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ABSTRACT

The analytical detection of sustained oscillations via Hopf bifurcations in ordinary differential equation systems of dimension higher than the second is a first concern in macro-economic dynamics. This paper aims to offer a unified perspective of the subject, by reviewing some useful result of the recent mathematical literature and by clarifying the relations between the body of stability theorems and the notions of simple and general Hopf bifurcations. A Liénard-Chipart version of the Routh-Hurwicz-type theorem by Liu (1994) is proven which appears of considerable usefulness in applications. Subsequently, the notion of stability boundary, which provides a powerful tool often making the detection of the bifurcation quite easy, is carefully discussed as well. Delay and heterogeneous systems seem to be areas which could benefit in a substantial manner from the results discussed here. A final illustration is given on two delay-models: a Solow-type demo-economic model and a Kaleckian extension of the Lotka-Volterra-Goodwin model.

1 Introduction

In the recent years there has been a renewed interest for economic growth and the economic business cycle. The development of the "endogenous growth theory" and of the "real business cycle theory", but also the renewed interest in Goodwin-type and keynesian macrodynamics, are main outcomes of this debate. As a starting point it is worth to remark that, from a theoretical point of view, there is only one mathematical notion of cycle which is relevant for applied modelers (independently on the actual context, physical, biological or economical): that of (asymptotically) stable limit cycle[1]. This is one of the reasons why the well known conservative Lotka-Volterra cycle, despite all its indisputable merits for the developments of the endogenous theory of fluctuations in applied sciences, is not a "good" fluctuation model.

The problem of the detection of stable limit cycles, i.e. of persistent periodic behaviours, in continuous-time systems, is intimately connected with the notion of Hopf bifurcation. Broadly speaking, two main tools are available, in the standard tool-box of modelers for the detection of limit cycles. As long as we are confined to 2-dimensional systems, the existence of closed orbits can be established via the Poincaré-Bendixson theorem. In those cases in which the Poincaré-Bendixson theory applies, the use of bifurcation theory usually does not provide further insights in models already known. Viceversa at dimension three or higher, the Poincare-Bendixson theorem cannot be applied anymore, and the Hopf theorem remains the key-tool to establish the existence of closed orbits.[2] The Hopf theorem can, in principle, establish the existence of (local) persistent periodic behaviours in whatever dimension. In general, given a dynamical system tuned by a paramenter μ, and having an isolated equilibrium E_1, a Hopf bifurcation occurs at E_1 when, a simple pair of complex eigenvalues of the linearised system "crosses the imaginary axis". To ascertain the existence of a Hopf bifurcation we therefore have to investigate the behaviour of eigenvalues as functions of μ, in order to see whether values of the μ parameter exist which give rise to the crossing.

As the dimension of the system to be analysed increases, the computation of the conditions stated by the Hopf theorem becomes more and more difficult, thereby making hard the analytical detection of the bifurcation. In particular, the "direct" criterion for a Hopf bifurcation, based on the explicit determination of the eigenvalues, is already nasty at dimensions three and four and totally impossible at dimension five or more. This makes it unavoidable to look for "indirect" criteria. Numerical algorithms can of course be used but they pay the price of loosing the possibility to provide economic interpretations of the bifurcation process. These difficulties seem to have been the responsible of the fact that most investigations have been so far based on oversimplified mathematical models, usually characterized by very low dimensions.[3] As far as

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[1] We only deal here with "regular" periodic oscillations. We are not interested, in this paper, in complex behaviours.

[2] Regarding the recent advances in the area of business cycle models (optimizing and non-optimizing) Semmler (1994) states, "This has been made possible by bifurcation theory ...".

[3] Even the more recent and influential textbooks in economic dynamics, despite their indisputable merits, have emphasised this difficulty. In fact, for what concerns the issue of the detection of limit cycles in higher-dimensional dynamical systems, Gabisch and Lorenz (1989, p. 166) state: "*The three-dimensional case is slightly*

we know, there are practically no macro-economic applications of the Hopf theorem to four (or higher) dimensional systems, unless under special forms of the Jacobian matrix. In some special (although relevant) cases, as that of the optimal economic control models with some rate of future discount, it has been possible (Dockner (1985)) to ascertain the existence of the Hopf bifurcation in a "direct" manner. This was due to the properties of the hamiltonian matrix (i.e. zero trace) which allows the explicit computation of eigenvalues. Many economic problems have been investigated, by ascertaining the Hopf bifurcation in such a direct manner (Wirl, 1991; Dockner and Feichtinger, 1991). Another instance of "direct" detection of a Hopf bifurcation is represented by the multisector neoclassical optimal growth model (Gandolfo 1996, ch. 25), where the bifurcation is found via a special triangular form of the Jacobian matrix. But in general we must resort to "indirect" methods to ascertain the Hopf bifurcation[4]. Indirect methods usually exploit the Routh-Hurwicz stability theorem (for instance Farkas and Kotsis (1992), or Asada and Semmler (1995), in 3-dimensional problems). Farkas and Kotsis (1992) is, to our knowledge, the only instance at dimension four. An important question is then: which are the relations between the appearance of a Hopf bifurcation and the Routh-Hurwicz criterion? Can stability criteria be used to detect Hopf bifurcations in arbitrarily large dimensions? These questions have been attacked in Liu (1994), who has clarified the relation between the Routh-Hurwicz theorem and the Hopf bifurcation.

Starting from Liu (1994), the present paper aims to provide a unifying perspective of the problem of the analytical detection of Hopf bifurcations, in order to make the Hopf theorem an operative tool also at dimensions greater than four. Following Liu (1994), we distinguish two types of Hopf bifurcation: the Simple Hopf bifurcation (SHB), which occurs when a pair of complex eigenvalues crosses the imaginary axis while all other eigenvalues have negative real parts, and the general Hopf bifurcation (GHB). By systematically using the notion of SHB several noteworthy facts are proven.

First, if all the coefficients a_i of the characteristic polynomial (CP) at the relevant equilibrium point are strictly positive, we remark that SHB and GHB are equivalent at dimensions smaller or equal four. We then show that, for instance, to ascertain the existence of a local limit cycle in a fourth dimensional system we only need to obtain the annihilation of a third-order (Routh-Hurwicz) determinant, which is surely a feasible task.

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more difficult but still analytically computable....In higher dimensional systems ($n \geq 4$) *the bifurcation values can often be calculated only by means of numerical algorithms*". Similarly Lorenz (1993, p. 101) notices: "*Though the applications of the Hopf bifurcation theorem (and expecially its existence part) are generally not restricted to low-dimensional dynamical systems, the conditions for the existence of the bifurcation can be shown to be fulfilled without difficulty only in two and three-dimesional cases. In higher-dimensional systems the bifurcation values can often be calculated only by means of numerical algorithms*". And Gandolfo (1996, p. 478-479): "*The huge value-added of the Hopf bifurcation theorem over the standard planar theory of oscillations lies in the fact that this theorem can be applied to higher-dimensional systems. However, also the application of the existence part of the Hopf bifurcation theorem often becomes analytically intractable for systems of dimension higher than the third...For fourth- and higher order equations, the problem becomes practically intractable from the algebraic point of view, except in particular cases*".

[4] We will not be involved with related topics, such as the stability of the emerging cycles.

Subsequently, by exploiting the Lienard-Chipart stability theorem we prove an extension of the theorem of Liu, which detects simple Hopf bifurcations via the Liènard and Chipart rather than the Routh-Hurwicz theorem. This extension appears to be of considerable usefulness in applications, as it shows that only $n/2$ conditions are needed to detect an SHB at dimension n.

But in effect we can go much further. We show that much simpler conditions can be obtained if we concentrate on Simple Hopf Bifurcations of systems which could be viewed as *parametric perturbations* of a known *stable* system. This case is so typical of the modeller's activity to be more than satisfactory for the goals of this paper. In fact, in this latter case (see MacDonald 1989), exactly as it happens while investigating the dependence of stability on one (or more) parameters, all what we need is to find, in the parameter space, the curves that bound regions of stability. We argue that, whenever we start from a point in the parameter space in which the system is stable, then, provided we can rule out zero-eigenvalues bifurcations, the points belonging to the locus $\triangle_{n-1} = 0$ ($\triangle_{n-1}$ is the higher order Routh-Hurwicz determinant) are SHB points. The extent of this simplification for the detection of the SHB is surprising: we do not need anymore to evaluate a large number of Routh-Hurwicz determinants, but only one, namely $\triangle_{n-1}$.

This simple methodology for detecting local limit cycles has wide applicability. In particular it is fruitful to study Hopf bifurcations in ordinary differential equation (ODE) problems arising from an underlying "reducible" distributed delay system with erlangian kernel. Distributed delay system with Erlangian kernels may be reduced (Mac Donald 1978, Farkas and Kotsis (1992)), to an autonomous ODE system. This new ODE system actually has a dimension which is greater of that of the corresponding delay system, but: a)it will preserve the equilibria of the underlying unlagged system; b)it adds to the original system new equations which are simple, at least for what regards the practical computations involved in local stability and Hopf bifurcation analyses.

