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Abstract

We provide a characterization of Pareto efficient allocations in a continuous time economy with agents' preferences represented by stochastic differential utilities. Taking the endowment process and the discounted weights as state variables, the vector of stochastic differential utilities can be viewed as the viscosity solution of a highly degenerate parabolic PDE.

1 Introduction

This paper is devoted to the characterization of Pareto efficient allocations in a continuous time economy under uncertainty, when the agent's preferences are represented by a stochastic differential utility (SDU). For a characterization of this type of utility functional we refer the reader to [Duffie, et al., 1992].

The topic was addressed in [Duffie, et al., 1994], where the authors showed that the classical result obtained in the finite dimensional case, that is a Pareto optimal allocation is characterized by the fact that the agents' marginal rates of substitution coincide, carries forward in the infinite dimensional setting. In the same paper, by extending some ideas employed in a deterministic setting ([Lucas and Stokey, 1984], [Epstein, 1987],[Dana and Le Van, 1990]) and in a discrete time stochastic environment ([Kan, 1995]) to a continuous time stochastic economy, the authors showed that a Pareto Optimal allocation can be viewed as a function of the trajectory of a dynamic system determined by the utility processes and the agents' discounted weights. The system is backward-forward, in the sense that the utility processes have a backward evolution,while the agents' weights a forward one.

In the present work, we use these results as a starting point to show that the vector given by the utility processes shows a functional link to the agents' discounted weights and to the economy endowment process, when the latter ones are taken as state variables. This function is characterized as the viscosity solution of a highly degenerate parabolic Partial Differential Equation.

The results we obtain are related to [Dumas, et al., 1997], where efficient allocations are characterized through the value function of a *social planner*.

Here we draw inspiration from the work [Schroder e Skiadas, 1999], to define the optimal consumption policy through a SDU. In their paper, Schroeder and Skiadas consider a single agent problem, which leads them to consider a system with only one forward equation (obtained from the first order conditions for the optimal process) and one backward equation (the recursive utility process). This problem associates with a parabolic partial differential equation, to exploit in order to solve the stochastic system with the aid of the theory developed in [Ma et al., 1994]. The critique to this approach carried forward by [Dumas, et al., 1997] is that it is not very tractable and does not account for the multidimensional case.

Instead here we address the multidimensional case, which we are able to characterize by means of a highly degenerate parabolic PDE. This approach was to some extent already indicated in the work of [Duffie, et al., 1994], even though no explicit method to solve the system was given. As a matter of fact, the structure of the equations gives rise to some technical problems as to the solvability. We prove that those can be avoided by restricting appropriately the time interval appropriately and, under this condition, we provide the PDE characterization. As for the complexity of computation, it seems to be comparable to that of [Dumas, et al., 1997].

This note is organized as follows. In the next section we recall the results obtained in [Duffie, et al., 1994], characterizing the Pareto Optimal allocations of the economy. In section 3 we analyze the dynamics of the Pareto efficient frontier and allocations.

2 The Economy and the Pareto Efficient allocations

Let $[0,T]$ be a finite time interval and $(\Omega, \mathcal{F}, \mu)$ a complete probability space, endowed with a filtration $\mathbf{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ of $\sigma-$algebras of $\mathcal{F}$, satisfying the usual hypotheses (see [Protter 1990]). Later on, we will specify this filtration as generated by a Brownian motion defined on our probability space.

For any $p \in [1,\infty)$, we denote by D_p the space of all processes $X : \Omega \times [0,T] \to \mathrm{R}$ measurable with respect to the predictable $\sigma-$algebra on $\Omega \times [0,T]$ generated by the $\mathbf{F}-$adapted, left-continuous processes, such that

$$||X||_p \equiv [E(\int_0^T ||X_t||^p dt)]^{1/p} < \infty.$$

This will be the space of the consumption process c. We may consider R^d-valued processes, rather than only real valued, but all the proofs remain the same and we prefer to keep the dimensions low for ease of exposition. Finally we denote by D_p^+ the positive cone of D_p. Again for simplicity, we will prove our results for $c \in D_2$, since the techniques are identical for any $p > 1$. The case $p = 1$ would actually require a separate treatment, as the spaces where the solution processes live change slightly. Nevertheless an adaptation of the methods goes through, so, when needed, we will point out the main differences rather than running a complete separate proof for this case.

We consider an economy with n agents. Given a consumption process $c \in D_p^+$, the $i-$th agent is characterized by a SDU $U^i(c) = V_0^i(c)$, where V^i solves the backward stochastic differential equation

(1) $$V_t^i = E(\int_t^T f^i(c_s, V_s^i)ds|\mathcal{F}_t), \quad t \in [0,T].$$

Each $f^i : \mathrm{R}_+ \times \mathrm{R} \to \mathrm{R}$ is a continuous function satisfying a growth condition in the first argument and uniformly Lipschitz in the second one. We assume that each f^i, and therefore U^i, is strictly increasing in the first argument. Thanks to [Duffie, et al., 1992], when $p > 1$ we know that a $p-$integrable solution to (1) in the space

$$\underline{\underline{S}}^p = \{X \text{ semimartingales } : E(\sup_{t\in[0,T]} |X_t|^p) < +\infty\}$$

exists, the case $p = 1$ was covered in [Antonelli, 1993].

