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Estimating the Drift Parameter in Diffusion Processes more Efficiently at Discrete Times: A Role of Indirect Estimation

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Abstract

The main aim of this paper is to illustrate how an indirect simulation based procedure might be exploited to improve the properties of a maximum liklihood estimate of the drift parameter in a class of diffusion processes sampled at discrete time. This is achieved through a series of Monte Carlo experiments conducted on the Square Root, the Ornstein-Uhlenbeck and a Geometric Brownian Motion process which are among the most widely used examples, especially in financial economics, of the class of processes considered.

Some key words: discrete maximum likelihood estimator, Euler scheme, exact solution, indirect estimation, stochastic differential equations.

1 Introduction

Consider the following one-dimensional Stochastic Differential Equation (SDE) model:

$$dX_t = k(\vartheta - X_t)dt + \sigma X_t^\beta dW_t \quad (1)$$

(1) implies a mean reversion towards the long-term mean, ϑ, with speed of adjustment given by k. σ is a scale parameter and β is the variance elasticity parameter which measures the sensitivity of relative changes to the level of the stochastic process X. k, ϑ and σ are strictly positive and $t \in [0, T]$. The parameter k also establishes the degree of convexity of the mean solution of the process. W_t is a scalar Wiener process.

The aim of this paper is to illustrate how simulation based indirect inference may be used along with a maximum likelihood estimator in an attempt to estimate more efficiently the drift parameter, k, when discrete data observed at equidistant time points t_0 , $t_1 = t_0 + \Delta$, $t_0 + 2\Delta$, ... , T are available. To do this, a Monte Carlo experiment is conducted on the Square Root (SR) process, for which $\beta = 0.5$, on a Geometric Brownian Motion (GBM) process, for which $\beta = 1.0$, and on the Ornstein-Ulhenbeck (O-U) process, for which $\beta = 0$, using the maximum likelihood estimator studied in Cleur and Manfredi (1999) and the simulation-based indirect estimator presented in Gourieroux et al. (1993). Various sample, or observation, steps Δ are considered in order to evaluate their importance on estimation. Although the three processes considered in this paper are extremely simple versions of the general class (1), the problems encountered when estimating them and the results reported below are common to the whole class.

2 Generating data from SDE processes

This paper reports results from various experiments carried out on simulated rather than real data. A word should be said on the generation of the required data. Suppose that data from the underlying continuous process are required at an observation step Δ. If the SDE has a so-called "exact solution", which could be in the form of either a conditional distribution function or a dynamic time domain model, such data may be generated "exactly" for any Δ. Unfortunately, very few SDE processes possess an exact solution. Thus, a widely used practice (see for ex. Bianchi and Cleur (1996), Ball and

Torous (1996), Shoji and Ozaki (1997) and Calzolari et al (1998)) consists in generating the process, using a low-order numerical scheme (see, for example, Kloeden et al. (1992) and Prakasa Rao (1999)) on a very fine grid of equi-distant time points t_0 , $t_1 = t_0+\delta$, $t_0+2\delta$, ... , T, where $\Delta > \delta$, so as to obtain a good approximation to the trajectory of the underlying continuous process and then sample the generated values at the required sample step size Δ. It is common to call δ the generation step. Most often the Euler scheme approximation has been used for such purposes, which for model (1) is given by

$$X_{t_i} = X_{t_{i-1}} + k(\vartheta - X_{t_{i-1}})\delta + \sigma\sqrt{\delta}X^{\beta}_{t_{i-1}}W_{t_i} \ , \tag{2}$$

Thus, for ex., in Ball and Torous (1996) daily interest rate data are generated from the SR process by setting δ to 1/360 which are then sampled every 30 observations in order to obtain so-called monthly data, i.e. data corresponding to time points with a sample step Δ=1/12, and in Calzolari et al. (1998) who consider the estimation of the SR and the O-U processes, in order to have data at time points t = 1, 2, ... , T, i. e. with an observation step Δ=1, the approximation is generated with $\delta = 1/20$ which is then sampled every 20 observations. The correctness of such an approach is supported by numerous empirical findings which suggest that when the generation step is small (in Cleur and Manfredi (1999), but also from unpublished results of the same authors, a value of 0.01 was found adequate although in Shoji and Ozaki (1997) a value of 0.005 is applied) the Euler scheme provides a good approximation to the underlying continuous process. If, on the other hand, the generation step is not sufficiently small, in Cleur and Manfredi (1999) it was shown that a higher order aprroximation such as the Taylor 1.5 strong order scheme should be used. This approach of approximating (generating) the process on a fine grid of time points and then resampling it is also applied in this paper although the SR and O-U processes have a so-called exact solution.