From the economic point of view, the use of "erlangian" distributed lags allows to represent in a fair manner two realistic elements, so far neglected in economics mostly because of the involved analytical complexity: the heterogeneity of agents and their tendency to react to economic impulses with different patterns of lag. In particular the equivalence (Invernizzi and Medio (1991)) between a single "representative" agent which reacts along a continuous gamma distributed lag and an indefinitely large number of agents reacting with different discrete lags whose lenghts are randomly distributed among agents according to a gamma distribution, permits to avoid the usual "rough" dynamic aggregation. In many cases, starting from an existing "roughly" aggregated model whose stability is known, we need to investigate whether stability is preserved under more general and realistic assumptions, such as heterogeneity and/or delayed responses of economic agents. The results presented here could permit to deal with this task, and to prove the existence of endogenous oscillations, with algebraic computations which are much simpler than usually believed (and often permit substantive economic interpretations).

These facts are illustrated by means of two models which provide nice instances of computable Hopf bifurcation: i) a demo-economic extension of the Solow's model (1956) in which the rate of change of the labour force is realistically related to the past fertility; ii) a 5-dimensional kaleckian version of the Goodwin's model. These illustrations show how the notion of stability boundary may be applied in order to detect SHB not only in the standard case in which the underlying unlagged model is (linearly) stable, but also in the case of neutral stability.

The present paper is organised as follows. In section two the Hopf theorem and the related notions of simple (SHB) and general (GHB) Hopf bifurcations are reviewed. In section three, after having reviewed the Routh-Hurwicz and Liénard-Chipart stability test, we review the Routh–Hurwicz type theorem for the detection of Hopf bifurcation by Liu (1994), and demonstrate that it can be simplified, by proving a Liénard-Chipart-type version of the same theorem. In section four we discuss the detection of simple Hopf bifurcations in relation to the notion of stability boundary. Section five is devoted to the economic illustrations. Conclusive remarks are left to the final section.

2 The Hopf theorem: simple and general Hopf bifurcations

A simplified formulation of the Hopf theorem is the following (for rigorous formulations and proofs see Guckenheimer and Holmes (1983), Marsden and McCracken (1976)).

THEOREM 1. *Let be given the dynamical system: $\dot{X} = f_\mu(X)$, parameterised by a "tuning" (scalar) parameter μ, and having an equilibrium E_1 at $(X_0(\mu), \mu)$. Let f be of class C^∞. The system has a Hopf bifurcation at $(X_0(\mu_0), \mu_0)$, if:*

1) the system has in (X_0, μ_0) a simple pair of purely imaginary eigenvalues[5] $\lambda(\mu), \bar{\lambda}(\mu)$, and no other eigenvalues have zero real parts.

2)the "bifurcating" pair $\lambda, \bar{\lambda}$ satisfies the "nonzero speed condition":

$$(d\,\mathrm{Re}(\lambda(\mu))/d\mu)_{\mu_0} = (d\,\mathrm{Re}(\lambda(\mu_0))/d\mu) \neq 0 \qquad (1)$$

As prescribed by theorem 1, the problem of the detection of the bifurcation is to be solved in two steps: first by checking for the existence of a pair of purely imaginary eigenvalues of the characteristic equation (and that no other eigenvalues have zero real parts); second by applying the "test of nonzero speed" (1), aimed to check that the involved pair of complex eigenvalues actually crosses the imaginary axis with nonzero speed. Notice (see Farkas (1995)) that the nonzero speed condition is actually not necessary for having a Hopf bifurcation. It is purely a genericity requirement.

[5]It is possible to show that a smooth curve of equilibria $(x(\mu), \mu)$ exists and satisfies $x(\mu_0) = x_0$. Moreover $\lambda(\mu), \bar{\lambda}(\mu)$ vary smoothly with μ.

The aforementioned form of the Hopf theorem is quite general. There are other forms (for instance Farkas, 1995) based on the stronger requirement that the remaining $(n-2)$ "non bifurcating" eigenvalues have negative real part. This latter formulation is closer to the "modeller view". Modelers in fact usually do as follows: given a nonlinear system having at least an equilibrium point E_1, they discuss the condition for the local stability of E_1 in terms of the parameter μ. The points in the parameter space that bound the stability region and in correspondence of which stability is lost due to the crossing of the imaginary axis by one (or more) complex pair are candidates to host a Hopf bifurcation. This approach is clearly special if compared to Theorem 1. The following definition due to Liu (1994) is useful in order to organise the present discussion:

DEFINITION 1. *(simple Hopf bifurcation) a dynamical system with an equilibrium point E_1 undergoes a simple Hopf bifurcation (SHB) at E_1 when a simple pair of complex conjugate eigenvalues of the jacobian matrix $J(E_1)$ passes through the imaginary axis while all other eigenvalues have negative real parts.*

The previous definition distinguishes *simple Hopf bifurcations* from other types of Hopf bifurcations, characterised by other eigenvalues on the right half plane (call them *General Hopf Bifurcation,* or GHB). From the point of view of modelers the SHB is the most relevant type of Hopf bifurcation. In fact it regards the case in which, for those parameter constellations for which the bifurcation is supercritical, the emerging periodic orbit will be asymptotically stable, and hence "observable", physically or numerically (Liu, 1994). Moreover it largely corresponds to the typical way of viewing the operating of the real world: we usually believe (more or less awarely) in a "stable" world but worry about the possibility that instability (mainly observable as fluctuations) occurs and hence seek the conditions needed to preserve stability. Finally, the notion of of SHB is also operative from the point of view of the practical detection of the bifurcation. In fact the detection of *simple Hopf bifurcations* at E_1 needs to check to existence of a loss of stability of E_1 having a special nature, i.e. a loss of stability driven by "movements" of a simple pair of complex eigenvalues. This naturally fills the bridge between Hopf bifurcation analysis and the body of theorems for ascertain local stability of dynamical systems in higher dimension, such as the criteria by Routh-Hurwicz, Liénard-Chipart and so on. As it is known the essence of these results is that they by-pass eigenvalues for ascertain stability: they are based on the pure inspection of some set of functions of the coefficients of the characteristic equation: the Routh-Hurwicz determinants.[6] Before continuing let us recall the conditions for a SHB:

$a1$) the system has in (X_0, μ_0) a simple pair of purely imaginary eigenvalues $\lambda, \bar{\lambda}$, and no other eigenvalues have zero real parts.

$a2$) the "nonzero speed condition" is satisfied:

[6] On the contrary the problem of the detection of GHB is not related in an evident way to the problem of stability.

3 Stability versus bifurcations: detection of SHB via Routh-Hurwicz-type theorems

3.1 Basic facts from stability theory

We recall those facts of stability theory which are essential for the subsequent discussion on SHB. Let $P_J(\lambda)$ be a characteristic polynomial (CP) ascertaing the local stability of an equilibrium point E_1 of an n-dimensional dynamical system:

$$\begin{aligned} P_J(\lambda) &= \det\left(J(E_1) - \lambda J\right) \\ &= a_0\lambda^n + a_1(\mu)\lambda^{n-1} + a_2(\mu)\lambda^{n-2} + \dots + a_{n-1}(\mu)\lambda + a_n(\mu) \end{aligned} \tag{2}$$

where $J(E_1)$ is the underlying jacobian matrix. We set for simplicity $a_0 = 1$, and wrote $a_i = a_i(\mu)$ to denote that the coefficients are functions of some (scalar) parameter μ. The equilibrium point E_1 is said to be locally asymptotically stable (LAS) (alternatively $P_J(\lambda)$ is said *linearly stable,* or also *strictly Hurwicz,* see MacDonald (1989), 60) if all its eigenvalues have negative real parts. The Routh-Hurwicz (RH) theorem (Gantmacher (1959), McDonald (1989)) provides a necessary and sufficient condition for the local stability of the polynomial $P_J(\lambda)$ giving also a practical stability test. Given the so called Routh table:

$$R = \begin{pmatrix} a_1 & a_3 & a_5 & a_7 & a_9 & \dots \\ a_0 & a_2 & a_4 & a_6 & a_8 & \dots \\ 0 & a_1 & a_3 & a_5 & a_7 & \dots \\ 0 & a_0 & a_2 & a_4 & a_6 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

the Routh-Hurwicz test states that $P_J(\lambda)$ is LAS iff the determinants Δ_i of the first n principal minors of the Routh table are strictly positive.

3.2 Necessary conditions for stability; the Liénard-Chipart conditions

An obvious necessary condition for stability is $a_n > 0$. As it holds: $\Delta_n = a_n\Delta_{n-1}$, then, provided $a_n > 0$, we only need to consider $(n-1)$ RH determinants. A more powerful necessary condition comes from the fundamental theorem of algebra which shows that in order E_1 be *LAS* all its coefficients a_i must be strictly positive. The proof is immediate (Gantmacher 1959): under stability $P_J(\lambda)$ may be written as a product of factors of the forms $(\lambda + u)$ or $\left(\lambda^2 + v\lambda + w\right)$, with $u > 0$, $v > 0$, $w > 0$.[7] Hence, the set of strictly Hurwicz CP's is a subset of the set of CP's with positive coefficients. This often neglected fact is of great help from many points of view.