We denote by $e \in D_p^+$ the total endowment and we denote the feasible allocation set by

$$\Delta = \{\mathbf{c} = (c^1, c^2, \ldots, c^n) \in (D_p^+)^{\otimes n} : \ e - \sum_{i=1}^n c^i \geq 0\}.$$

Definition 2.1 : *A feasible allocation* $\mathbf{c}$ *is said Pareto Optimal if there is no other feasible allocation* $\hat{\mathbf{c}}$ *such that* $U^i(\hat{c}^i) \geq U^i(c^i)$, $i = 1, \ldots, n$, *with strict inequality for at least one agent.*

Given a set of positive weights $\alpha \in \mathrm{R}_+^n$, *we define* $U_\alpha : (D_p^+)^{\otimes n} \to \mathrm{R}$ *as*

$$U_\alpha(\mathbf{c}) = \sum_{i=1}^n \alpha_i U^i(c^i). \tag{2}$$

An allocation in $(D_p^+)^{\otimes n}$ *is* $\alpha-$**efficient**, *if it maximizes* U_α *over* Δ.

In [Duffie, et al., 1994, Proposition 1] it is shown that if the aggregators f^i are all concave, then $\mathbf{c}$ is Pareto Optimal if and only if there exists some non zero α such that $\mathbf{c}$ is α-efficient. The classical result establishing a one to one connection between a set of weights and a Pareto Optimal allocation is therefore confirmed in this setting.

The first order conditions characterizing Pareto optimality are obtained by imposing that the gradient of U_α is negative along any feasible direction. More precisely, given a fixed consumption process $\overline{c} \in D_p^+$, we define the set of feasible directions $F(\overline{c}) = \{h \in D_p : \overline{c} + h \in D_p^+\}$ and the Gateaux derivative of U^i at $\overline{c}$ in direction h as the linear functional

$$\nabla U^i(\overline{c})h = \lim_{\epsilon\to 0} \frac{U^i(\overline{c} + \epsilon h) - U^i(\overline{c})}{\epsilon}, \quad h \in F(\overline{c}),$$

whenever this limit exists.

Moreover we say that $\nabla U^i(\overline{c})$ admits a Riesz representation if there exists a process $\pi_t^i(\overline{c}) \in D_q$, with $\frac{1}{p} + \frac{1}{q} = 1$, such that

$$\nabla U^i(\overline{c})h = E(\int_0^T h_t \pi_t^i(\overline{c})dt) \quad \text{for} \quad h \in F(\overline{c}).$$

From now on, we assume that f^i are differentiable in both variables in the interior of the domain.

[Duffie, et al., 1994, Proposition 2] guarantees the existence of the Gateaux derivative of U^i and of its Riesz representation, if $p \geq 2$ and either f^i_c satisfies a uniform growth condition in the first argument, or if f^i is concave and $\overline{c}$ is bounded away from zero. Besides, it is straightforward to prove that

$$\pi^i_t(\overline{c}) = \exp\{\int_0^t f^i_v(\overline{c}_s, V^i_s)ds\} f^i_c(\overline{c}_t, V^i_t).$$

Restricting our attention to Pareto optimal allocations $(c^1, c^2, \ldots, c^n)$, so that the consumption process for each agent is bounded away from zero, then the allocation is $\alpha-$efficient and therefore the following first order conditions are verified

$$\alpha_i \pi^i(c^i)_t = \alpha_j \pi^j(c^j)_t \qquad \text{a.e. } (t, \omega), \quad i, j = 1, \ldots, n. \tag{3}$$

In [Duffie, et al., 1994] a set of sufficient conditions ensuring that the Pareto Efficient allocation is bounded away from zero is provided. The set includes Inada conditions and calls for an endowment process bounded away from zero and strictly positive weights.

Extending the reasoning developed in a deterministic setting in [Lucas and Stokey, 1984, Epstein, 1987, Dana and Le Van, 1990], in [Duffie, et al., 1994] it is shown that the Pareto Optimal allocation associated with a given vector of weights α can be viewed as a function of the solution of a backward-forward stochastic differential system, where the backward components are the agents' utility processes (1) and the forward ones are the agents' discounted weights.

In other words, for any vector of Pareto Optimal consumption processes $\mathbf{c} = (c^1, \ldots, c^n)$ and vector of weights $\alpha = (\alpha_1, \ldots, \alpha_n)$, we define the vector of the discounted weight processes $\Lambda = (\lambda^1, \ldots, \lambda^n)$ by

$$\lambda^i_t = \alpha_i \exp(\int_0^t f^i_v(c^i_s, V^i_s)ds), \qquad i = 1, \ldots, n.$$

With this notation, the first order conditions (3) become

$$\lambda^i_t f^i_c(c^i_t, V^i_t) = \lambda^j_t f^j_c(c^j_t, V^j_t) \qquad a.e.\ (t, \omega),\ i, j = 1, \ldots, n. \tag{4}$$