3 Estimation Methods

The estimation, using maximum likelihood and indirect inference procedures, of the drift parameter in the O-U and SR processes for known σ and data available on a fine grid of time points with $\delta = \Delta = 0.01$ was considered in Cleur and Manfredi (1999). In the present paper, σ is taken as unknown and, hence, the problem of estimating it along with the remaining parameters of the underlying process from data available at various time steps $\Delta > \delta$ must be examined.

3.1 An Indirect Estimator

Roughly speaking, given an observed series from which a certain model has to be estimated, the indirect estimator consists in simulating a series of data from that model such that the difference between the real data and the simulated data is as small as possible according to some statistical criterion (see, for example, Duffie and Singleton (1993), Gallant and Tauchen (1996), Gourieroux et al. (1993)). The indirect estimator used in this paper was presented in Gourieroux et al. (1993) and has been widely experimented (for details, see also Bianchi and Cleur (1996), Broze et al. (1995), Calzolari et al. (1998), Cleur and Manfredi (1999) and Pagan et al. (1997)).

3.2 A Discrete Maximum Likelihood Estimator

When the continuous process is observed at equidistant time points t_0 , $t_1 = t_0 + \Delta$, $t_0 + 2\Delta$, ... , T, various approaches are available for obtaining Discrete Maximum Likelihood Estimators (DMLE henceforth) of k and ϑ for given values of σ and β. In Cleur (2000), it was shown that the maximum liklihood estimator used in Cleur and Manfredi (1999), the maximum likelihood ratio estimator of Yoshida (1992) (see also Shoji (1997 and Shoji and Ozaki (1997)) and the maximum likelihood estimator derived from the Euler scheme approximation produce very similar results even for processes observed over relativly short time intervals although they are only asymptotically equivalent (see, for example Shoji (1997)). Thus, in this paper, the first of the three methods mentioned was chosen for the computational illustrations for the simple fact that the explicit expressions for the estimates of k were already available from Cleur and Manfredi (1999) thereby rendering unneccesary the maximization of a likelihood.

Now, in the SR, the GBM and the O-U processes considered, β is fixed and hence does not have to be estimated, but at least a preliminary estimate of σ must be available in order to compute the maximum likelihood estimate. When the discretization step is sufficiently small, the following quadratic variation estimate has often been proposed in the literature on SDEs (see, for ex., Pedersen (1995), Polson and Roberts (1994), Prakasa Rao (1999), Shoji and Ozaki (1996)) and will be used in this paper

$$\hat{\sigma}^2 = \frac{1}{T} \sum_{t_i \leq T} \frac{(X_{t_i} - X_{t_{i-1}})^2}{X_{t_{i-1}}^{2\beta}} \tag{3}$$

The behaviour of the maximum liklihood estimator of the drift parameter is highly dependent on σ as underlined in Prakasa Rao (1999) (Chapter 3) and it is this aspect of the estimation of a SDE process which is studied in this paper.

4 Simulation Experiments

For the SR, the GBM and O-U processes, a Monte-Carlo experiment is carried out by generating the data using the Taylor 1.5 strong order scheme with the following parameter values: $k = 0.8$, $\vartheta = 0.10$, $\sigma = 0.06$, and generation step $\delta = 0.01$. These data are subsequently sampled by taking every fifth, tenth, twenyfifth, fiftieth and hundreth generated value; in other words, the effects of five different sample steps, i.e. $\Delta = 0.05$, 0.10, 0.25, 0.50 and 1.00, are evaluated. It was decided to apply a low-order numerical scheme approximation for the generation of the data instead of the exact solutions when these were available mainly for uniformity with other published works. Processes of length T = 100 and T = 2000 are considered. Each combination of the above parameter values, series lengths and observation steps is replicated 10,000 times.