[7] This result gives a simple test: if only one of the coefficients is positive then the system is unstable.

Obviously the converse is not true: positive coefficients are not sufficient to imply stability: this is why we need RH-type theorems. As a simple instance of the previous statement consider the 3-rd order polynomial:

$$P_J(\lambda) = \lambda^3 + a_1(\mu)\lambda^2 + a_2(\mu)\lambda + a_3(\mu)$$

The RH theorem gives the stability conditions: $\Delta_1 > 0$; $\Delta_2 > 0$; $\Delta_3 > 0$. Clearly $a_3 > 0$ (as also implied by $\Delta_3 = a_3\Delta_2$). Moreover $\Delta_1 = a_1$ implying the positivity of the second coefficient. Finally, from $\Delta_2 = a_1a_2 - a_3 > 0$ it follows: $a_2 > \frac{a_3}{a_1} > 0$ implying the positivity of a_2 as well. This shows that in the set of the parameter space in which the CP is LAS , all the coefficients a_i are "forced" to be strictly positive.

REMARK 1. *If all the coefficients a_i are strictly positive Descartes rule tells us that there are no positive real roots. Hence if there are real roots these are always negative. This implies in turn that if a given CP has only real roots, then the positivity of the coefficients becomes an IFF condition for stability, rather than simply a necessary condition.*

When all (or at least some) the coefficients of the CP are positive then the n conditions of the RH theorem are no longer independent and the RH test may be replaced by the more "economical" Liénard-Chipart (LC) test. The LC conditions may be espressed in any one of the following four alternative versions (Gantmacher, 1959):

a)$a_n > 0$; $a_{n-2} > 0$; ...; $\Delta_1 > 0$; $\Delta_3 > 0$; ...

b)$a_n > 0$; $a_{n-2} > 0$; ...; $\Delta_2 > 0$; $\Delta_4 > 0$; ...

c)$a_n > 0$; $a_{n-1} > 0; a_{n-3} > 0$; ...; $\Delta_1 > 0$; $\Delta_3 > 0$; ...

d)$a_n > 0$; $a_{n-1} > 0; a_{n-3} > 0$; ...; $\Delta_2 > 0$; $\Delta_4 > 0$; ...

The implications of the fact that the necessary condition $a_i > 0$ for all i is satisfied are twofold. On the algorithmic side the analysis of stability is sharply simplified: we use the LC rather than RH conditions.[8] On the theoretical side we arrive to the true core of the stability problem. We may in fact look at the stability conditions in the following way. A given CP of degree n may admit real roots (at most n) and complex roots (occurring in pairs): at most $n/2$ pairs if n is even and, at most $(n-1)/2$ if n is odd. The role of the stability condition is that of "controlling" the activity of these roots. We may be specific on the way the stability conditions perform this control: from the necessary condition we know that if $a_i > 0$ for all i, then, provided all the eigenvalues are real, then they would all be negative, ensuring stability. This shows that the necessary condition "controls" the activity of the real roots in the sense that if the roots of the CP are real then the necessary condition $a_i > 0$ becomes necessary and sufficient. Viceversa the necessary condition does not control the activity of the complex roots, thereby calling for extra conditions given by the LC determinants. Hence, to ensure stability in

[8] In the end we need the examination of only $n/2-1$ (n even) or $(n-1)/2$ (n even) extra conditions. For instance for $n = 6$, provided $a_i > 0$ we need to examine: $\Delta_3 > 0, \Delta_5 > 0$. From the practical point of view it is highly recommendable to study in advance the conditions for the positivity of the coefficients of the CP.

presence of the necessary condition, which rules out possible disturbances to stability caused by real eigenvalues, we need a set of independent conditions the role of which is to ensure that all the possible complex pairs have negative real parts. Hence, we need *at most* $n/2$ conditions if n is even and $(n-1)/2$ if n is odd, which is exactly the number of LC conditions.

3.3 Routh–Hurwicz and Liénard-Chipart-type theorems for the detection of Simple Hopf Bifurcations

The relation between the RH theorem and the SHB was used by modelers since time ago (McDonald 1989). The following result by Liu (1994) is a Routh-Hurwicz-type theorem for the detection of a SHB which states this relation in formal terms:

THEOREM 1. *(Liu 1994) The conditions* $a1), a2)$ *(see section 2) for a SHB at the point* μ_0 *are equivalent to the following conditions:*

$$\begin{array}{l} b1)\ \Delta_1(\mu_0) > 0\,;\quad \Delta_2(\mu_0) > 0 ... \Delta_{n-2}(\mu_0) > 0\,;\quad \Delta_{n-1}(\mu_0) = 0 \\ b2)\ \left(\frac{d\Delta_{n-1}}{d\mu}\right)_{\mu=\mu_0} \neq 0 \end{array} \qquad (3)$$

We sketch the basic ideas of the proof by Liu, by showing that the conditions $b1, b2)$ are necessary and sufficient for $a1)$, i.e. to ensure that a simple complex pair passes through the imaginary axis while all other eigenvalues have negative real parts. The condition $b2)$ defines the test for nonzero speed.

Only if part. Let us assume that $a1), a2)$ hold, i.e. that a simple complex pair passes through the imaginary axis at μ_0 while all other eigenvalues have negative real parts and prove that $b1)$ holds. The condition $a1)$ implies that in some neighborhood of μ_0 $P_J(\lambda)$ may be factored as the product of two polynomials:

$$P_J(\lambda) = \left(\lambda^2 + \alpha(\mu)\lambda + \beta(\mu)\right) Q(\lambda, \mu) \qquad (4)$$

where the first factor depends on the bifurcating pair of eigenvalues (hence it satisfies $\beta(\mu) > 0$, $\alpha^2(\mu) - 4\beta(\mu) < 0$, $\alpha(\mu_0) = 0$, $(d\alpha(\mu)/d\mu)_{\mu_0} \neq 0$), while $Q(\lambda, \mu)$ has order $(n-2)$ and all its eigenvalues have negative real part. Hence all the RH determinants D_i, i=1,..,$n-2$) of $Q(\lambda, \mu)$ are strictly positive: $D_1 > 0, D_2 > 0, .., D_{n-2} > 0$. The relations between the coefficients a_i of $P_J(\lambda)$ and q_i of $Q(\lambda, \mu)$ in (4) is:

$$a_i(\mu) = \beta(\mu) q_i(\mu) + \alpha(\mu) q_{i-1}(\mu) + q_{i-2}(\mu) \qquad (5)$$

By exploiting (5) it may be shown that at the bifurcation point μ_0 it holds for the first (n-2) Routh-Hurwicz determinants of $P_j(\lambda)$:

$$\begin{array}{l} \Delta_1(\mu_0) = \beta(\mu_0) D_1(\mu_0) > 0 \\ \Delta_2(\mu_0) = \beta^2(\mu_0) D_2(\mu_0) > 0 \\ \vdots \\ \Delta_{n-2}(\mu_0) = \beta^{n-2}(\mu_0) D_{n-2}(\mu_0) > 0 \end{array} \qquad (6)$$

while:

$$\Delta_{n-1} = 0 \tag{7}$$

showing that $b1$) holds.

If part. Let us assume that $b1,2$) holds and prove that $a1$) holds.

If $b1$ holds, this means that there is a right or left neighbourhood of the point μ_0 in which the RH conditions for the local stability of E_1 hold (implying all the eigenvalues of $J(E_1)$ have negative real parts), while this is not true on the other side of μ_0 (thanks to $b2$)). Hence at μ_0 there exists at least one pair of complex eigenvalues having zero real part (notice that this does not depend on real eigenvalues as $a_n(\mu_0) > 0$). To complete the proof of the existence of a simple Hopf bifurcation we have to exclude the possibility that more than one purely immaginary pair exists at μ_0. If this were possible, it would imply that not all the D_i are strictly positive, and hence, in turn, that not all the Δ_i in (5) are positive, having therefore found a contradiction with $b1$). Hence $a1$) holds.

The result by Liu is remarkable in that it fills the bridge between the body of theorems for the local stability of equilibria and the notion of Hopf bifurcation. His criterion for the detection of a Simple Hopf bifurcation is "isomorphic" to the Routh-Hurwicz criterion. Its usefulness is evident by itself. For instance we do not need to check for the presence of complex eigenvalues as a necessary condition for a bifurcation. This fact was repeatedly stressed in Lorenz (1993 and 1994) and Gabisch and Lorenz (1989), who suggest, at the third order, to study the sign of the discriminant involved in the resolvent formulas of the characteristic equation. Such a Sysiphus fatigue (totally impossible at dimensions higher than four due to the lack of resolvent formulas) is bypassed by the Liu theorem.