Following [Duffie, et al., 1994], we assume that each f^i is C^3 in the interior of its domain and that the Hessian matrix associated with $f^i(\cdot, v)$ is everywhere negative definite for all $v \in \mathrm{R}$. Under those assumptions, the implicit function theorem gives the existence of a C^2

function $\gamma : \mathrm{R}_+ \times \mathrm{R}^n_+ \times \mathrm{R}^n \to \mathrm{R}^n_+$, such that the Pareto Optimal allocation $\mathbf{c}$ associated with α can be written as

$$c^i = \gamma^i(e(\omega,t), \lambda^1(\omega,t), \ldots, \lambda^n(\omega,t), V^1(\omega,t), \ldots, V^n(\omega,t)), \quad i = 1, \ldots, n.$$

Summarizing, the $\alpha-$efficient allocation $\mathbf{c}$ can be characterized as $\mathbf{c}_t = \gamma(e_t, \Lambda_t, \mathbf{V}_t)$, where the couple of vectors $(\Lambda, \mathbf{V})$ solves the following stochastic integral system in $(\mathrm{R}^n \times \mathrm{R}^n)$

$$\lambda^i_t = \alpha_i \exp\{\int_0^t f^i_v(\gamma^i(e_s, \Lambda_s, \mathbf{V}_s), V^i_s)ds\} \tag{5}$$

$$V^i_t = E\Big(\int_t^T f^i(\gamma^i(e_t, \Lambda_t, \mathbf{V}_t), V^i_s)ds\,|\mathcal{F}_t\Big), \tag{6}$$

for $i = 1, \ldots, n$. In conclusion, as we already said, we know that the Inada conditions apply, that the endowment process is bounded away from zero and that f^i is regular, hence a solution to this system exists, see [Duffie, et al., 1994, Proposition 4].

3 The Dynamics of the Efficient Frontier

In this section our goal is to show that the Pareto efficient frontier $(V^1(\omega,t), \ldots, V^n(\omega,t))$ obtained as the solution of the system (5)-(6) can be characterized by means of the viscosity solution of a Partial Differential Equation. In some sense this means to revert to perspective and to concentrate on the system directly.. We first prove that the solution of (5)-(6) shows continuous dependence on the parameters, when restricting suitably the time interval

We need to assume that

A1. *the filtration* $\mathbf{F}$ *is generated by a one-dimensional Brownian motion* W*, augmented of the* $P-$*null sets and made right continuous to satisfy the "usual hypotheses". Besides the endowment process* e *satisfies the SDE*

$$e_s = x + \int_0^s \mu(r, e_r)dr + \int_0^s \sigma(r, e_r)dW_r, \tag{7}$$

with coefficients μ, $\sigma : [0,T] \times \mathrm{R} \longrightarrow \mathrm{R}$ *uniformly Lipschitz with constant* $k_1 > 0$.

As mentioned before, we are also taking the process e bounded away from zero, that is $e_s > \delta > 0$ all (s, ω). An example of such process might be $e_t = \mathcal{E}(W)_t + \delta$ ($\mathcal{E}$ denotes the stochastic exponential, see [Protter 1990]), with δ a strictly positive constant. Clearly the process is bounded away from 0 and it verifies the equation

$$de_t = (e_t - \delta)dW_t \qquad e_0 = 1 + \delta,$$

that has uniformly Lipschitz coefficients.

Furthermore we recall that we based the existence of the functions γ^i on the Inada conditions, thus we know that $\lim_{c\to 0} f^i_c(c,v) = \infty$ and $\lim_{c\to\infty} f^i_c(c,v) = 0$, $\forall\, v \in \mathrm{R}$. These imply that also the Pareto efficient allocations (and hence the γ^i's) are bounded away from zero, so when the functions f^i are acting on those, they are actually Lipschitz (globally) in the first argument. Therefore, from now on, we assume

A2. *there exists a constant* $\gamma_0 > 0$ *such that*

$$\gamma^i(e, \mathbf{l}, \mathbf{v}) > \gamma_0 \quad i = 1, \dots, n,$$

for all $(e, \mathbf{l}, \mathbf{v}) \in \mathrm{R}_+ \times \mathrm{R}^n \times \mathrm{R}^n$*;*

A3. *the coefficients* f^i *are all differentiable and there exists a constant* $k > 0$ *such that*

$$|f^i_v(c,v)| \le k \quad \textit{all } (c,v) \in \mathrm{R}\times\mathrm{R}, \qquad |f^i_c(c,v)| \le k \quad \textit{all } (c,v) \in [\gamma_0, +\infty) \times \mathrm{R}.$$

Remark 3.1 : *Last assumption implies that for a fixed* $v \in \mathrm{R}$*, as a function of* c*, each* $f^i(\cdot, v)$ *is bounded on* $[\gamma_0, +\infty)$*, let us denote by* $M(v)$ *the bound common to all the functions. Moreover, when* $c_s > \gamma_0$*, for all* s, ω*, we have that*

$$\begin{aligned} |V^i_t| &\le E\Big(\int_t^T |f^i(c_s, V^i_s)| ds | \mathcal{F}_t\Big) \le E\Big(\int_t^T \Big[k|V^i_s| + |f^i(c_s, 0)|\Big] ds | \mathcal{F}_t\Big) \\ &\le E\Big(\int_t^T \Big[k|V^i_s| + M(0)\Big] ds | \mathcal{F}_t\Big) \end{aligned}$$

which implies by the stochastic Gronwall's inequality

$$(8) \qquad |V^i_t| \le \frac{\mathrm{e}^{k(T-t)} - 1}{k} M(0) \qquad i = 1, \dots, n.$$

Also, let us remark that the forward components, due to the assumption A.2, are such that

$$\lambda^i_t = \alpha_i \exp\{\int_0^t f^i_v(c^i_s, V^i_s) ds\} \in [\alpha_i \mathrm{e}^{-kT}, \alpha_i \mathrm{e}^{kT}]$$

for all (t, ω). Besides the weights α_i are all strictly positive, hence there exist constants $\alpha_0, A_0 > 0$ such that $\alpha_0 < \alpha_i < A_0$, for all $i = 1, \dots, n$.

