the wage is on average exceeding the stable disequilibrium wage. This implies that (on average) an excess supply of labor over time. Their suggestion is therefore that the model endogenously display a Phillips-curve-type pattern, but also that in this case the revealed negative relationship between wage dynamics and unemployment which is persistent in the long period could not be interpreted by the policy-maker as a locus of equilibrium trade-offs between wage inflation and unemployment between which to choose. In fact the Phillips curve would only be "a by-product of price dynamics in a non-convex economy". But we recall that in the CHL model there is no WE, and therefore their reinterpretation of the Phillips curve could be ascribed to the specificity of their economic assumptions (non-convexity, no market equilibrium, etc.); in contrast, in our model there is a WE which can be destabilised by various factors, among which a low flexibility in the firms' employment decisions and a high speed of wage adjustment. Out of equilibrium, our economy can show a stable erratic pattern, the main economic features of which are: 1) a " dynamical Phillips-curve reminiscence", and 2) a behaviour of wage and employment which lie systematically respectively above and below the levels predicted by the Walrasian Equilibrium.

6. Conclusions

This paper has showed that regular fluctuations and chaotic behaviour of wages and employment may be a robust outcome of a market economy when consumption and leisure are low substitutes. In particular when the unique walrasian equilibrium of this economy is destabilised, then the economic variables (the wage, and the demand and and supply of labor) evolve toward a stable attracting region within which their motion is chaotic. In this region we

[9] This results seems mainly due, as already pointed out, to the special discontinuous production function, according to which a sufficiently high wage leads to zero production and, consequently, to a zero labor demand.

have shown that both the average wage and the average unemployment are larger than those existing in the Walrasian Equilibrium, suggesting possible empirical implications in terms of a 'reminiscence' of a Phillips curve.

In conclusion we summarise the following main results: 1) in the chaotic set there always is either excess labor supply or excess demand for labor along with wage changes. The wage always moves in opposite direction to the unemployment, but this relationship is persistent in the long-run, then it is not only a transitory disequilibrium phenomenon for a wage adjustment process converging to the equilibrium as postulated by the neoclassical interpretation of the Phillips curve; 2) the economy is driven away from the WE and trapped in a stable 'disequilibrium' wage set in which excess demand for (or even supply of) labor lead to wage changes which are: i) persistent in the long-period, and ii) of erratic time pattern, and finally where iii) on average the employment is largely smaller than that predicted by the WE.

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APPUNTI di RECUPERO

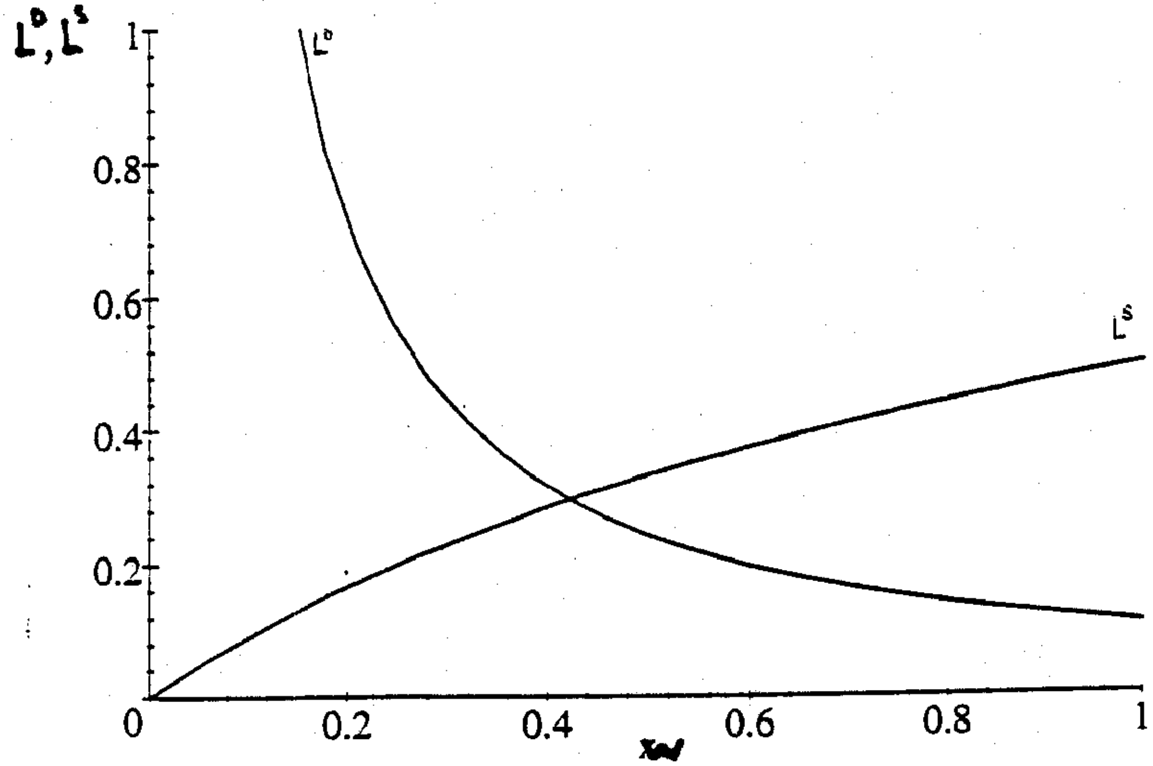
A little more flesh could be added to the bare bones of the theory expressed by eq (.). The addition of still more flesh would imply considering the possibility that......, as well as the possibility that the wages do not immeditaely adjust to labour market disequilibrium.

That these modifications do not alter the basic results (findings, outcomes) reached (obtained, attained, shown) in the present essay (paper, work) is shown in Fanti (1999).