As pointed out in section three, the RH theorem is not the most economic IFF condition for stability, which is actually given by the Liénard-Chipart conditions. The theorem by Liu can be reformulated by substituting to the whole structure of the Routh-Hurwicz conditions, the corresponding LC conditions. A simplified version assuming the strict positivity of all the coefficients a_i is the following:

THEOREM 2. *(LC version of the Liu theorem) Provided $a_i > 0$, the requirements $a1$), $a2$) for a SHB are equivalent to one or another of the following two sets of conditions:*

i)

$c1$) $\Delta_2(\mu_0) > 0$; $\Delta_4(\mu_0) > 0$; $...\Delta_{n-3}(\mu_0) > 0$; $\Delta_{n-1}(\mu_0) = 0$
$c2$) $\left(\frac{d\Delta_{n-1}}{d\mu}\right)_{\mu_0} \neq 0$

ii)

$c1bis$) $\Delta_3(\mu_0) > 0$; $\Delta_5(\mu_0) > 0$; $...\Delta_{n-3}(\mu_0) > 0$; $\Delta_{n-1}(\mu_0) = 0$
$c2bis$) $\left(\frac{d\Delta_{n-1}}{d\mu}\right)_{\mu_0} \neq 0$

To prove theorem 3 we exploit the relation between the Routh-Hurwicz and Liénard-Chipart conditions.

Only if part. It is trivial that if $b1$) holds ($\Delta_1(\mu_0) > 0$; ...$\Delta_{n-2}(\mu_0) > 0$; $\Delta_{n-1}(\mu_0) = 0$) both $c1$) and $c1bis$) necessary hold "a fortiori".

If part. Let us assume that $c1$) (or $c1bis$)) holds, and prove that $b1$) holds, so that $a1$) also holds. If $c1$ holds, this means that there is a right or left neighbourhood of the point μ_0 in which the LC conditions for the local stability of E_1 hold (hence it holds either $\Delta_{n-1}(\mu) > 0, \Delta_{n-3}(\mu) > 0, ..$ or: $\Delta_n(\mu) > 0, \Delta_{n-2}(\mu) > 0, ...$), while this is not true on the other side of μ_0. Let us suppose without loss of generality that it is a left neighborhood $V = (\mu_0 - d, \mu_0)$. Clearly if the LC conditions hold in V, the RH conditions hold as well in V, i.e. in V it holds $\Delta_i(\mu) > 0,\ i = 1, .., n$. But at the point μ_0 it holds $\Delta_{n-1}(\mu_0) = 0$, while all the other RH determinants $\Delta_i(\mu_0),\ i = 1, .., n-2$ are strictly positive. Hence at the point μ_0, $b1$) holds, implying that $a1$) also holds.

The latter theorem is particularly useful since it considerably reduces the computations involved in the detection of SHB, as shown in the following:

EXAMPLE 1. *(detection of simple Hopf bifurcations at low dimensions when the necessary condition $a_i > 0$ is satisfied) At dimension four (Fanti and Manfredi 1998a) a SHB simple occurs when:*

$$\Delta_3 = 0 \ ; \ \left(\frac{d\Delta_3(\mu)}{d\mu}\right)_{\mu_0} \neq 0$$

At dimension five the LC stability conditions take the two possible forms: a)$\Delta_2 > 0$; $\Delta_4 > 0$; *b)*$\Delta_3 > 0$; $\Delta_5 > 0$. *Theorem 3 (let us work with the simpler set of conditions) leads to the conditions:*

$$\Delta_2 > 0 \ ; \ \Delta_4 = 0 \ ; \ \left(\frac{d\Delta_4(\mu)}{d\mu}\right)_{\mu_0} \neq 0$$

At dimensions six a SHB occurs when:

$$\Delta_5 = 0 \ ; \ \Delta_3 > 0 \ ; \ \left(\frac{d\Delta_5(\mu)}{d\mu}\right)_{\mu_0} \neq 0$$

and so on.

4 Boundaries of stability and Hopf bifurcations: "physiology" of the bifurcation process

As already pointed out the two theorems of the previous section were implicitly used in applied work since time ago. This implicit use was passing through the notions of stability boundary and of "stability switch" indicator, clearly elucidated in Hahn (1967) or in the nice synthesis on stability analysis by Mc Donald (1989). The present section essentially relies on McDonald (1989): we extend his discussion on stability to the issue of the detection of SHB bifurcations. This allows us: 1) to define a tool-box which, under some conditions often met in macro-economic models, permits to detect SHB with an amazingly simpler effort, even compared to the theorems of the previous section (the economic consequences of this fact are deeply illustrated in the next section); 2) to clarify the distinction between SHB and GHB. To begin let us observe that, although Theorem 2 permits to substantially simplify the analytical detection of a SHB, compared to Theorem 1, its usefulness rapidly decays as the dimension of the system to be analysed increases. Despite this, following McDonald (1989, ch. 4): "... *a major simplification comes about if we realise that in practise what matters is to find, in the space of the involved parameters, the points, curves, or surfaces, that bound regions of stability. Typically, one can start from some point in the parameter space in which stability is known to prevail. Then, since the eigenvalues depend continuously on the parameters, a change of stability can only happen by way of the appearance of a zero (real) eigenvalue* ($\lambda = 0$)[9] *or of a purely imaginary pair* $\lambda = \pm i\omega$."

The previous considerations, though concerned with stability analysis, straightforwardly extend to SHB. The key-tools at this stage are the "stability switch" indicators, i.e functions which changes their sign in correspondence of the stability boundary. The simplest indicator of the appearance of a zero eigenvalue is $a_n = (-1)^n \det J = (-1)\prod_j \lambda_j$ (the stability boundary corresponds to $a_n = 0$). This case is not of interest for us as we are not concerned with stability losses caused by real eigenvalues: in fact we always assume $a_n > 0$. The simplest indicator of the appearance of a purely immaginary pair is Δ_{n-1}, as showed by Orlando's formula (Gantmacher (1959)):

$$\Delta_{n-1} = (-1)^{\frac{n(n-1)}{2}} \prod_{i=j}^{n} \prod_{j=i+1}^{n} (\lambda_i + \lambda_j) \tag{8}$$

(8) shows that the equality $\Delta_{n-1} = 0$ holds in one and only one of the following cases: $i) P_j(\lambda)$ has a zero real root with algebraic multiplicity at least two; $ii) P_j(\lambda)$ has (at least) two real roots of identical absolute value but opposite sign; $iii) P_j(\lambda)$ has (at least) a purely imaginary pair; $iv) P_j(\lambda)$ has two complex pairs having opposite real parts and immaginary parts of identical absolute value.

REMARK 2. *If the necessary condition for stability is satisfied ($a_i > 0$ for all i) then non negative real roots are impossible and cases i) and ii) are ruled out. Hence only possibilities iii) and iv) remain.*

Let us now, still following Mc Donald (1989), follow an "operative" approach, in which stability is always the starting point of the story. Let us therefore consider an "initial" parameter constellation μ_S in which the system is stable, i.e. in which $a_i > 0$ for all i holds, and all the relevant LC determinants are strictly positive. In this case losses of stability can only occur through the crossing of the imaginary axis by one (or more) previously stable complex pair (the

[9] *In the bifurcation jargon this problem leads to another type of bifurcation, the fold bifurcation, see for instance Guckenheimer and Holmes (1984), and Lorenz (1993) for economic applications.*

aforementioned case *iii*)). Hence to detect how stability can be lost due to movements of the system parameter(s) from the "initial" constellation μ_S, we only need to find the position of μ_S with respect to the locus defined by $\Delta_{n-1} = 0$: no instabilities due to the crossing of the imaginary axis by a complex pair may occur without first causing $\Delta_{n-1} = 0$. But this makes it easy the problem of the detection of instability. Mac Donald (1989, 74) is enlightening on this point. Let us consider fig. 1a,1b reporting two distinct forms of the locus $\Delta_{n-1} = 0$ for a two-dimensional parameter space labeled, just to fix the ideas, as (p, q). Let us assume that in both cases the point P represents the "initial" parameter constellation μ_S at which stability prevails and that $a_n > 0$ in the whole parameter space. Case 1a is unambiguous: the whole external region is a region of stability, while the whole inner region is an instability region. In this case the investigation of the stability of the system is complete with the only knowledge of the behaviour at P. Viceversa case 1b is ambiguous: instability certanly prevails in the area comprisen between the two curves but the crossing of the inner curve does not necessarily implies a reswitch to stability (in other terms the inner curve is not necessarily a true stability boundary, although satisfying $\Delta_{n-1} = 0$). In case 1b the characterisation of stability is complete only once we have checked the stability nature of the points inside the inner curve.