In conclusion, by continuity, without loss of generality we may assume that

A4. $\gamma^i(e, \mathbf{l}, \mathbf{v})$ *and* $f^i_v(\gamma^i(e, \mathbf{l}, \mathbf{v}), v_i)$ *are Lipschitz in all the arguments with constant* $K > 0$*, when* $(e, \mathbf{l}, \mathbf{v}), \in [\delta, +\infty) \times [\alpha_0 \mathrm{e}^{-kT}, A_0 \mathrm{e}^{kT}]^{\otimes n} \times \mathrm{R}^n$.

Let us take $t, x, \mathbf{y} = y_1, \ldots, y_n$ varying in $[0,T] \times \mathrm{R} \times \mathrm{R}^n$ and consider, for $i = 1, \ldots, n$, the following flows associated with our equations

$$\begin{aligned}
(9) \qquad e_s^{t,x} &= x + \int_t^s \mu(r, e_r^{t,x})dr + \int_t^s \sigma(r, e_r^{t,x})dW_r, \qquad e_t^{t,x} = x \\
(10) \qquad \lambda_s^{i,t,x,\mathbf{y}} &= y_i + \int_t^s \lambda_r^{i,t,x,\mathbf{y}} f_v^i(\gamma^i(e_r^{t,x}, \Lambda_r^{t,x,\mathbf{y}}, \mathbf{V}_r^{t,x,\mathbf{y}}), V_r^{i,t,x,\mathbf{y}})dr \qquad \lambda_t^{i,t,x,\mathbf{y}} = y_i \\
(11) \qquad V_s^{i,t,x,\mathbf{y}} &= E\Big(\int_s^T f^i(\gamma^i(e_r^{t,x}, \Lambda_r^{t,x,\mathbf{y}}, \mathbf{V}_r^{t,x,\mathbf{y}}), V_r^{i,t,x,\mathbf{y}})dr | \mathcal{F}_s\Big).
\end{aligned}$$

For any fixed $t_1, t_2 \in [0,T]$, $x_1, x_2 \in [\delta, +\infty)$ and vectors $\mathbf{y^1} = (y_1^1, \ldots, y_n^1), \mathbf{y^2} = (y_1^2, \ldots, y_n^2)$ in $[\alpha_0 \mathrm{e}^{-kT}, A_0 \mathrm{e}^{kT}]^{\otimes n}$, we denote

$$e_\cdot^j = e_{\cdot \vee t_j}^{t_j, x_j}, \quad \lambda_\cdot^{i,j} = \lambda_{\cdot \vee t_j}^{i,t_j,x_j,\mathbf{y^j}}, \quad V_\cdot^{i,j} = V_{\cdot \vee t_j}^{i,t_j,x_j,\mathbf{y^j}}, \qquad i = 1, \ldots, n, \quad j = 1, 2,$$

where $s \vee t$ stands for $\max(s,t)$.

Theorem 3.2 : *Under assumptions* **A1.** – **A4.**, *for T small enough, the flows (9), (10), (11) are continuous in $(t,x,\mathbf{y}) \in [0,T] \times [\delta, +\infty) \times [\alpha_0 \mathrm{e}^{-kT}, A_0 \mathrm{e}^{kT}]^{\otimes n}$. More specifically, for given t_1 and x_1, there exists a constant C_1 depending only on k_1, t_1, x_1, $\mu(r,0)$, $\sigma(r,0)$, such that*

$$(12) \qquad E(\sup_{s \in [0,T]} |e_s^2 - e_s^1|^2) \leq C_1(|x_2 - x_1|^2 + |t_2 - t_1|).$$

Further, if T is such that

$$(13) \qquad T(k + nK)\max(k, \mathrm{e}^{kT}) < 1,$$

then there exists a constant C_2, depending only on $k, k_1, K, T, t_1, x_1, \mathbf{y^1}$ and $M(0)$ such that

$$(14) \quad E(\sup_{s \in [0,T]} [\,||\Lambda_s^2 - \Lambda_s^1|| + ||\mathbf{V}_s^2 - \mathbf{V}_s^1||\,]^2) \leq C_2(|x_2 - x_1|^2 + ||\mathbf{y}^2 - \mathbf{y}^1||^2 + |t_2 - t_1|).$$

