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First and second order optimality conditions in vector optimization

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Abstract

In this paper we suggest a general approach in studying optimality for a multiobjective problem. First and second order optimality conditions are firstly achieved by means of suitable tangent sets; the obtained results are specified for an unconstrained problem and for a problem whose feasible region is expressed by means of functional constraints. Furthermore, the role played by generalized concavity and by second order regularity conditions is pointed out in order to achieve first order sufficient optimality conditions and in order to obtain second order optimality conditions in a dual form involving multipliers, respectively.

Keywords Vector Optimization, Generalized concavity, optimality conditions.
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1 Introduction

Optimality conditions for a scalar optimization problem have been studied extensively in the literature. In these last years particular attention has been devoted to second order optimality conditions and different approaches have been suggested in order to obtain results in a primal form and in a dual form involving multipliers, with respect to any feasible region, non necessarily expressed by functional constraints [2, 5, 8, 11, 12, 14, 16, 17, 19, 25, 26].
With respect to a multiobjective problem, while first order necessary and/or sufficient optimality conditions have been extensively studied by several authors, little work has been concerned with second order ones. In this paper we suggest a general approach which extends to vector case the one given in [14] and which point out the role of generalized concavity in order to achieve sufficient condition.
More precisely, in Section 3, first order optimality conditions are expressed

by means of the Bouligand tangent cone, while second order optimality conditions are established by introducing suitable second order tangent sets. The obtained results are specified for an unconstrained problem (Section 4) and for a problem whose feasible region is expressed by means of functional constraints (Section 5). At last, the role played by second order regularity is pointed out in order to obtain second order optimality conditions in a dual form involving multipliers.

2 Preliminaries

In this section we introduce notations, definitions and some known results which will be used throughout the paper.

- $\Re^k$ is the k-dimensional Euclidean space;
- $int\Re^k_+$ ($Fr\Re^k_+$) denotes the interior (the boundary) of the Paretian cone $\Re^k_+ = \{x = (x_1, ..., x_k) \in \Re^k : x_i \geq 0, i = 1, ..., k.\}$;
- x^T denotes the transpose of x;
- if K is a finite set we denote with $\mid K \mid$ the cardinality of K.

Let $\psi : \Re^n \to \Re^l$ be any twice continuously differentiable vector function. We denote by

- $\psi'(x)$ the Jacobian of ψ at x, whose i-th row is $\psi_i'(x)$, the gradient of ψ_i at x;
- $\psi_i''(x)$ the Hessian matrix of ψ_i at x;
- $\psi''(x)(d,d)$ the vector whose i-th component is $d^T\psi_i''(x)d$.

In order to obtain optimality conditions expressed by means of multipliers, we will utilize some known separation theorems. Let W be a linear subspace of $\Re^t$ and a an element of $\Re^t$ (when $\Re^t = \Re^s \times \Re^m$, an element w of W will be denoted by $w = (w_1, w_2)$, $w_1 \in \Re^s$, $w_2 \in \Re^m$).

Theorem 2.1 *i) $W \cap int\Re^t_+ = \emptyset$ if and only if*

$$\exists \gamma \in \Re^t_+ \setminus \{0\} : \gamma^T w = 0, \ \ \forall w \in W \tag{2.1}$$

ii) $W \cap \Re^t_+ = \{0\}$ if and only if

$$\exists \gamma \in int\Re^t_+ : \gamma^T w = 0, \ \ \forall w \in W \tag{2.2}$$

iii) $(W + a) \cap \Re^t_+ = \emptyset$ if and only if

$$\exists \gamma \in \Re^t_+ \setminus \{0\} : \gamma^T w = 0, \ \ \forall w \in W, \ \gamma^T a < 0 \tag{2.3}$$

Theorem 2.2 *i) $W \cap (int\Re^s_+ \times int\Re^m_+) = \emptyset$ if and only if*

$$\exists \alpha \in \Re^s_+, \exists \beta \in \Re^m_+, (\alpha, \beta) \neq 0 : \alpha^T w_1 + \beta^T w_2 = 0, \ \ \forall (w_1, w_2) \in W \tag{2.4}$$

ii) $W \cap (int\Re^s_+ \times \Re^m_+) = \emptyset$ if and only if

$$\exists \alpha \in \Re^s_+ \setminus \{0\}, \exists \beta \in \Re^m_+ : \alpha^T w_1 + \beta^T w_2 = 0, \ \ \forall (w_1, w_2) \in W \tag{2.5}$$

iii) $W \cap (\Re^s_+ \times \Re^m_+) = \{0\}$ *if and only if*

$$\exists \alpha \in int\Re^s_+, \exists \beta \in int\Re^m_+ : \alpha^T w_1 + \beta^T w_2 = 0, \ \forall (w_1, w_2) \in W \qquad (2.6)$$