FIG. 4 – Plot of the *time path* of the variables L, S, w : regular oscillations (g=1.066) (time plotted 0 –5200; the variable w is rescaled).

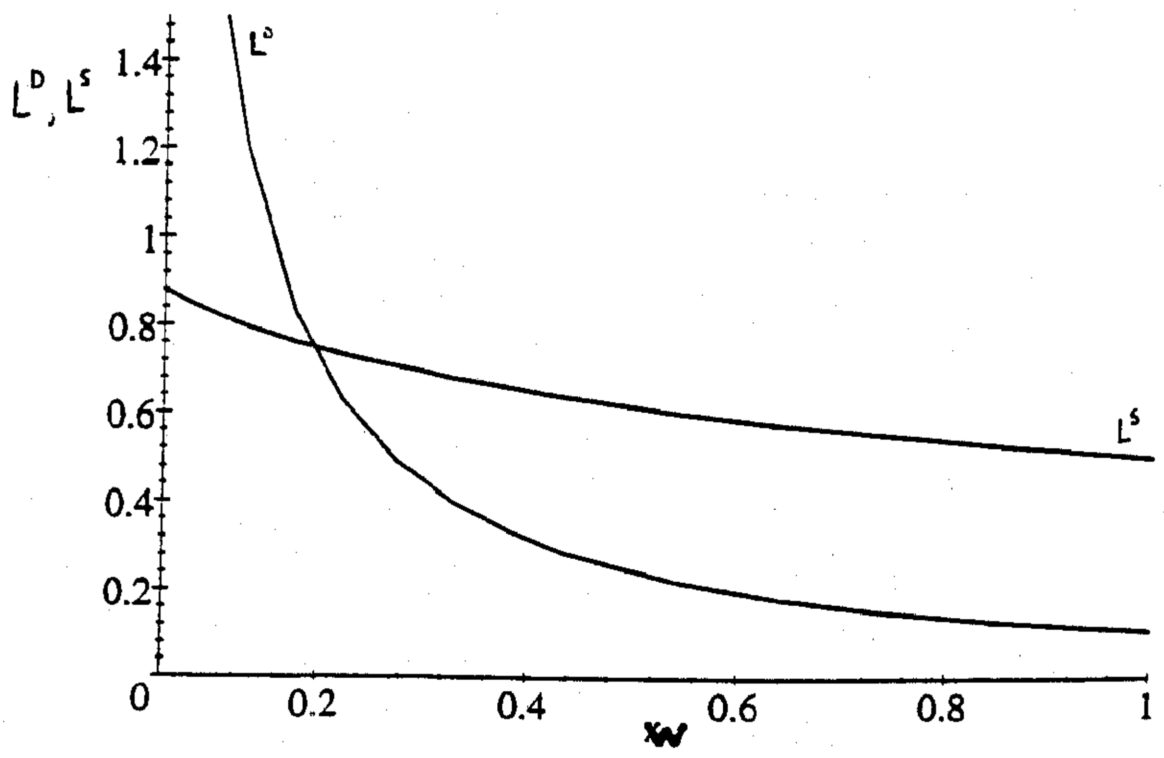
FIG. 8 – Plot of the *time path* of the variables L, S, w : irregular oscillations (g=0.55) (time plotted 0 -5200).

FIG. 11 – An enlarged window of the plot of the *time path* of the variables L, S, w : irregular oscillations (g=0.55) (time plotted 3100-4600).