Proof: Without loss of generality we may assume $t_1 \leq t_2$. By the Lipschitz property of μ and σ, it is easy to verify that

$$\begin{aligned}
|e_s^2 - e_s^1|^2 &\leq 5|x_2 - x_1|^2 + 5(s \vee t_2 - t_2)\int_{t_2}^{s \vee t_2} k_1^2 |e_r^2 - e_r^1|^2 dr + [\int_{t_1 \wedge s}^{t_2 \wedge s} |\mu(r, e_r^1)|dr]^2 \\
&+ 5[\int_{t_2}^{s \vee t_2} |\sigma(r, e_r^2) - \sigma(r, e_r^1)|dW_r]^2 + [\int_{t_1 \wedge s}^{t_2 \wedge s} |\sigma(r, e_r^1)|dW_r]^2,
\end{aligned}$$

taking expectations and applying Doob's inequality we get

$$\begin{aligned}
E(\sup_{0 \leq s \leq t} |e_s^2 - e_s^1|^2) &\leq 5|x_2 - x_1|^2 + 5k_1^2(|t - t_2| + 1)\int_{t_2}^{t \vee t_2} E(\sup_{0 \leq s \leq r} |e_s^2 - e_s^1|^2)dr \\
&+ 5|t_2 - t_1|(1 + |t_2 - t_1|)\left[E(\sup_{t_1 \leq s \leq t_2} |e_s^1|^2) + \max_{0 \leq r \leq T}(|\mu(r,0)|^2 + |\sigma(r,0)|^2)\right]
\end{aligned}$$

and using Gronwall's inequality, we are able to derive (12).

Let us look at the other two equations. We remind the reader that in $\mathbb{R}^n$ the euclidean norm is equivalent to the norm $||\mathbf{x}|| = \sum_{i=1}^{n} |x_i|$ which we will use to prove (14), as the calculations follow more smoothly.

Let us analyze the forward components. Keeping in mind the previous remarks, exploiting our hypotheses, for each i we have

$$\begin{aligned}
|\lambda_s^{i,2} - \lambda_s^{i,1}| &\le |y_i^2 - y_i^1| + \int_{t_2}^{s\vee t_2} |\lambda_r^{i,2} f_v^i(\gamma^i(e_r^2, \Lambda_r^2, \mathbf{V}_r^2), V_r^{i,2}) - \lambda_r^{i,1} f_v^i(\gamma^i(e_r^1, \Lambda_r^1, \mathbf{V}_r^1), V_r^{i,1})|dr \\
&+ \int_{s\wedge t_1}^{s\wedge t_2} |\lambda_r^{i,1} f_v^i(\gamma^i(e_r^1, \Lambda_r^1, \mathbf{V}_r^1), V_r^{i,1})|dr \\
&\le |y_i^2 - y_i^1| + k\int_{t_2}^{s\vee t_2} |\lambda_r^{i,2} - \lambda_r^{i,1}|dr \\
&+ K\mathrm{e}^{kT} \int_{t_2}^{s\vee t_2} \Big[\sum_{i=1}^{n} |\lambda_r^{i,2} - \lambda_r^{i,1}| + \sum_{i=1}^{n} |V_r^{i,2} - V_r^{i,1}| + |e_r^2 - e_r^1|\Big] dr \\
&+ k\mathrm{e}^{kT}|s\wedge t_2 - s\wedge t_1|.
\end{aligned}$$

As for the backward components, we have that by the martingale representation theorem, there exist two predictable vector processes $\mathbf{Z}_r^1$ and $\mathbf{Z}_r^2$ such that

$$V_s^{i,j} = \int_{s\vee t_i}^{T} f^i(\gamma^i(e_r^j, \Lambda_r^j, \mathbf{V}_r^j), V_r^{i,j})dr - \int_{s\vee t_i}^{T} Z_r^{i,j} dW_r, \quad \text{with} \quad E\left(\int_0^T |Z_r^{i,j}|^2 dr\right) < +\infty$$

for each $i = 1, \dots, n$ and $j = 1, 2$.

Taking conditional expectations with respect to $\mathcal{F}_s$, the martingale parts disappear and by the Lipschitz property of the coefficients, we have

$$\begin{aligned}
|V_s^{i,2} - V_s^{i,1}| &\le E\Big(\int_{s\vee t_2}^{T} |f^i(\gamma^i(e_r^2, \Lambda_r^2, \mathbf{V}_r^2), V_r^{i,2}) - f^i(\gamma^i(e_r^1, \Lambda_r^1, \mathbf{V}_r^1), V_r^{i,1})|dr \\
&+ \int_{s\vee t_1}^{s\vee t_2} |f^i(\gamma^i(e_r^1, \Lambda_r^1, \mathbf{V}_r^1), V_r^{i,1})|dr|\mathcal{F}_s\Big) \\
&\le E\Big(\int_{s\vee t_2}^{T} k\Big[|\gamma^i(e_r^2, \Lambda_r^2, \mathbf{V}_r^2) - \gamma^i(e_r^1, \Lambda_r^1, \mathbf{V}_r^1)| + |V_r^{i,2} - V_r^{i,1}|\Big] dr \\
&+ \int_{s\vee t_1}^{s\vee t_2} [k|V_r^{i,1}| + |f^i(\gamma^i(e_r^1, \Lambda_r^1, \mathbf{V}_r^1), 0)|]dr|\mathcal{F}_s\Big) \\
&\le E\Big(\int_{s\vee t_2}^{T} \Big[kK(|e_r^2 - e_r^1| + \sum_{i=1}^{n} |\lambda_r^{i,2} - \lambda_r^{i,1}| + \sum_{i=1}^{n} |V_r^{i,2} - V_r^{i,1}|) + k|V_r^{i,2} - V_r^{i,1}|\Big] dr \\
&+ \int_{s\vee t_1}^{s\vee t_2} [k|V_r^{i,1}| + M(0)]dr|\mathcal{F}_s\Big) \\
&\le E\Big(\int_{s\vee t_2}^{T} \Big[kK(|e_r^2 - e_r^1| + \sum_{i=1}^{n} |\lambda_r^{i,2} - \lambda_r^{i,1}| + \sum_{i=1}^{n} |V_r^{i,2} - V_r^{i,1}|) + k|V_r^{i,2} - V_r^{i,1}|\Big] dr
\end{aligned}$$

$$+ e^{kT} M(0)|t_2 - t_1| |\mathcal{F}_s\Big),$$

where we used inequality (8) in the last passage.