Fig. 1 (reprinted from Mac Donald 1989) The notion of

stability boundary; a) the unambiguous case; b) an ambiguous case

The previous considerations on stability losses straightforwardly extend to the detection of Hopf bifurcations. In fact, as long as we start from "initial" parameter constellations in which the system is stable, we are necessarily concerned with stability losses caused by the crossing of the imaginary axis by previously stable complex pairs[10]. Clearly the equality $\Delta_{n-1} = 0$ in general cannot discriminate whether this occurs because just one previously stable pair, rather than more than one, crosses the boundary. In other terms the only use of $\Delta_{n-1} = 0$ cannot discriminate between a SHB and a GHB unless in very simple cases. For instance, in the "trivial" three-dimensional case GHB are not possible. Similarly, at dimension four, the positivity of the coefficients of the CP implying $\Delta_1 = a_1 > 0$, again prevents the possibility of a GHB.

In higher order cases to make sure that we are faced with a SHB we should control that all the remaining LC determinants remain strictly positive for those parameter values causing $\Delta_{n-1} = 0$. But of course this is a little worrying fact: the simultaneous crossing by several roots is certainly a "less likely" event (less generic). From the practical point of view this dramatically reduces the complexity of the problem of the detection of the bifurcation, even compared to the LC test developed in theorem 3. The example of fig. 1a is enlightening on this profile: provided the initial point P is stable, all the points of the curve are Hopf bifurcation points.

[10]Of course we should also apply the test for nonzero speed, in order to ensure the existence a Hopf bifurcation, but in the spirit of footnote five this is again a minor problem.

Clearly, if the "initial" parameter constellation μ_0 is not a stable one the condition $\Delta_{n-1} = 0$ is less powerful. In the example of fig. 1b all the points on the external curve are Hopf bifurcation points. If stability prevails in the inner region as well (as previously pointed out we need further information on the stability behaviour), and provided $a_n > 0$ still holds everywhere, then also the points on the inner curve are Hopf bifurcation points (for instance at dimension three this new Hopf bifurcation would be caused by a re-switch of stability due to the activity of the same complex pair). In the event the inner region is not a stable one, the equality $\Delta_{n-1} = 0$ still detects a Hopf bifurcation, namely a GHB, only provided that the possibilities *i*), *ii*) (for instance by assuming $a_i > 0$ for all i) and *iv*) aforementioned, are ruled out. The nature of this GHB may be very wide (especially at "really" high dimensions) depending on how many pairs have negative real parts and how many have positive real parts.[11]

The present approach is particularly useful for systems for which the "initial" parameter constellation μ_0 in which the system is stable, "naturally" exists and is identifiable. This situation is not unfrequent: modellers usually study the effects of "complications" on known models (which are very often stable), and these complications are usually obtained by adding extra-terms depending on some extra-parameters ϑ. These enlarged models usually reduce to the old simpler model when $\vartheta = 0$ or so. Hence we often know a "natural" initial parameter constellation. But parameter perturbations of this type usually inflence not only the stability conditions, but also the structure of equilibria, and this is a further complication. From this latter point of view, a remarkable case in which a natural "initial" parameter constellation in which the system is stable exists, is that of delay systems. Standard formulations of delayed models (i.e. by introducing delays in a previously unlagged model) usually do not change the equilibria: the effects of the delay are purely on stability.

Now, a standard question in science is the following. Consider a dynamical system having a stable equilibrium in absence of time-delays. Which is the action on stability played by the introduction of delays? In this case the "natural" initial parameter constellation μ_0 in which the system is stable corresponds to the case in which the delay is absent. This corresponds to the case in which the parameter tuning the delay, let us call it T, is set equal to zero. These aspects are illustrated in the next section by means of some higher order ODE models derived from an underlying distributed delay model.

[11] Fig. 1 reports quite abstract forms of possible bifurcations: their meaning is intended to be purely exemplifying. The next section reports two sound economic illustrations.

5 Economic illustrations

5.1 The system is stable in absence of the delay: insights from a delayed Solow-type model

Here we consider the following delayed Solow- model introduced elsewhere (Fanti and Manfredi, 1999a):

$$\dot{k} = sk^{\alpha} - \delta k - \left(\int_{-\infty}^{t} n\left(k^{\alpha}(\tau)\right) G(t-\tau) d\tau \right) k \tag{9}$$

where k =capital-labour ratio, k^{α} the output per unit of labour under a Cobb-Douglas production function ($0 < \alpha < 1$), s=the saving rate ($0 \le s \le 1$), δ =rate of capital depreciation. Compared to th standard Solow's model, the constant exogenous rate of growth of the supply of labour (usually denoted by n), has been replaced by an integral term dependent on the past income k^{α} through a prescribed map n (for simplicity we assume that n is linear.) The purpose of this term is that of mimicking the effects of past wage-related fertility, along a malthusian mechanism, on the current rate of change of k through the delayed entrance of individuals into the labour force (for details see Fanti and Manfredi, 1999a). As pointed out in the literature time-delays represent good approximations of the age structure mechanisms (Mac Donald 1978, Manfredi and Fanti 1999a) in that they permit simpler representations of age structure while often preserving the same richness of dynamical results.

Finally, the function $G(t-\tau)$ is the delaying kernel, usually taken as a probability density function. The dynamical properties of (9) crucially depend on the structure of the delaying kernel G. When G belongs to the erlangian family the equation (9) may be reduced to an ODE system via the so called linear trick (Mac Donald 1978). A kernel is erlangian-type with parameters (r, a) when its density function follows an erlangian density[12] (r, a):

$$G_{r,a}(x) = \frac{r^{a}}{(r-1)!} x^{r-1} e^{-ax} \qquad x > 0;\ a > 0,\ r = 1, 2, ... \tag{10}$$

By varying the parameter r the erlangian family describes a flexible family of density functions: for $r = 1$ we have the classical exponentially fading memory (i.e. a negative exponential kernel), while for $r = 2, 3$ and so on we have typical "humped" memories. In particular the mean delay of an erlangian density (r, a) is given by: $T = r/a$, while its variance is $Var = r/a^2$. Moreover, under (10) when we let $T \to 0$ in (9), the unlagged formulation is recovered (for instance Invernizzi and Medio, 1991). The passage to ODE's under erlangian kernels is obtained by introducing the new variable:

$$S(t) = \int_{-\infty}^{t} V(\tau) G(t-\tau) d\tau \tag{11}$$

[12] More generally we define as erlangian a kernel which is a linear combination of erlangian densities.

A time differentiation of (11) transforms[13] the system (9) into its "augmented" ODE form, in which the delay is replaced by a a "cascade" of r adaptive equations characterised by the same speed of adjustment a.

It is easy to show that for $r = 1$ the positive equilibrium E_1 of the model (9) (we recall that (9) has the same equilibria of the unlagged Solow's model), remains LAS independently on the delay. More interesting things arise for $r = 2$. In this case the model takes the form

$$\begin{aligned} \dot{Z} &= \alpha Z \left(s Z^{\frac{\alpha-1}{\alpha}} - nX \right) \\ \dot{X} &= a(R - X) \\ \dot{R} &= a(Z - R) \end{aligned} \tag{12}$$

where $Z = k^{\alpha}$. The third order characteristic polynomial at E_1: $P(X) = X^3 + a_1 X^2 + a_2 X + a_3$ has the coefficients:

$$a_1 = 2\beta + (1-\alpha)\, nZ_1 \ ; \ a_2 = 2\beta\,(1-\alpha)\, nZ_1 + \beta^2 \ ; \ a_3 = \beta^2 nZ_1$$

which are strictly positive. The necessary condition for stability is therefore always satisfied. The stability boundary is defined by the locus: $\Delta_2 = a_1 a_2 - a_3 = 0$, i.e.:

$$2a^2 + nZ_1\,(4 - 5\alpha)\, a + 2\left((1-\alpha)\, nZ_1\right)^2 = 0 \tag{13}$$

Fig. 2, which depicts the locus $\Delta_{n-1} = 0$ for the unique positive equilibrium of model (12). The necessary condition is always satisfied. The $\Delta_{n-1} = 0$ locus is represented in the 2-dimensional parameter space (n, T), where T is the average delay ($T = 2/a$) and n is the fertility rate in the population. All the points of the n-axis ($T = 0$) can be chosen as our "initial parameter constellation": as well known the positive equilibrium in the classical unlagged Solow's model is (globally) stable. Hence in all the points of the (n, T) space below the line L_1 stability prevails. The line L_1 necessarily is a locus of SHB points. The region comprisen between the lines L_1 and L_2 is an instability region (in Fanti and Manfredi (1999) it is shown that stable limit cycles exist in the whole region). It is of interest, at this stage, to understand the role played by the L_2 curve. As the necessary condition is satisfied in the whole parameter space, also the locus L_2 is a stability boundary, at which a stability reswitch occurs, due to a further crossing of the previously unstable pair. Hence, the acknowledgement of a realistic pattern of change of the labour supply leads to a stability reswitching, i.e. to two distinct bifurcation values of the delay. The smaller bifurcating delay occurs on a typically demographic time scale, whereas the larger one occurs on a very long time scale. This result, which appears a counterintuitive consequence of the mathematical analysis of the model, reveals the existence of an unexpected influence of the fertility behaviour of generations more ancient than that of the parents (we called it a "supergenerational echoe"), which could be an interesting issue for demo-economists.