The starting value for each replication, X_0, is always set to the long-term mean of the process, i.e. $X_0 = \vartheta = 0.1$.

The parameters are estimated using the procedures outlined in the previous Section.

Since the behaviour of the maximum liklihood estimator of the drift parameter in the three processes considered was very similar, to save space only the results for the SR process are reported.

4.1 Estimation of the SR process with known σ

In this Section, the importance of having a good preliminary estimate of σ in the DMLE is evidenced by first considering the performance of this estimator in the special case when σ is known. These results will then serve as a benchmark for evaluating the effects of an estimate of σ.

The estimation of ϑ does not pose any problems as seen from the published literature (see, for example, Calzolari et al. (1998) and Cleur and Manfredi (1999)) and will therefore not be the object of any further interest in this paper.

Tables 1 and 2 summarize the results when the generated process is sampled at regular intervals, and k is calculated from the expressions in Cleur and Manfredi (1999). The standard errors reported in brackets refer to the 10,000 estimates, i.e. what is commonly known as the Monte Carlo standard error, so that if confidence intervals are desired, these should be divided by $\sqrt{10000} = 100$.

Table 1. DMLE of the SR process. T = 100
True Values: k = 0.8, $\vartheta = 0.10$. $\sigma = 0.06$ taken as known,
$\delta = 0.01$ (standard errors in brackets). 10,000 replications

	$\Delta = 0.01$	$\Delta = 0.05$	$\Delta = 0.10$	$\Delta = 0.25$	$\Delta = 0.50$	$\Delta = 1.00$
$\hat{k}$	.8403	.8403	.8407	.8406	.8414	.8499
	(.1319)	(.1319)	(.1339)	(.1339)	(.1377)	(.1521)

Table 2. DMLE of the SR process. T = 2000
True Values: k = 0.8, $\vartheta = 0.10$. $\sigma = 0.06$ taken as known,
$\delta = 0.01$ (standard errors in brackets). 10,000 replications

	$\Delta = 0.01$	$\Delta = 0.05$	$\Delta = 0.10$	$\Delta = 0.25$	$\Delta = 0.50$	$\Delta = 1.00$
$\hat{k}$	.8018	.8019	.8019	.8019	.8019	.8022
	(.2852E-1)	(.2856E-1)	(.2862E-1)	(.2829E-1)	(.2910E-1)	(.3168E-1)

The differences in the estimates of k for varying sample steps, Δ, are very limited thereby suggesting that, when σ is known, it does not really matter whether the process is observed on a fine or relatively large grid of time points.This result has not been particularly evidenced in the existing literature although it may be argued that σ is generally unknown.

Table 1 conveys the message that, even if σ were known, maximum likelihood methods will not be able to produce unbiased estimates of the drift parameter for short series of data.

4.2 Estimation of the SR process with σ unknown

4.2.1 The Maximum Likelihood Estimate

As mentioned above, in the published literature the quadratic variation estimator of σ (3) has often been suggested for obtaining maximum likelihood estimates of the drift parameter in SDE processes. A simulation experiment, based on 10,000 replications was carried out in order to evaluate the bahaviour of this estimator for data sampled at fixed time intervals. The results for the SR process are reported in Table 3.

Table 3. Quadratic Variation Estimate of σ in the SR process
True values: k = 0.8, $\vartheta = 0.10$, $\sigma =$ 0.06, $\delta = 0.01$
(standard errors in brackets). 10,000 replications

T	$\Delta = 0.01$	$\Delta = 0.05$	$\Delta = 0.10$	$\Delta = 0.25$	$\Delta = 0.50$	$\Delta = 1.00$
100	.5988E-1	.5941E-1	.5885E-1	.5710E-1	.5455E-1	.4994E-1
	(.4230E-3)	(.9802E-3)	(.1351E-2)	(.2129E-2)	(.2869E-2)	(.4017E-2)
2000	.5989E-1	.5943E-1	.5886E-1	.5722E-1	.5469E-1	.5010E-1
	(.9451E-4)	(.2178E-3)	(.9451E-4)	(.4687E-3)	(.6534E-3)	(.9149E-3)

The underestimation of the true value of σ increases with the sample step, but the bias does not appear to depend from the length, T, of the observed process.