L^D, L^S
1
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1
xw
L^D
L^S

FIG. 1



L^D, L^S
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1
xw
L^D
L^S

FIG. 2

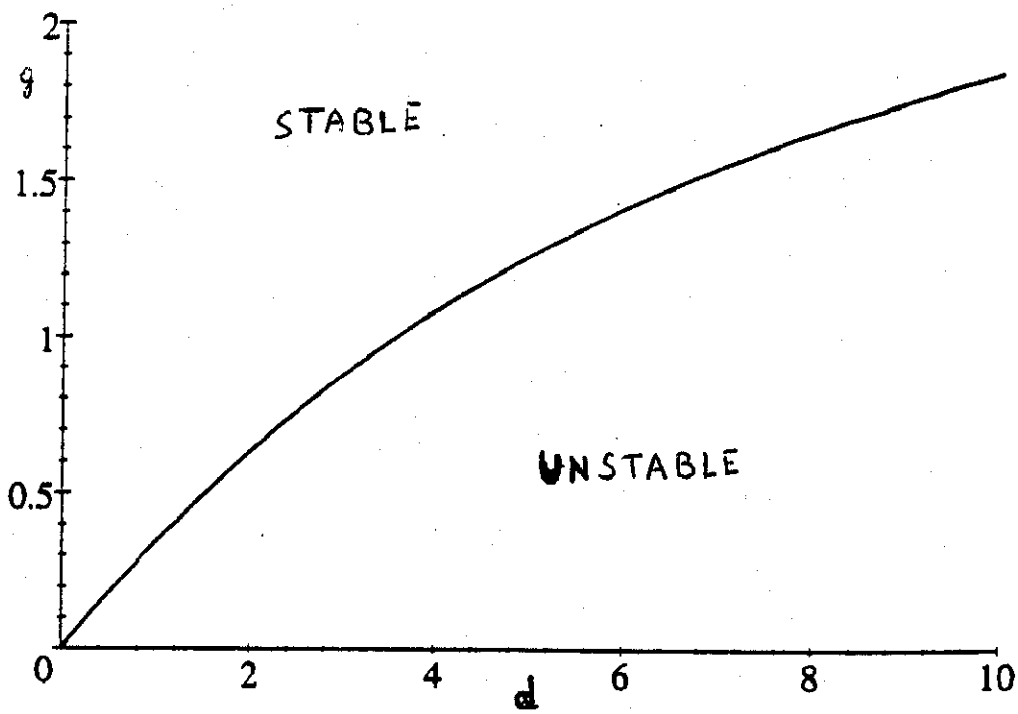


FIG. 3

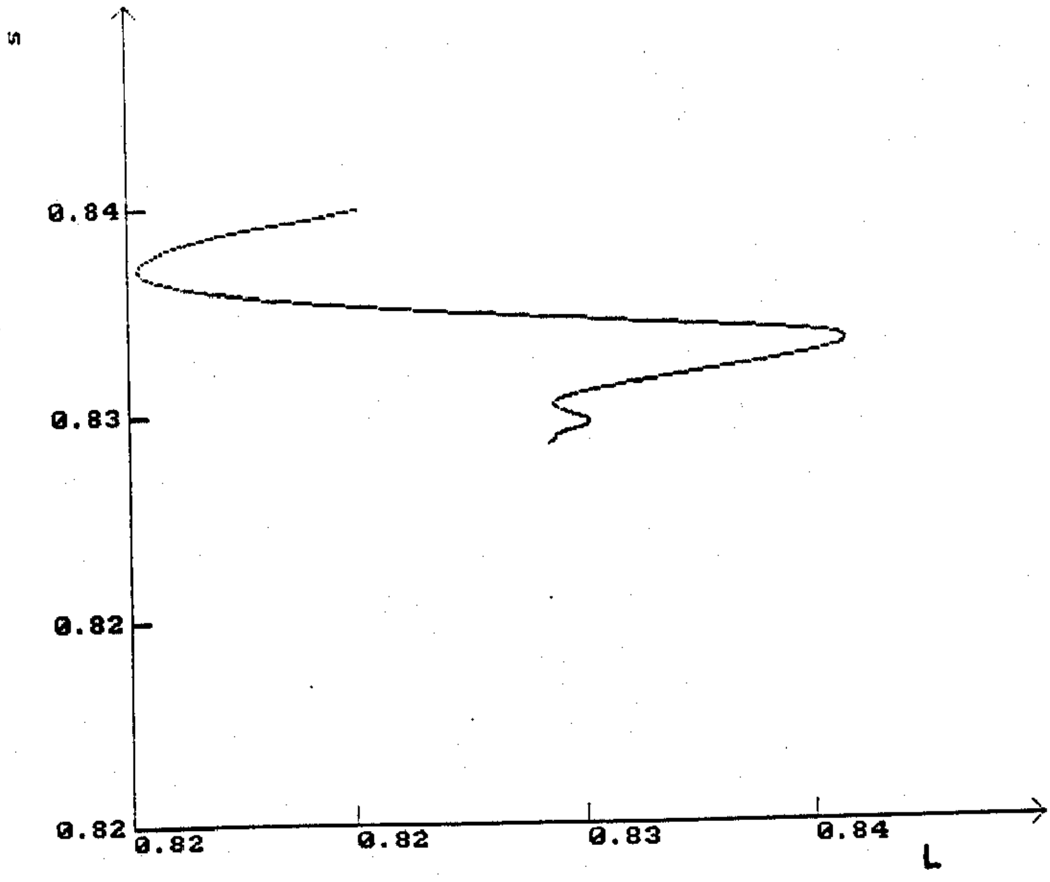

S
0.84
0.83
0.82
0.82
0.82
0.82
0.83
0.84
L

FIG. 4