Summing the components for $i = 1, \ldots, n$ we may summarize the above inequalities as

$$\begin{aligned}
||\Lambda_s^2 - \Lambda_s^1|| &\leq ||y^2 - y^1|| + k \int_{t_2}^{s\vee t_2} ||\Lambda_r^2 - \Lambda_r^1|| dr \\
&\quad + nK e^{kT} \int_{t_2}^{s\vee t_2} \Big[||\Lambda_r^2 - \Lambda_r^1|| + ||\mathbf{V}_r^2 - \mathbf{V}_r^1|| + |e_r^2 - e_r^1|\Big] dr + nk e^{kT} |t_2 - t_1| \\
||\mathbf{V}_s^2 - \mathbf{V}_s^1|| &\leq E\Big(\int_{s\vee t_2}^{T} \Big[nkK(|e_r^2 - e_r^1| + ||\Lambda_r^2 - \Lambda_r^1|| + ||\mathbf{V}_r^2 - \mathbf{V}_r^1||) + k||\mathbf{V}_r^2 - \mathbf{V}_r^1||\Big] dr |\mathcal{F}_s\Big) \\
&\quad + n e^{kT} M(0)|t_2 - t_1|.
\end{aligned}$$

Summing the two inequalities together, we finally obtain

$$\begin{aligned}
||\Lambda_s^2 - \Lambda_s^1|| + ||\mathbf{V}_s^2 - \mathbf{V}_s^1|| &\leq ||y^2 - y^1|| + R_1|t_2 - t_1| \\
&\quad + R_2 E\Big(\int_{s\vee t_2}^{T} \Big[|e_r^2 - e_r^1| + ||\Lambda_r^2 - \Lambda_r^1|| + ||\mathbf{V}_r^2 - \mathbf{V}_r^1||\Big] dr |\mathcal{F}_s\Big),
\end{aligned}$$

where $R_1 = n \max(k, M(0)) e^{kT}$ and $R_2 = (k + nK)\max(k, e^{kT})$. Squaring both sides, applying Cauchy-Schwarz inequality and Doob's inequality, we obtain

$$\begin{aligned}
&E\Big(\sup_{0\leq s\leq T} \Big[||\Lambda_s^2 - \Lambda_s^1|| + ||\mathbf{V}_s^2 - \mathbf{V}_s^1||\Big]^2\Big) \\
\leq\ & \frac{4||y^2 - y^1||^2 + 4R_1^2|t_2 - t_1|^2 + 4T^2 R_2^2 E(\sup_{0\leq s\leq T} |e_s^2 - e_s^1|^2)}{(1 - R_2 T)^2}
\end{aligned}$$

which gives our thesis, by virtue of our hypotheses, condition (13) and inequality (12). □

By hypothesis, all the coefficients occurring in the previous equations are deterministic and differentiable. By the standard technique of time shift and because of Blumenthal's 0-1 law, it is possible to show that the functions

$$\phi(t, x) = e_t^{t,x}, \qquad \psi^i(t, x, y_1, \ldots, y_n) = \lambda_t^{i,t,x,y_1,\ldots,y_n}, \qquad \theta^i(t, x, y_1, \ldots, y_n) = V_t^{i,t,x,y_1,\ldots,y_n}$$

are all deterministic. Proposition 3.2 tells us that these functions are locally Lipschitz in $x, y_1, \ldots, y_n$ and Hölder of order $\frac{1}{2}$ in t, consequently their derivatives are defined a.s. and bounded on compacts.

Our next goal is to prove that

$$\theta(t, x, \mathbf{y}) = \theta(t, x, y_1, \ldots, y_n) = (\theta^1(t, x, y_1, \ldots, y_n), \ldots, \theta^n(t, x, y_1, \ldots, y_n))^*$$

where $*$ denotes the transpose, is a viscosity solution of a system of degenerate semilinear parabolic PDE. First we would like to remind the notion of viscosity solution for second order operators. (For a detalied study of viscosity solutions we refer the reader to [Fleming, Soner, 1993])

Definition 3.3 : *Let $L = L(t, \theta, D\theta, D^2\theta)$ be an elliptic (possibly degenerate) operator and let us consider the PDE problem in a certain domain $\mathcal{O} \subseteq [0, T] \times \mathrm{R}^m$*

$$(15) \qquad \begin{cases} -\dfrac{\partial \theta}{\partial t} - L(t, \theta, D\theta, D^2\theta) = 0 \\ \\ \theta(t, x) - g(x) = 0 \qquad (t, x) \in \partial\mathcal{O}. \end{cases}$$