Fig. 2. The locus $\Delta_{n-1} = 0$ for the positive equilibrium of a 3-dimensional Solow's model

[13] A further formal requirement is needed, in order to make compatible the "distributed" initial condition of the original IDE system, with the "concentrated" initial condition of the ODE system.

5.2 The system is only neutrally stable in absence of the delay: a Goodwin-Kalecki-type model

We discuss now the bifurcation problem arising in a delayed Goodwin-type model expanded to take into account of kaleckian effects. Its economic foundations and results are discussed in Manfredi and Fanti (1999b). This example is of interest in that it shows how to treat the case in which the system is only neutrally stable in absence of the delay. The structure of the model is given by the following integro-differential (IDE) system:

$$\begin{aligned} \frac{\dot{V}(t)}{V(t)} &= -(\alpha+\gamma)+\rho U \\ \frac{\dot{U}(t)}{U(t)} &= (c+k)\,m(1-V)-(\alpha+n)-km\left(1-\int_{-\infty}^{t} V(\tau)G(t-\tau)d\tau\right) \end{aligned} \tag{14}$$

In (14) $U=U(t)$=employment rate at time t, defined as the ratio between the total labour force actually employed $L(t)$ and the supply of labour $N(t)$, $V=V(t)$ = the distributive share of labour, given by the ratio $w(t)L(t)/Q(t)$, where w is the real wage and Q the total product. V can be expressed also as: $V=w/A$ where A is the average productivity of labour. Obviously: $1-V$ =the profit share. Moreover m=the (constant) output-capital ratio, α=the (exogenous) rate of growth of the average productivity of labour, n=the (exogenous) rate of growth of the supply of labour, c=the saving rate of the capitalists; finally γ,ρ are characteristic parameters of the (linear) Phillips curve governing the labour market ($0<\gamma<\rho$). All the aforementioned parameters are those characteristic of the original Goodwin's model. The model (14) embeds kaleckian effects via the lagged term, embedding past profitability; in particular $k>0$ and the delaying kernel G tune the "rashness" of investors. When $c=1, k=0$ (14) collapses in the classical Goodwin's (1967) model, which, provided a positive equilibrium exists, exhibits the classical Lotka-Volterra-Goodwin (LVG) conservative oscillations. If we keep $c=1$ the model (14) has the same equilibria of the original Goodwin's model[14], namely the zero equilibrium $E_0=(0,0)$, and the positive equilibrium $E_1=(U^*,V^*)=(\gamma/\rho,(m-\alpha-n)/m)$. E_1 is economically meaningful provided that $m-\alpha-n<m$.

The dynamical properties of (14) essentially depend on the structure of the delaying kernel G. For $r=1$, i.e. the case of the exponentially fading memory, the original system becomes:

$$\begin{aligned} \frac{\dot{V}(t)}{V(t)} &= -(\alpha+\gamma)+\rho U \\ \frac{\dot{U}(t)}{U(t)} &= (c+k)\,m(1-V)-(\alpha+n)-km\,(1-S) \\ \dot{S} &= a(V-S) \end{aligned} \tag{15}$$

It is easy to show that in the model (15) (which preserves the equilibria of (14)) the positive equilibrium E_1 is always LAS independently on the delay. In other terms a kaleckian exponentially fading memory always stabilizes the conservative center of the Goodwin's model. By passing we notice that this is a nice instance of the fact that delays can also be stabilising, and not only destabilising, as often claimed in the literature (Farkas and Kotsis 1992). It is of interest

[14]Notwithstanding the introduction of the distinction between rash and cautious behaviours of the capitalists.

to check whether this stability is preserved under different forms of the delaying kernel. In many cases systems which are stable under an exponentially fading memory are destabilised under the simplest type of "hump" memory, i.e. under a kernel erlangian $(2, a)$. An instance of such effect is Fanti and Manfredi (1998a). This effect is usually explained with the strongly different qualitative action played by a humped memory as opposite to an exponentially fading one.

5.3 The effect of the simplest humped memory

Under the simplest humped memory, i.e. a kernel erlangian $(2, a)$, the system (14) becomes:

$$\begin{array}{l} \frac{\dot{V}(t)}{V(t)} = -(\alpha+\gamma)+\rho U \\ \frac{\dot{U}(t)}{U(t)} = (c+k)\, m(1-V) - (\alpha+n) - km\,(1-S) \\ \dot{S} = a(Z-S) \\ \dot{Z} = a(V-Z) \end{array} \tag{16}$$

(having the same equilibria of (14) and (15)). The local stability analysis at E_1 leads to the characteristic poynomial:

$$P_{E_1}(X) = X^4 + 2aX^3 + \left(a^2 + Bk\right) X^2 + 2aB\,(1+k)\, X + Ba^2$$

whose coefficients are always positive (we denoted $B = m\rho U^* V^*$). The LC test for stability requires $\Delta_1 > 0$ (always satisfied as $\Delta_1 = a_1$) and $\Delta_3 > 0$. But:

$$\Delta_3 = 4a^4 Bk > 0$$

Hence E_1 remains stable independently on the delay in the simplest humped case as well. It is therefore of interest to look for the possibility that destabilisation is caused by delays of higher order.

5.4 Effects of more concentrated humped memories

Let us consider the effects of the next element of the erlangian family. Under a kernel erlangian $(3, a)$, the reduced ODE system has the form:

$$\begin{array}{l} \frac{\dot{V}(t)}{V(t)} = -(\alpha+\gamma)+\rho U \\ \frac{\dot{U}(t)}{U(t)} = (1+k)\, m(1-V) - (\alpha+n) - km\,(1-S) \\ \dot{S} = a(Z-S) \\ \dot{Z} = a(W-Z) \\ \dot{W} = a(V-W) \end{array} \tag{17}$$

The jacobian at E_1 is:

$$J(E_1) = \begin{pmatrix} 0 & \rho V & 0 & 0 & 0 \\ -(1+k)\,mU & 0 & kmU & 0 & 0 \\ 0 & 0 & -a & a & 0 \\ 0 & 0 & 0 & -a & a \\ a & 0 & 0 & 0 & -a \end{pmatrix}$$

The corresponding fifth order characteristic polyomial has the coefficients:

$$a_1 = 3a;\ a_2 = B\,(1+k) + 3a^2;\ a_3 = a\left(a^2 + 3B\,(1+k)\right);\ a_4 = 3Ba^2\,(c+k);\ a_5 = Ba^3$$

which are always positive. We quickly have:

$$\Delta_4 = Bka^6\left(24a^2 - B\,(8+9k)\right)$$

Hence, the condition $\Delta_{n-1} = \Delta_4 = 0$ gives:

$$a = \frac{\sqrt{B\,(8+9k)}}{24} = \frac{\sqrt{(m-\alpha-n)\,(\alpha+\gamma)\,(8+9k)}}{24} \tag{18}$$

to which corresponds the mean delay T=$3/a$. The curve (18) expresses in a sharp manner the bifurcation curve.

Can we now directly assure that the locus (18) is a Hopf bifurcation locus? A difficulty which apparently prevents the use of the sole condition $\Delta_{n-1} = 0$ is the fact that the "natural initial parameter constellation", corresponding to the case of no-delay ($T = 0$), corresponds to the original Goodwin's model, for which the E_1 equilibrium is not (linearly) stable, but only neutrally stable. The difficulty is solved as follows. It is immediate to check that the system (17) is the ODE system that would have been obtained by delaying (14) by an erlangian kernel $G_{2,a}$. In general we may say that the application of a delaying kernel $G_{r,a}$ is equivalent to r sequential applications of a kernel $G_{1,a}$: see the appendix for details.[15]

In other words: from the practical point of view of the analysis of stability and bifurcation, we do not need to necessarily refer the "natural" initial parameter constellation in which the system is stable, to the original unlagged system. Let us reconsider our problem. We have to perform the stability analysis of the fifth order system (17) obtained by delaying with a kernel $G_{3,a}$ the original conservative Goodwin's system. But the stability analysis of (17) is equivalent, for instance to the stability analysis of the system (15), which is stable, when the S variable therein is furtherly delayed by $G_{r,2}$. It is also equivalent to the stability analysis of (16), which is stable, when the S variable therein is furtherly delayed by $G_{r,1}$. This implies that both systems (15) or (16) provide an initial parameter constellation in which the system is stable. Therefore, once the stability boundary(16) is crossed, a unique switch between stability and instability

[15] This result may be furtherly extended: the application of a delaying kernel $G_{r,a}$ is equivalent to the sequential application of a kernel $G_{s,a}$, followed by that of a kernek $G_{r-s,a}$ (s in an integer $\leq r$).

occurs: this implies, without the need for any further inquiry, that all the points of the stability boundary are SHB points.