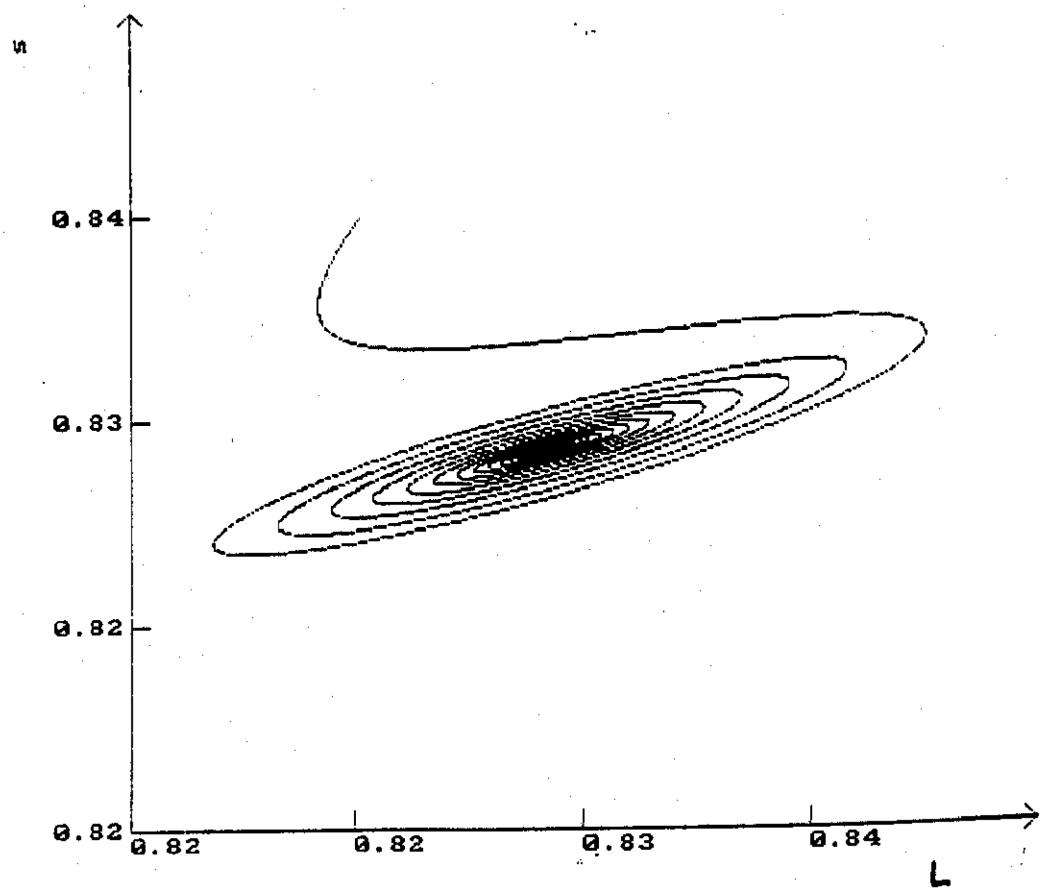

S
0.84
0.83
0.82
0.82
0.82
0.82
0.83
0.84
L

FIG. 5

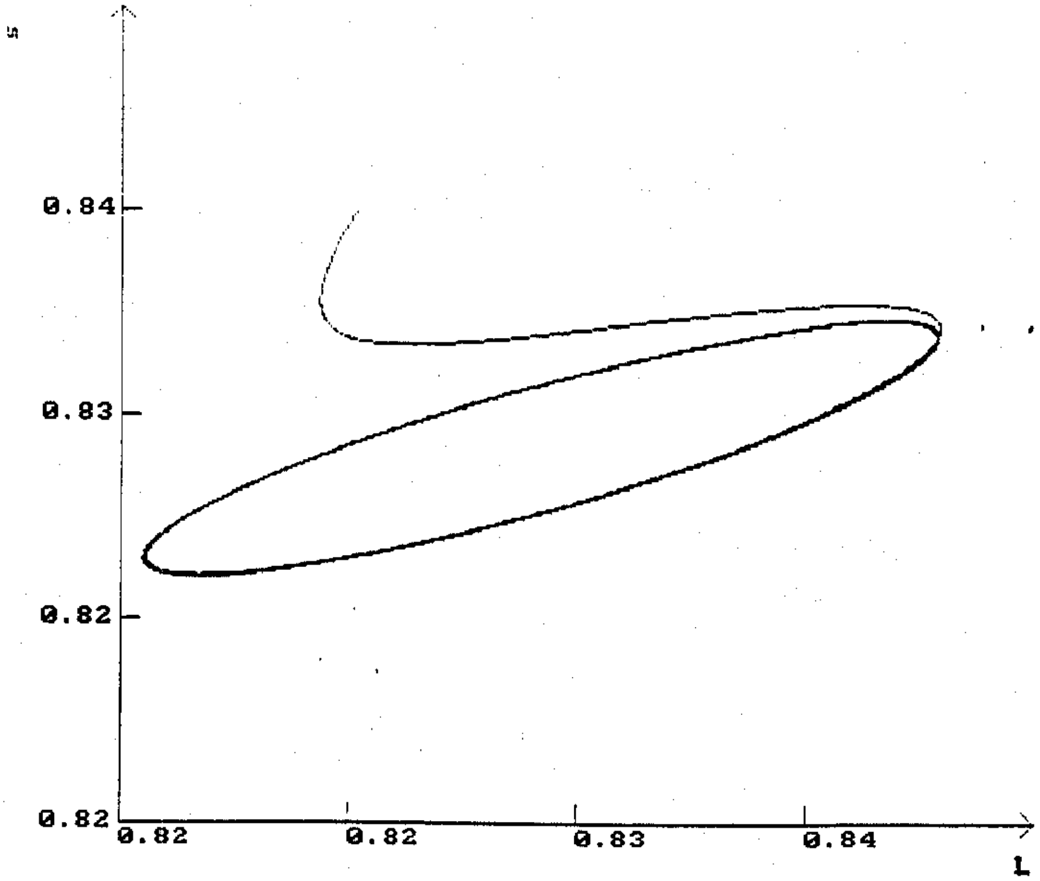

S
0.84
0.83
0.82
0.82
0.82
0.82
0.83