Tables 4 and 5 report the estimates of k calculated from the expressions in Cleur and Manfredi (1999) with σ substituted by the corresponding quadratic variation estimate defined in (3) and reported in Table 3. As could be anticipated, the substitution of a badly biased estimate of σ has notable repercussions on the estimate of k.

Table 4. DMLE of the SR process. T = 100
True Values: k = 0.8, $\vartheta = 0.10$, $\sigma = 0.06$, $\delta = 0.01$
(standard errors in brackets). 10,000 replications

	$\Delta = 0.01$	$\Delta = 0.05$	$\Delta = 0.10$	$\Delta = 0.25$	$\Delta = 0.50$	$\Delta = 1.00$
$\hat{k}$	.8371	.8241	.8089	.7616	.6934	.5795
	(.1314)	(.1293)	(.1289)	(.1199)	(.1090)	(.9329E-1)

Table 5. DMLE of the SR process. T = 2000
True Values: k = 0.8, $\vartheta = 0.10$, $\sigma = 0.06$, $\delta = 0.01$
(standard errors in brackets). 10,000 replications

	$\Delta = 0.01$	$\Delta = 0.05$	$\Delta = 0.10$	$\Delta = 0.25$	$\Delta = 0.50$	$\Delta = 1.00$
$\hat{k}$	.7988	.7868	.7718	.7297	.6658	.5588
	(.2844E-1)	(.2807E-1)	(.2719E-1)	(.2570E-1)	(.2373E-1)	(.2056E-1)

4.2.2 An Indirect Estimate

In recent years, indirect simulation-based estimation procedures have been increasingly applied. Such procedures, based on repeated approximations to the underlying model to be estimated, are computationally intensive, but have provided very promising results in the estimation of SDE processes (see, for example, Bianchi and Cleur (1996), Broze et al. (1995), Calzolari et al. (1998), Cleur and Manfredi (1999) and Gourieroux et al (1993)). In this Section, for completeness in the presentation, the capability of the indirect estimation procedure defined in Gourieroux et al. (1993) in correcting for the heavy bias due to the sampling of the underlying process is evidenced. Table 6 summarizes the results for the SR process when T=100; we can expect a general improvement for larger T. Calibration, which is an integral part of the procedure, was carried out on the DMLEs of k, ϑ and σ although we report results only for k.

For details on the computational procedure followed, the reader may refer to Bianchi and Cleur (1996) or Cleur and Manfredi (1999). Identical results were obtained by calibrating the estimates of a "naive" model defined by the Euler scheme approximation (see Bianchi and Cleur (1996), Calzolari (1998)).

Table 6. Indirect Estimates of the SR process. T = 100
True Values: k = 0.8, $\vartheta = 0.10$, $\sigma = 0.06$, $\delta = 0.01$.
(standard errors in brackets). 10,000 replications

	$\Delta = 0.05$	$\Delta = 0.10$	$\Delta = 0.25$	$\Delta = 0.50$	$\Delta = 1.00$
$\hat{k}$	.8038	.8046	.8062	.8078	.8072
	(.1874)	(.1938)	(.2019)	(.2224)	(.2616)

The overall capability of the indirect estimate to correct for bias is clearly evident in Table 6. It may be noted that, although the bias in $\hat{k}$ increases by almost 50% when passing from $\Delta = 0.05$ to $\Delta = 1.0$, in absolute terms it remains small and insignificant. This suggests that if we are prepared to apply a simulation-based indirect estimation procedure, the continuous diffusion process does not have to be observed on a fine grid of time points which, from a practical point of view, is encouraging.

On the other hand, the standard errors in Table 6 are markedly high and this is a defect inherent in any indirect estimation procedure; i.e., the indirect estimator could be very inefficient.