Just to check this point let us use "in toto" the theorem 2 of section 3 and compute Δ_2. We have:

$$\Delta_2 = a_1 a_2 - a_3 = 3a\left(B\left(1+k\right)+3a^2\right) - a\left(a^2+3B\left(1+k\right)\right) = 8a^3$$

which is always satisfied for $a > 0$. We can thus say that all the points of the line $a = (24)^{-1}\sqrt{B\left(8+9k\right)}$ are, on the basis of the discussion in section 3, *simple Hopf bifurcation points* for E_1.

Getting back to more substantive facts, we remark that:

a) our bifurcation toolkit allows a clear interpretation of the bifurcation process and the role played by the rashness parameters with respect to the original Goodwin's ones. In other words: the bifurcation process in our 5-dimensional model (14) is perfectly interpretable in terms of the underlying economic theory.

b) we have showed that, although the simplest type of humped memory is not able to destabilise the basic model, this can be caused by a sufficiently concentrated humped memory. This result is of interest by itself in that (see Manfredi and Fanti 1999b) it proves that (distributed) delays may not only stabilise a conservative LVG system for slightly concentrated delays, and destabilise it for highly concentrated memories, but they may also lead the conservative LVG system to persistently oscillate (Manfredi and Fanti 1999).

c) the analysis of the conservative LVG model has showed that, compared to the stability boundary analysis, we do not necessarily need that the underlying undelayed system is stable. Also conservative systems, in which the LV family constitute a leading model in applied mathematics, seem to be easily tractable by the same toolkit.

6 Conclusions

The detection of the existence of endogeneous stable fluctuations (or, mathematically speaking, of stable limit cycles) is a first concern in macro-dynamics. This problem is intimately related to the notion of Hopf bifurcation.

Practically all the papers concerned with the detection of Hopf bifurcations in the macroeconomic literature are concerned with systems of very low dimension. The number of papers investigating dynamical systems of dimension four is very small. One of the possible reason for this state of things is the lack of a unified treatment of the subject.

This paper aims to provide this unified treatment. We first try to clarify the different typologies of detectable Hopf bifurcations (i.e. the notions of Simple and General Hopf bifurcation). Starting from this distinction we then discuss how the stability theorems, such as Routh-Hurwicz and Lienard-Chipart, can be used to detect Simple Hopf bifurcations. A Liénard-Chipart-type result for the detection of Hopf bifurcations is given which appears of considerable usefulness in applications.

Moreover, by relying on the notion of "stability boundary", we show that in some cases the conditions for the detection of the existence of the Hopf bifurcation can be stated in an astonishingly parsimonious way, especially compared to the standard "belief". This result appears to be of a critical usefulness to treat special classes of problems. One of these is the class of distributed delay problems reducible to ODE's, which are more and more common in macrodynamics, for instance when we postulate the existence of delayed, but heterogeneous, reactions of the economic agents.

Our economic illustrations finally show how the notion of stability boundary can be used to detect SHB, both in the case of an underlying stable unlagged system (the Solow-type example), and in the case of an underlying neutrally stable unlagged system (the LVG example).

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8 Appendix: some relations between delayed systems

Here we show that the application of a delaying kernel $G_{r,a}$ is equivalent to r sequential applications of a kernel $G_{1,a}$.[16] This result is the deterministic counterpart of the know probability theorem defining the erlangian density $G_{r,a}$ as the sum of r independent and identically distributed exponential densities $G_{1,a}$. We prove the result for $r = 2$. The completion of the proof follows by induction. Let us consider the quantity

$$S_1(t) = \int_{-\infty}^{t} X(\tau) G_{1,a}(t-\tau)\, d\tau = \int_{-\infty}^{t} X(\tau) a e^{-a(t-\tau)} d\tau$$

obtained by delaying the quantity $X(t)$ through a kernel $G_{1,a}$. Let us now consider further applications of the delay same operator, i.e. consider:

$$S_2(t) = \int_{-\infty}^{t} S_1(\tau) a e^{-a(t-\tau)} d\tau$$

We have:

$$\begin{aligned} S_2 &= \int_{-\infty}^{t} S_1(\tau) a e^{-a(t-\tau)} d\tau = \int_{-\infty}^{t} \left(\int_{-\infty}^{\tau} X(u) a e^{-a(\tau-u)} du \right) a e^{-a(t-\tau)} d\tau = \\ &= \int_{-\infty}^{t} \int_{-\infty}^{\tau} X(u) a e^{-a(\tau-u)} a e^{-a(t-\tau)} du d\tau \end{aligned}$$

By interchanging the order of integration we get:

$$\begin{aligned} S_2 &= \int_{-\infty}^{t} \int_{-\infty}^{\tau} X(u) a e^{-a(\tau-u)} a e^{-a(t-\tau)} du d\tau = \\ &= \int_{-\infty}^{t} X(u) \left(\int_{u}^{t} a^2 e^{-a(\tau-u)} e^{-a((t-u)+(u-\tau))} d\tau \right) du = \\ &= \int_{-\infty}^{t} X(u) a^2 e^{-a(t-u)} \left(\int_{u}^{t} e^{-a(\tau-u)} e^{-a(u-\tau)} d\tau \right) du = \end{aligned}$$

[16] This result may be furtherly extended: the application of a delaying kernel $G_{r,a}$ is equivalent to the sequential application of a kernel $G_{s,a}$, followed by that of a kernek $G_{r-s,a}$ (s in an integer $\leq r$).

$$
\begin{aligned}
&= \int_{-\infty}^{t} X(u)a^2 e^{-a(t-u)} \left(\int_{u}^{t} d\tau \right) du = \\
&= \int_{-\infty}^{t} X(u)a^2 \, (t-u) \, e^{-a(t-u)} du = \\
&= \int_{-\infty}^{t} X(u) G_{2,a} du
\end{aligned}
$$

which proves our statement.

9 Appendix. Multiple convolutions

By definition the convolution of two functions f_1, f_2 defined on the real line is

$$f_{1,2}(t) = f_1 * f_2 = \int_0^t f_1(\tau) f_2(t-\tau) d\tau$$

Let us now consider the convolution of $f_{1,2}(t)$ with a third function:

$$\begin{aligned}
(f_1 * f_2) * f_3 &= f_{12} * f_3 = \int_0^t f_{12}(u) f_3(t-u) du = \\
&= \int_0^t \left(\int_0^u f_1(\tau) f_2(u-\tau) d\tau \right) f_3(t-u) du = \\
&= \int_0^t \int_0^u f_1(\tau) f_2(u-\tau) f_3(t-u) d\tau du
\end{aligned}$$

By interchanging the order of integration we find:

$$\begin{aligned}
f_1 * f_2 * f_3 &= \int_0^t f_1(\tau) \left(\int_\tau^t f_2(u-\tau) f_3(t-u) du \right) d\tau \\
&= \int_0^t f_1(\tau) K(t,\tau) d\tau
\end{aligned}$$

Notice that the delaying kernel K in (3) is not of the difference type:

$$K(t,\tau) = \int_\tau^t f_2(u-\tau) f_3(t-u) du$$

Let us introduce the change of variable $(u-t) = a$ with $du = da$. The new integration limits are: $(0, t-\tau)$. We obtain:

$$\begin{aligned}
K(t,\tau) &= \int_\tau^t f_2(u-\tau) f_3(t-u) du = \\
&= \int_0^{t-\tau} f_2(a) f_3 \left[(t-\tau) - (u-\tau) \right] du = \\
&= \int_0^{t-\tau} f_2(a) f_3 \left[(t-\tau) - a \right] da = K * (t-\tau)
\end{aligned}$$

which is really a difference kernel. This finally proves that:

$$\begin{aligned}
f_1 * f_2 * f_3 &= \int_0^t f_1(\tau) K(t-\tau) du = \\
&= \int_0^t \left(\int_0^u f_1(\tau) f_2(u-\tau) d\tau \right) f_3(t-u) du = \\
&= \int_0^t f_1(\tau) \left[\int_0^{t-\tau} f_2(a) f_3 \left[(t-\tau) - a \right] da \right] d\tau
\end{aligned}$$

i.e exactly that, as expected, the 3-level convolution can be expressed as a convolution between the first variable and the convolution of the remaining two.