0.84
L

FIG. 6

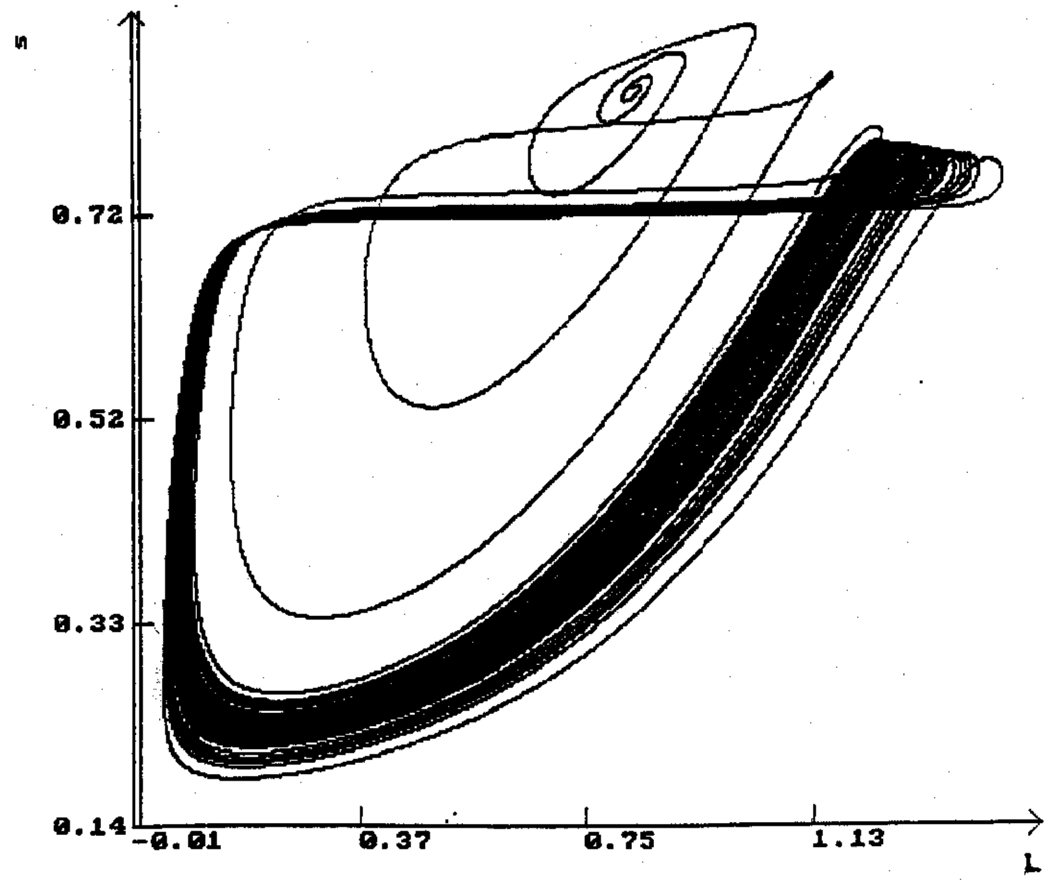

S
0.72
0.52
0.33
0.14
-0.01
0.37
0.75
1.13
L

FIG. 7

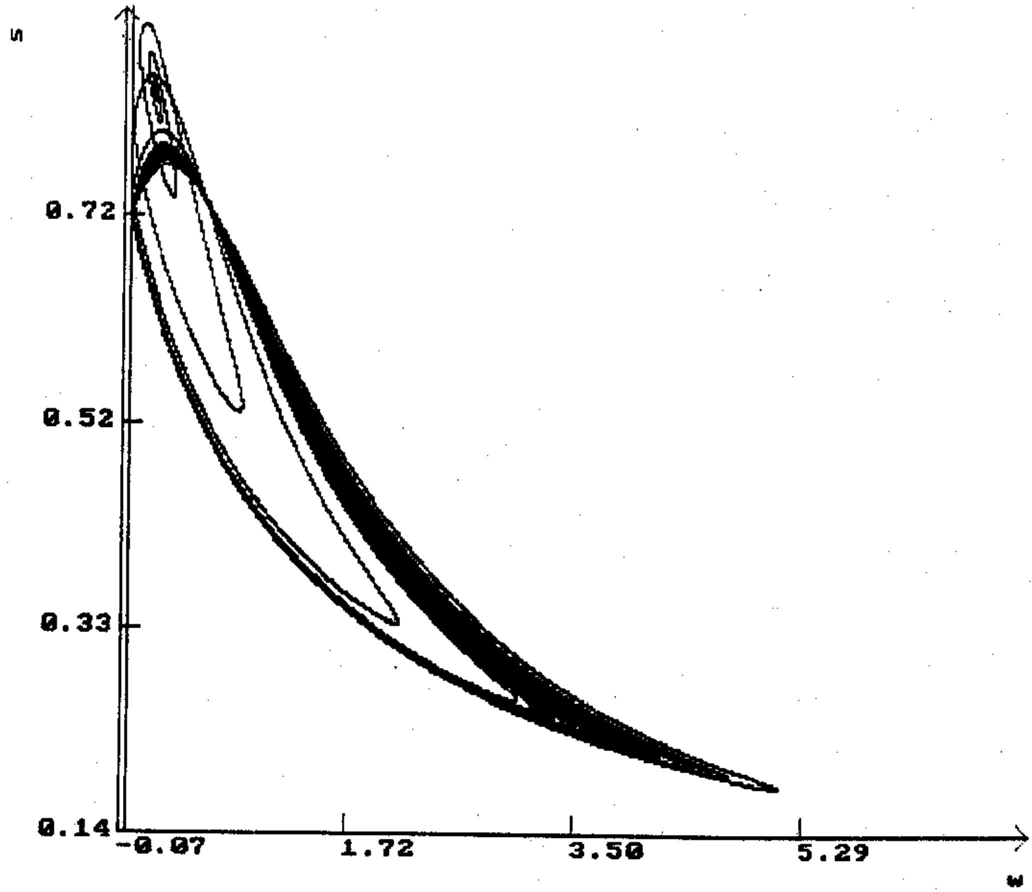

s
0.72
0.52