5 Improving the DMLE

The above results indicate a badly biased maximum likelihood estimator and a bias correcting, but inefficient, simulation-based indirect estimator when the sample, or observation, step is large. Clearly, a solution which reduces, not only the bias, but keeps low the standard errors of the estimate would be most welcome. Calzolari et al. (1998) use control variates in an attempt to reduce this inefficiency in the indirect estimate. The solution works reasonably well, but requires additional computations. An alternative solution, if one is interested in estimating the drift parameter, is prompted by the above results themselves.The DMLE of the drift coefficient, $\hat{k}$, was obtained conditional on a preliminary quadratic variation estimate of σ. It was noted that when the sample step is large, this preliminary estimate was heavily biased and, as a result, so too was $\hat{k}$. Hence, in practice, if the real data are available at a large observation step they cannot be used to obtain a good estimate of σ needed for estimating the drift parameter using maximum likelihood techniques. However, since the indirect estimates are obtained from a series of simulated data which are initially generated on a very fine grid of time points, with generation step δ, before being sampled at the same observation step as the original real data and before proceeding to the calibration phase, suggests that the whole series of simulated values, before sampling, might be used to obtain the required preliminary estimate of σ. If such a procedure is followed, for a small T the resulting DMLE will, in any case, be biased as is evident from Table 1, but for a large T there could be very marked improvements in terms of bias over the DMLE reported in Table 5 and in terms of efficiency in terms of the indirect estimator. We note from Table 7 that the proposed procedure appears to provide some promising results when the sampling step is not too large, say ≤ 0.5; the variance of the DMLE with respect to the variance of the indirect estimate passes from approximately 50% when $\Delta = 0.01$, to approximately 67% when $\Delta = 0.5$. Overall, the results for the DMLE are very close to those of Table 2 where the true value of σ was used.

Table 7. DMLE[a] and Indirect Estimate of the SR process. True values: k = 0.8, $\vartheta = 0.10$, $\sigma = 0.06$, $\delta = 0.01$ T = 2,000 (standard errors in brackets). 10,000 replications

	$\Delta = 0.01$	$\Delta = 0.05$	$\Delta = 0.10$	$\Delta = 0.25$	$\Delta = 0.50$	$\Delta = 1.00$
DMLE	.8018	.8018	.8019	.8019	.8023	.8037
	(.2867E-1)	(.2965E-1)	(.3079E-1)	(.3447E-1)	(.4139E-1)	(.5762E-1)
Indirect	.8002	.8003	.8000	.8002	.8005	.8018
	(.3986E-1)	(.4060-1)	(.4141E-1)	(.4430E-1)	(.5005E-1)	(.6374E-1)

[a] initial estimate of σ obtained from the simulated data used in the indirect procedure

A comparison of Tables 5 and 7 clearly illustrates the advantage in applying a combination of the indirect and maximum likelihood procedures in estimating the drift parameter of the SDE models considered here.

Since the indirect estimation procedure used provides also an estimate of σ, one might be led into using this estimate in the same manner as the quadratic variation estimate above, but the results obtained are slightly less encouraging than those reported in Table 7 for the simple reason that the indirect estimate of σ suffers from a relatively larger variance with respect to the quadratic variation estimate.

6 Conclusions

In order to estimate the drift parameter of a SDE model from sampled discrete data using standard maximum likelihood methods a preliminary quadratic variation estimate of the diffusion coefficient, σ, is often used. This preliminary estimate is heavily biased when the discrete data are observed at large time intervals and, as a result, so too is the maximum likelihood estimate of the drift parameter. Simulation based methods like the one defined in Gourieroux et al (1993) contemporaneously correct for the bias in all the parameters of the SDE model as evidenced in various Monte Carlo studies among which Bianchi and Cleur (1996), Broze et al. (1995), Calzolari et al. (1998) and Cleur and Manfredi (1999). Indirect estimators, however, suffer from relatively large sampling variations and hence the desire for more efficient estimates remains. A possible solution, proposed in this paper and which produces interesting results, could involve the use of an indirect estimate of σ in calculating a maximum likelihood estimate of the drift. The gain in terms of efficiency is comparable to that produced by the control variate approach proposed in Calzolari et al (1998) and has the advantage of being computationally less expensive.

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