$\theta \in \mathcal{C}(\overline{\mathcal{O}})$ is said to be a viscosity sub- (resp. super-) solution of (15) if for any function $\varphi \in \mathcal{C}^{1,2}(\overline{\mathcal{O}})$, taken any $(\bar{t}, \bar{x}) \in \overline{\mathcal{O}}$, which is a global maximum (resp. minimum) point for $\theta - \varphi$, we have

$$(16) \qquad \begin{cases} -\dfrac{\partial \varphi}{\partial t}(t, \bar{x}) - L(\bar{t}, \bar{x}, \theta(\bar{t}, \bar{x}), D\varphi(\bar{t}, \bar{x}), D^2\varphi(\bar{t}, \bar{x})) \leq (\text{resp. } \geq) \, 0 \\ \\ \theta(\bar{t}, \bar{x}) - g(\bar{x}) \leq (\text{resp. } \geq) 0 \qquad \text{whenever} \quad (\bar{t}, \bar{x}) \in \partial\mathcal{O}. \end{cases}$$

θ is said to be a solution of (15) if it is both a viscosity sub and super-solution.

Remark 3.4 : *By the previous proposition, we have that the functions $\theta^i(t, x, \mathbf{y}) = V_t^{i,t,x,\mathbf{y}}$ are indeed continuous in $[0, T] \times [\delta, +\infty) \times [\alpha_0 \mathrm{e}^{-kT}, A_0 \mathrm{e}^{kT}]^{\otimes n}$, answering the first condition of viscosity solutions.*

Theorem 3.5 : *Under Assumptions* **A1.** – **A4.** *and condition (13), the vector function $\theta(t, x, \mathbf{y})$ is a viscosity solution of the PDE problem in $[0, T] \times [\delta, +\infty) \times [\alpha_0 \mathrm{e}^{-kT}, A_0 \mathrm{e}^{kT}]^{\otimes n}$,*

$$(17) \begin{cases} -\theta_t^i - \dfrac{\sigma^2(t, x)}{2}\theta_{xx}^i + \mu(t, x)\theta_x^i - \displaystyle\sum_{j=1}^{n} y_j f_v^j(\gamma^j(x, \mathbf{y}, \theta), \theta^j)\theta_{y_j}^i - f^i(\gamma^i(x, \mathbf{y}, \theta), \theta^i) = 0 \\ \\ \theta^i(T, x, \mathbf{y}) = 0, \end{cases}$$

for $i = 1, \ldots, n$.

Proof: First of all, we would like to remark that theoretically we are missing the boundary conditions in problem (17), but we could actually consider it in the whole of $\mathrm{R} \times \mathrm{R}^n$, since our solution processes automatically live in the considered region.

By construction, the processes $e_s^{t,x}, \Lambda_s^{t,x,\mathbf{y}}$ and $\mathbf{V}_s^{t,x,\mathbf{y}}$ have all continuous paths and are adapted with respect to the filtration generated by the Brownian motion. Therefore, because of the Markov property and the pathwise uniqueness of the solution, it is possible to show that actually $\mathbf{V}_s^{t,x,\mathbf{y}} = \theta(s, e_s^{t,x}, \Lambda_s^{t,x,\mathbf{y}})$ a.s.

To show our statement, we need to prove that θ is both a sub and a super-solution of (17). As a matter of fact, we only show the sub-solution inequality, since the proof of the other one goes along the same lines.

Let us consider a point $(t, x, \mathbf{y}) \in [0, T] \times \mathrm{R} \times \mathrm{R}^n$ and a family of smooth functions φ^i, $i = 1, \ldots, n$, such that

$$0 = \theta^i(t, x, \mathbf{y}) - \varphi^i(t, x, \mathbf{y})$$

is a global maximum for $\theta^i - \varphi^i$ (without loss of generality we can assume this maximum to be zero).

This means that for any stopping time τ, necessarily

$$\theta^i(\tau, e_\tau^{t,x}, \Lambda_\tau^{t,x,\mathbf{y}}) - \varphi^i(\tau, e_\tau^{t,x}, \Lambda_\tau^{t,x,\mathbf{y}}) \le 0. \tag{18}$$

From now on, for ease of writing we will omit the superscripts. Because of their regularity, we can apply Itô's formula to the functions φ^i's in the interval $[t, \tau]$, obtaining

$$\begin{aligned}\varphi^i(\tau, e_\tau, \Lambda_\tau) &= \varphi^i(t, x, \mathbf{y}) + \int_t^\tau \sigma(r, e_r)\varphi_x^i(r, e_r, \Lambda_r)dW_r \\ &+ \int_t^\tau \Big[\varphi_t^i(r, e_r, \Lambda_r) + \frac{\sigma^2}{2}(r, e_r)\varphi_{xx}^i(r, e_r, \Lambda_r) + \mu(r, e_r)\varphi_x^i(r, e_r, \Lambda_r)\Big]dr \\ &+ \int_t^\tau \Big[\sum_{j=1}^n \lambda_r^j f_v^j(\gamma^j(e_r, \Lambda_r, \mathbf{V}_r), V_r^j)\varphi_{y_j}^i(r, e_r, \Lambda_r)\Big]dr.\end{aligned}$$