0.33
0.14
-0.07
1.72
3.50
5.29
w

FIG. 8

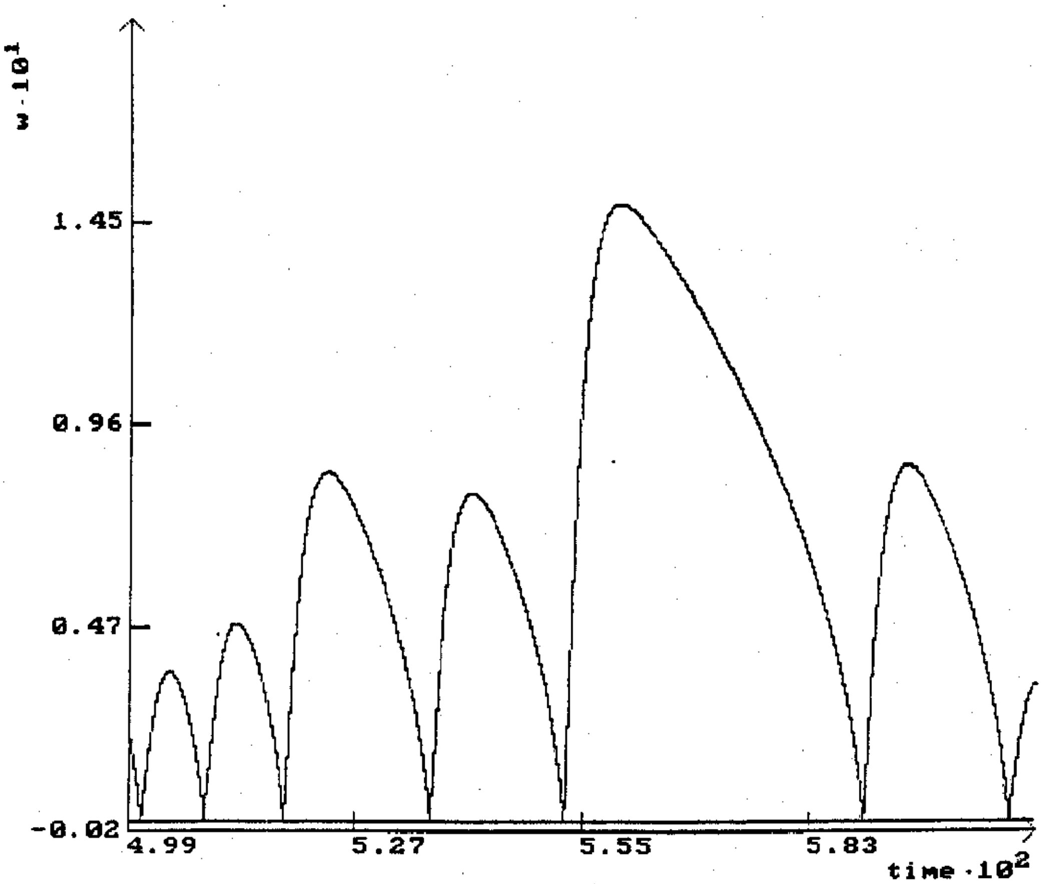
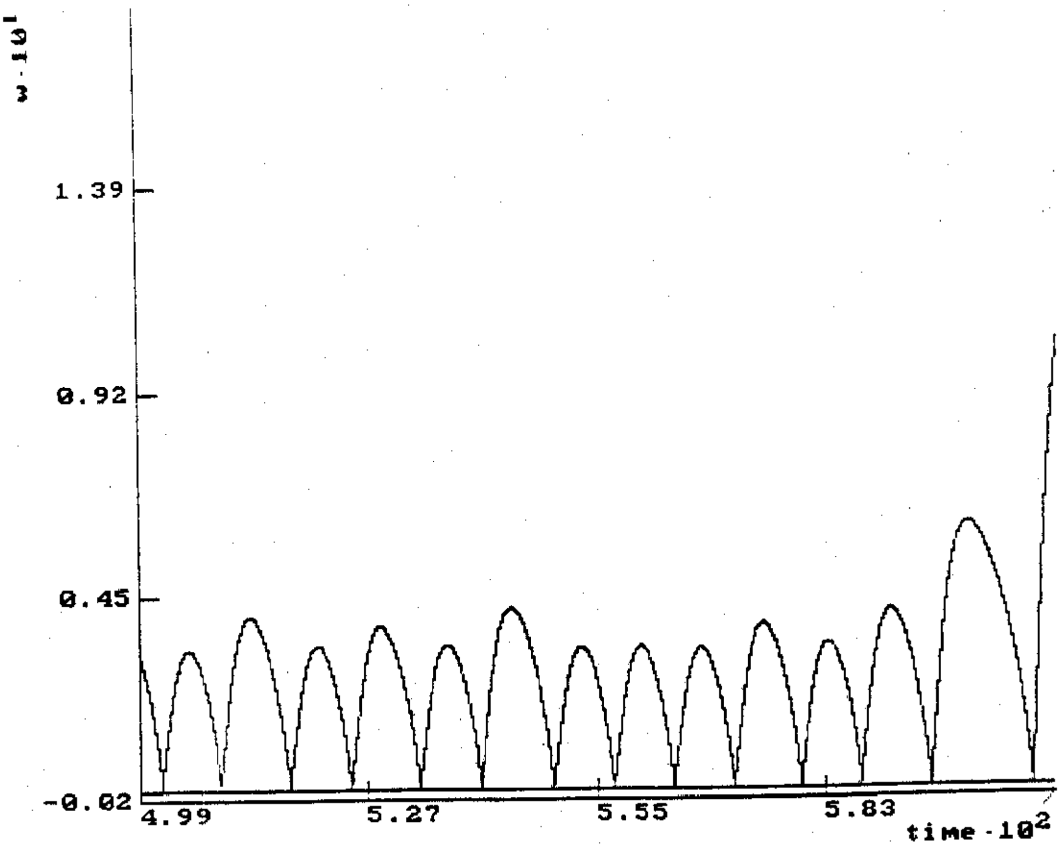
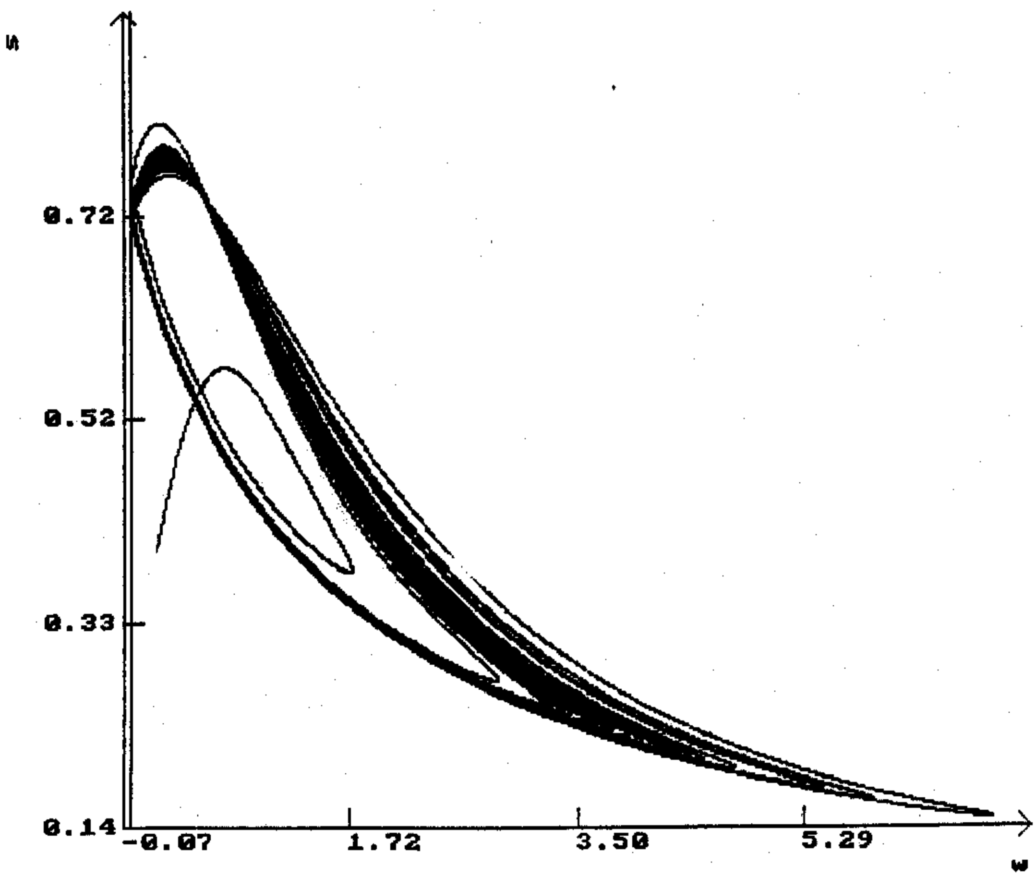