The general expression for an n-th order convolution is then:

$$f_1 * f_2 * f_3 = \int_0^t f_1(\tau)K(t-\tau)du =$$
$$= \int_0^t \left(\int_0^u f_1(\tau)f_2(u-\tau)d\tau\right) f_3(t-u)du = \int_0^t f_1(\tau)\left[\int_0^{t-\tau} f_2(a)f_3\left[(t-\tau)-a\right]da\right]d\tau$$

9.1 A Goodwin-type model with profit-sharing

Here we consider a Goodwin type model (Fanti and Manfredi 1998b) in which it is assumed that the rate of change of the wage-share also depends on the lagged profit:

$$\begin{array}{l} \frac{\dot V}{V} = -(\alpha+\gamma) + \rho U + \varepsilon m\left(1 - \int_{-\infty}^{t} V(\tau)G(t-\tau)\,d\tau\right) \\ \frac{\dot U}{U} = m - \alpha - n - mV \end{array} \tag{19}$$

The parameter $\varepsilon > 0$ reflects the action of the profit-sharing effect on the rate of change of the wage-share. Fanti and Manfredi (1998b) showed that under the action of the simplest humped memory (erlangian $(2,a)$) persistent oscillations appear in (19) via a Hopf bifurcation of the unique positive equilibrium E_1. Here we consider the problem of the detection of Hopf bifurcations under a kernel erlangian $(3,a)$.

$$\begin{array}{l} \frac{\dot V}{V} = -(\alpha+\gamma) + \rho U + \varepsilon m(1-S) \\ \frac{\dot U}{U} = m - \alpha - n - mV \\ \dot S = a(Z-S) \\ \dot Z = a(W-Z) \\ \dot W = a\,(V-W) \end{array} \tag{20}$$

Notice that in this case the underlying unlagged model:

$$\begin{array}{l} \frac{\dot V}{V} = -(\alpha+\gamma) + \rho U + \varepsilon m(1-V) \\ \frac{\dot U}{U} = m - \alpha - n - mV \end{array} \tag{21}$$

has a unique positive equilibrium E_1 which, provided it exists, it is always LAS. The underlying unlagged model correponds to the case $T = 2a^{-1} = 0$, where T is the mean-delay. Hence we know the stability "status" of the model at a prescribed "natural" initial parameter constellation μ_0 and can apply the considerations of the previous section. If we study the stability the equilibrium point E_1 we get the characteristic polynomial with the coefficients:

$$\begin{array}{rcl} a_1 &=& 3a;\;\; a_2 = 3a^2 + mU\rho V;\;\; a_3 = a\left(a^2 + 3mU\rho V\right) \\ a_4 &=& a^2 mV\left(3U\rho + a\varepsilon\right)\;;\;\; a_5 = mU\rho V a^3 \end{array}$$

which are all positive. Hence the condition $\Delta_{n-1} = 0$ gives:

$$\Delta_4 = \det \begin{pmatrix} 3a & a\left(a^2+3mU\rho V\right) & mU\rho V a^3 & 0 \\ 1 & 3a^2+mU\rho V & a^2 mV\left(3U\rho + a\varepsilon\right) & 0 \\ 0 & 3a & a\left(a^2+3mU\rho V\right) & mU\rho V a^3 \\ 0 & 1 & 3a^2+mU\rho V & a^2 mV\left(3U\rho + a\varepsilon\right) \end{pmatrix} > 0$$

i.e.:

$$\Delta_4 = mVa^7\left(8\varepsilon a^2 - 9mV\varepsilon^2 a - 24mVU\rho\varepsilon\right) = 0$$

The locus defining the stability boundary is:

$$8\varepsilon a^2 - 9mV\varepsilon^2 a - 24mVU\rho\varepsilon = 0$$

which is a simple line in the (a, ε) plane. Therefore once the stability boundary is crossed a unique switch between stability and instability occurs: this implies, without the need for any further inquiry, that all the points of the stability boundary are SHB points. Just to check this point let us use "in toto" theorem three and compute Δ_2. We find:

$$\Delta_2 = 3a\left(3a^2 + mU\rho V\right) - a\left(a^2 + 3mU\rho V\right) = 8a^3$$

confirming our previous finding.[17]

[17] It is easy to check that the condition for the crossing with nonzero speed is satisfied.

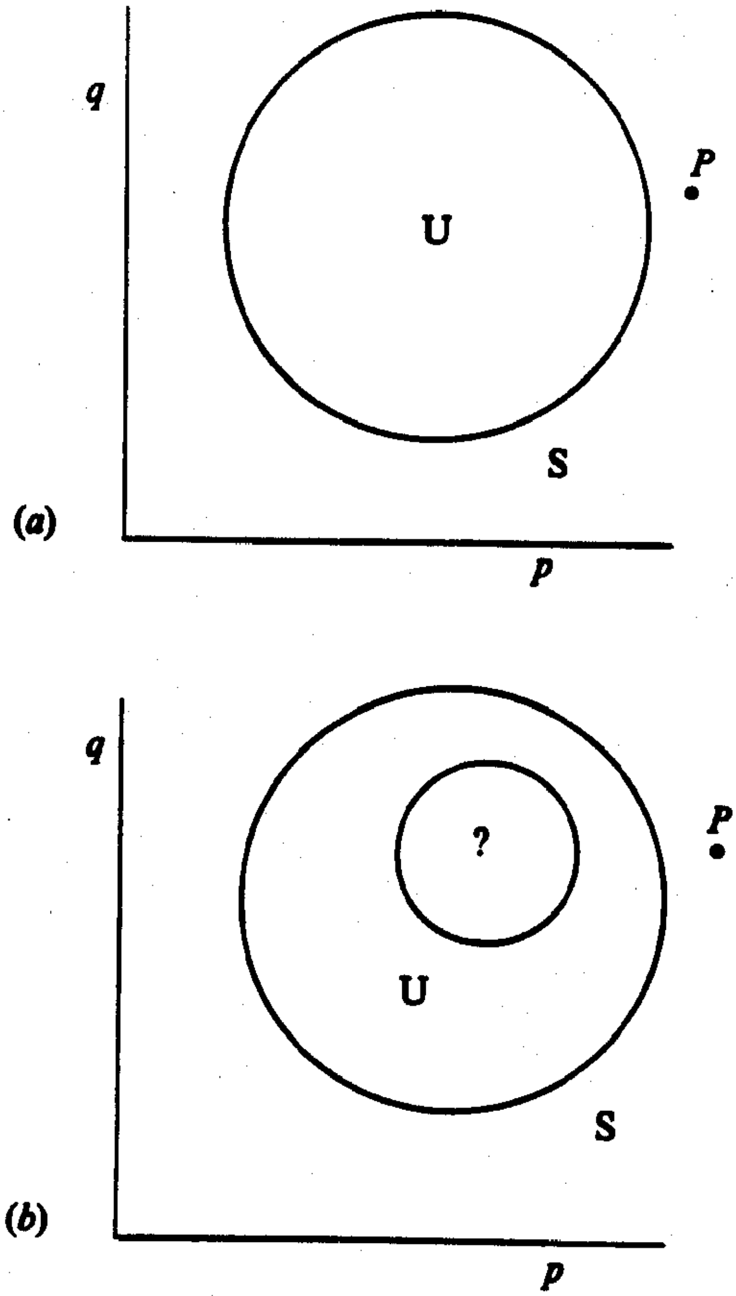

Fig. 1

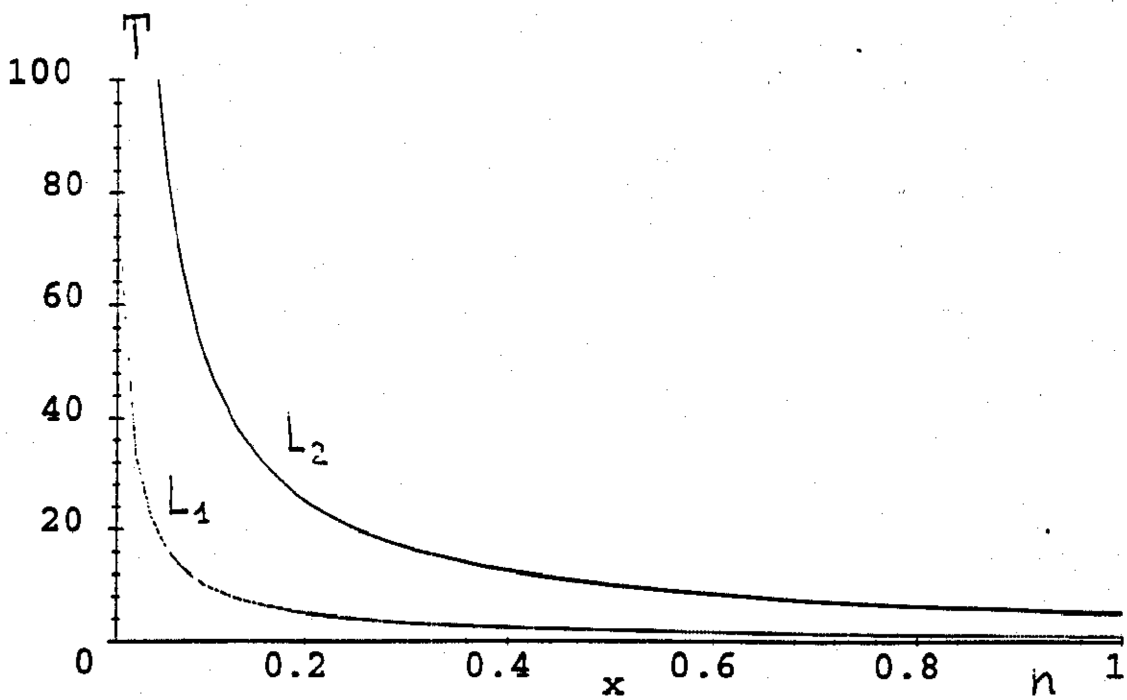

Fig. 2