On the other hand, because of (6), using the martingale representation theorem, we have

$$\begin{aligned}\theta^i(t, x, \mathbf{y}) = V_t^i &= V_\tau^i + \int_t^\tau f^i(\gamma^i(e_r, \Lambda_r, \mathbf{V}_r), V_r^i)dr - \int_t^\tau Z_r^i dW_r \\ &= \theta^i(\tau, e_\tau, \Lambda_\tau) + \int_t^\tau f^i(\gamma^i(e_r, \Lambda_r, \mathbf{V}_r), V_r^i)dr - \int_t^\tau Z_r^i dW_r,\end{aligned}$$

for each $i = 1, \ldots, n$. Substituting these last two equalities in (18), we obtain

$$\begin{aligned}0 &\ge \theta^i(\tau, e_\tau, \Lambda_\tau) - \varphi^i(\tau, e_\tau, \Lambda_\tau) \\ &= \theta^i(t, x, \mathbf{y}) - \varphi^i(t, x, \mathbf{y}) + \int_t^\tau \Big[Z_r^i - \sigma(r, c_r)\varphi_x^i(r, e_r, \Lambda_r)\Big]dW_r \\ &- \int_t^\tau \Big[\varphi_t^i(r, e_r, \Lambda_r) + \frac{\sigma^2}{2}(r, e_r)\varphi_{xx}^i(r, e_r, \Lambda_r) + \mu(r, e_r)\varphi_x^i(r, e_r, \Lambda_r)\Big]dr \\ &- \int_t^\tau \Big[\sum_{j=1}^n \lambda_r^j f_v^j(\gamma^j(e_r, \Lambda_r, \mathbf{V}_r), V_r^j)\varphi_{y_j}^i(r, e_r, \Lambda_r) + f^i(\gamma^i(e_r, \Lambda_r, \mathbf{V}_r), V_r^i)\Big]dr.\end{aligned}$$

By the uniqueness of paths, we know that $\mathbf{V}_r = \theta(r, e_r, \Lambda_r)$, therefore substituting in the former expression we obtain

$$\begin{aligned}
&\int_t^T \Big[Z_r^i - \sigma(r, c_r)\varphi_x^i(r, e_r, \Lambda_r)\Big] dW_r \\
&- \int_t^T \Big[\varphi_t^i(r, e_r, \Lambda_r) + \frac{\sigma^2}{2}(r, e_r)\varphi_{xx}^i(r, e_r, \Lambda_r) + \mu(r, e_r)\varphi_x^i(r, e_r, \Lambda_r)\Big] dr \\
&- \int_t^T \sum_{j=1}^n \lambda_r^j f_v^j(\gamma^j(e_r, \Lambda_r, \theta(r, e_r, \Lambda_r)), \theta^j(r, e_r, \Lambda_r))\varphi_{y_j}^i(r, e_r, \Lambda_r) \\
&- \int_t^T f^i(\gamma^i(e_r, \Lambda_r, \theta(r, e_r, \Lambda_r)), \theta^i(r, e_r, \Lambda_r)) dr \le 0.
\end{aligned}$$

Taking expectations, the martingale parts give no contribution and we can summarize the inequality by writing

(19) $$E\Big(\int_t^T \Sigma^i(r, e_r, \Lambda_r) dr\Big) \le 0,$$

where $\Sigma^i(\cdot,\cdot,\cdot) = -\varphi_t^i - L^i(\cdot,\cdot,\cdot,\theta(\cdot,\cdot,\cdot),\varphi^i(\cdot,\cdot,\cdot))$ and

$$\begin{aligned}
L^i(t, x, \mathbf{y}, \theta(t, x, \mathbf{y}), \varphi^i(t, x, \mathbf{y})) &= \frac{1}{2}\sigma^2(t, x)\varphi_{xx}^i(t, x, \mathbf{y}) + \mu(t, x)\varphi_x^i(t, x, \mathbf{y}) \\
&+ \sum_{j=1}^n y_j f_v^j(\gamma^j(x, \mathbf{y}, \theta(t, x, \mathbf{y})), \theta^j(t, x, \mathbf{y})\varphi_{y_j}^i(t, x, \mathbf{y}) \\
&+ f_v^i(\gamma^i(x, \mathbf{y}, \theta(t, x, \mathbf{y})), \theta^i(t, x, \mathbf{y})
\end{aligned}$$

To say that θ is a subsolution of (17) means that we must verify that $\Sigma^i(t, x, \mathbf{y}) \le 0$, each i. By contradiction we assume there exists an $\epsilon_i > 0$ such that $\Sigma^i(t, x, \mathbf{y}) > \epsilon_i$ and we define the stopping time

$$\tau_1^i = \inf\{s > t : \Sigma^i(s, e_s, \Lambda_s) \le \frac{\epsilon_i}{2}\} \wedge T.$$

Since $\Sigma^i(t, x, \mathbf{y}) > \epsilon_i$, we have $\tau_1 > t$ a.s. Inequality (19) holds for any stopping time, therefore also for τ_1^i and we have

$$0 < \frac{\epsilon_i}{2}(\tau_1^i - t) < E\Big(\int_t^{\tau_1^i} \Sigma^i(s, e_s, \Lambda_s) ds\Big) \le 0$$

which is a clear contradiction, hence we proved that θ is a subsolution of (17).

Analogously we can prove that θ is a viscosity super-solution of (17) and complete the proof.

□

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