w·10¹
1.45
0.96
0.47
-0.02
4.99
5.27
5.55
5.83
time·10²

FIG. 9a



ω·10¹
1.39
0.92
0.45
-0.02
4.99
5.27
5.55
5.83
time·10²

FIG. 9b



0.72
0.52
0.33
0.14
-0.07
1.72
3.50
5.29

FIG. 10

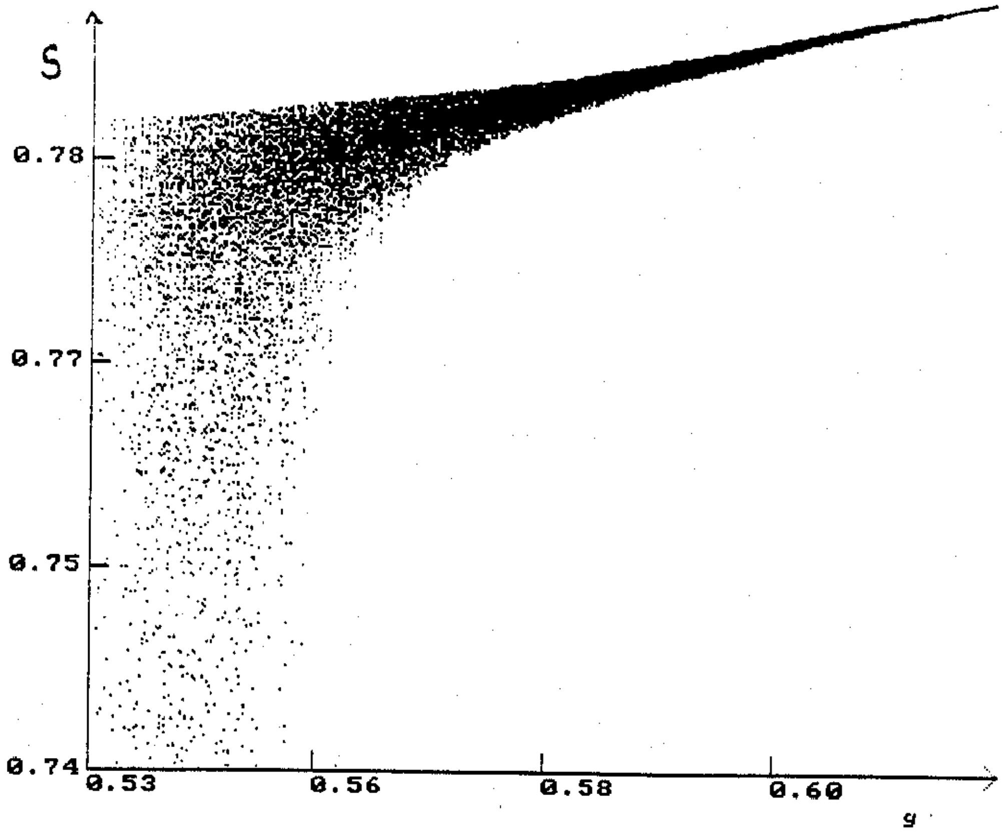

S
0.78
0.77
0.75
0.74
0.53
0.56
0.58
0.60
g

FIG. 11

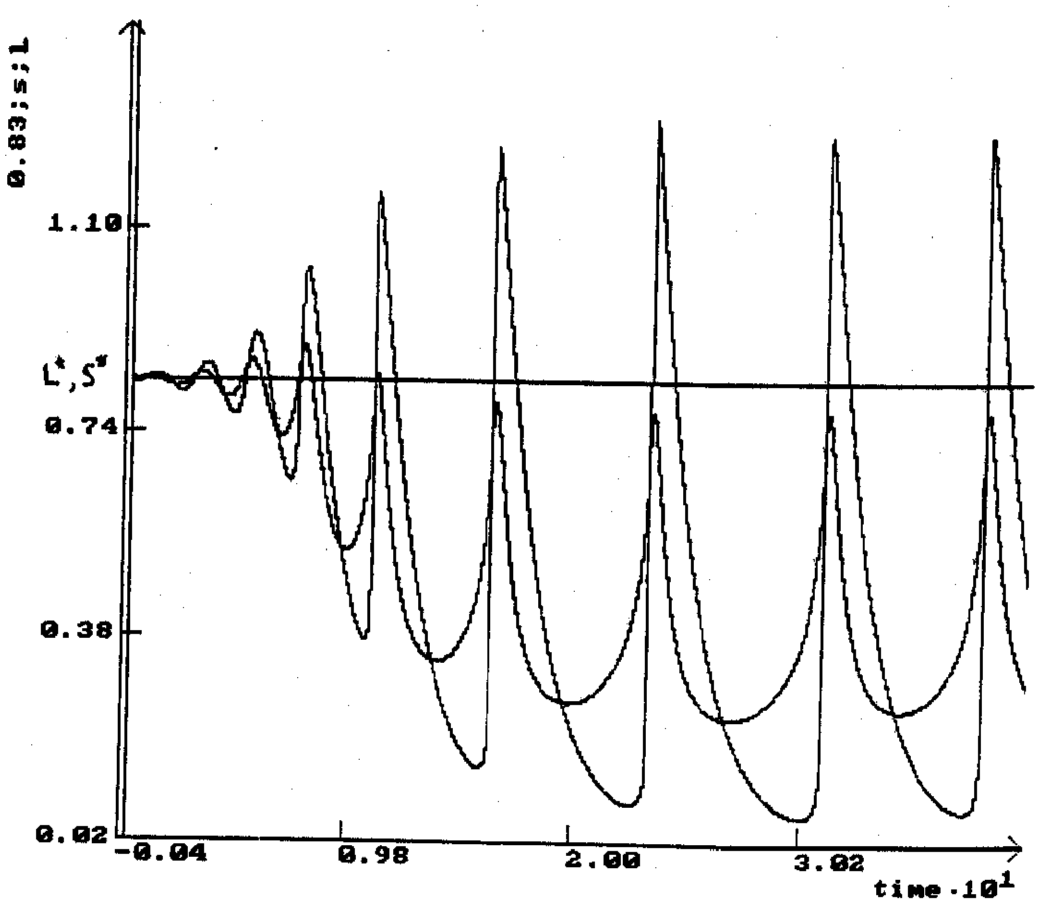

0.83;s;L
1.10
L*,S*
0.74
0.38
0.02
-0.04
0.98
2.00
3.02
time·10¹

FIG. 12

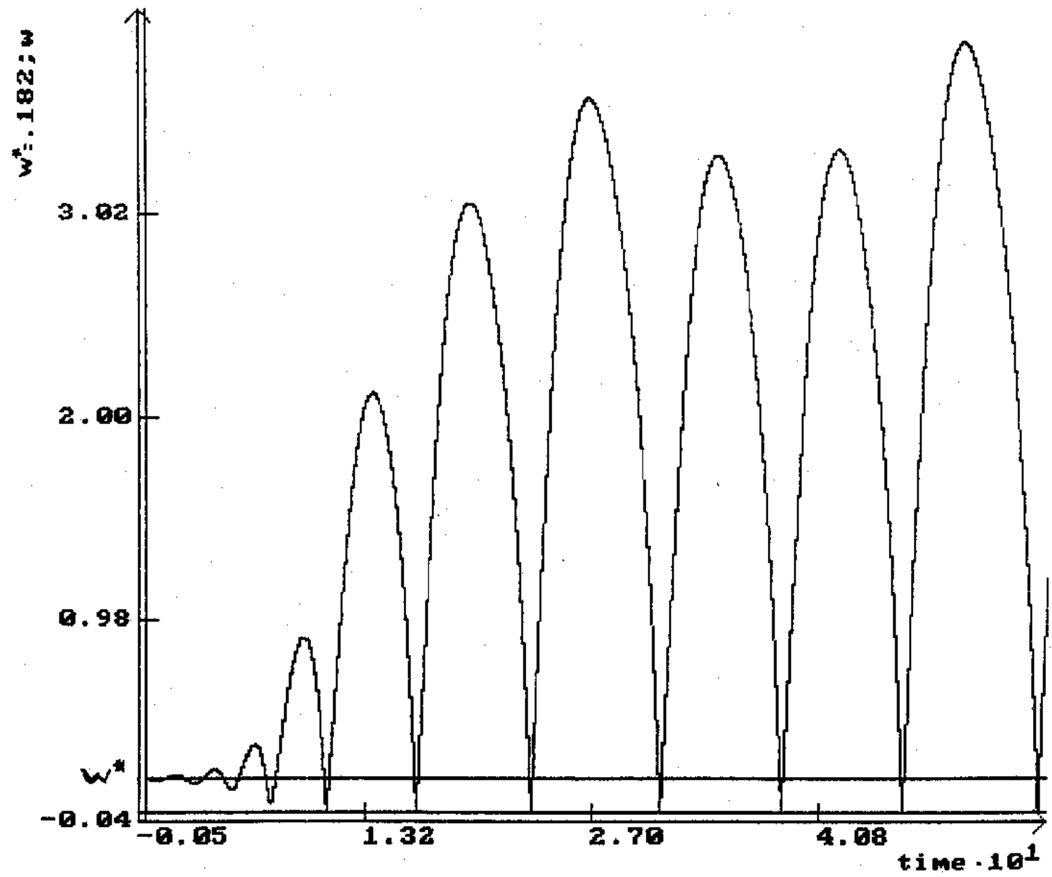


FIG. 13