Now we consider the following vector optimization problem:

$$P : max \ F(x), x \in S \subseteq X$$

where X is an open set of $\Re^n$, $F : X \to \Re^s$, and S is any subset of X. A point x_0 is said to be a Pareto efficient point or Pareto point for short for problem P if

$$F(x) \notin F(x_0) + \Re^s_+ \setminus \{0\}, \forall x \in S \qquad (2.7)$$

If (2.7) is verified in a suitable neighbourhood I of x_0, x_0 is said to be a local Pareto efficient point.
In finding conditions under which first order necessary optimality conditions become sufficient too, an important role is played by the concept of generalized concavity at a point introduced by Mangasarian [22] for a scalar optimization problem.
As is known, there are different ways in generalizing to the vector case the definitions of generalized concave functions given in the scalar case [7, 10, 15, 18, 20]. For the aim of this paper we limit ourselves to consider the following classes of generalized concave vector valued functions.
Assume that S is star shaped at $x_0 \in S$, that is $x \in S$ implies

$$[x_0, x] = \{tx + (1-t)x_0 : t \in [0,1]\} \subset S$$

Definition 2.1 The function F is said to be $\Re^s_+$-quasiconcave at x_0 (with respect to the star shaped set S at x_0) if:

$$x \in S, F(x) \in F(x_0) + \Re^s_+ \Rightarrow F(x_0 + \lambda(x - x_0)) \in F(x_0) + \Re^s_+, \forall \lambda \in [0,1]$$

Definition 2.2 Let F be differentiable at x_0 ; F is said to be $\Re^s_+$-weakly pseudoconcave at x_0 (with respect to the star shaped set S at x_0) if:

$$x \in S, F(x) \in F(x_0) + \Re^s_+ \setminus \{0\} \Rightarrow F'(x_0)d \in \Re^s_+ \setminus \{0\}, d = \frac{x - x_0}{\| x - x_0 \|}$$

.

Definition 2.3 Let F be differentiable at x_0 ; F is said to be $\Re^s_+$-pseudoconcave at x_0 (with respect to the star shaped set S at x_0) if:

$$x \in S, F(x) \in F(x_0) + \Re^s_+ \setminus \{0\} \Rightarrow F'(x_0)d \in int\Re^s_+, d = \frac{x - x_0}{\| x - x_0 \|}$$

Let us note that when $s = 1$, Definition 2.1 is the ordinary definition of quasiconcave function at a point x_0, while Definitions 2.2 and 2.3 collapse to the ordinary definition of a pseudoconcave function at x_0 [22].

Remark 2.1 If F is differentiable at x_0 and $\Re^s_+$-quasiconcave at x_0, then $F(x) \in F(x_0) + \Re^s_+$ implies $F'(x_0)d \in \Re^s_+$, $d = \frac{x-x_0}{\|x-x_0\|}$; the converse is not true [15].

Remark 2.2 It follows immediately from the given definitions that a linear function is $\Re^s_+$-weakly pseudoconcave but it is not $\Re^s_+$- pseudoconcave; furthermore if S is a convex set and F is a $\Re^s_+$-concave function (i.e. every component of F is concave), then F is $\Re^s_+$-weakly pseudoconcave but not necessarily $\Re^s_+$- pseudoconcave.

Remark 2.3 Let us note that the previous definitions do not imply componentwise quasiconcavity or pseudoconcavity. With this regard it is sufficient to observe that if at least one component of F has a strict local maximum point at x_0, then F verifies Definitions 2.1, 2.2 and 2.3 without any other requirement on the other components of F.

3 First and second order optimality conditions: a general approach

In this section we state some necessary and/or sufficient optimality conditions for problem P by means of a general approach involving the directions belonging to the tangent cone to the feasible region at x_0.
We recall that the Bouligand tangent cone to the set S at $x_0 \in S$, is the set:

$$S'(x_0) = \{d : \exists\{\alpha_n\} \subset \Re, \exists\{x_n\} \subset S, \alpha_n \to +\infty, x_n \to x_0, with \ \alpha_n(x_n - x_0) \to d\}.$$

Let us note that $S'(x_0) = \{0\}$ if and only if x_0 is an isolated point and in such a case x_0 is obviously a Pareto efficient point for problem P. For this reason throughout the paper it will be assumed that $S'(x_0) \neq \{0\}$.
The optimality conditions that we are going to establish in this section and in the following ones involve convergent or divergent sequences. With this regard, from now on, to simplify notations and to save words, we take advantage of the fact that, for any bounded (unbounded) sequence in a finite dimensional normed space, there exists a convergent (divergent in norm) subsequence. Namely, when we say that a bounded sequence z_n converges to z, we mean that if this sequence is not convergent, then it is replaced by its appropriate subsequence and this subsequence is again denoted by z_n. Similar abuse of language and notation is applied to unbounded sequences, when we say that z_n diverges to infinity.
A necessary first order optimality condition is stated in the following theorem.

Theorem 3.1 *Consider problem P where F is differentiable at x_0. If x_0 is a local Pareto efficient point for P then*

$$F'(x_0)d \notin int\Re^s_+, \ \ \forall d \in S'(x_0) \tag{3.1}$$

Proof Let $d \in S'(x_0)$; then there exist $\{x_n\} \subset S$, $\{\alpha_n\} \subset \Re$, $x_n \to x_0$, $\alpha_n \to +\infty$, such that $\alpha_n(x_n - x_0) \to d$. By Taylor's expansion we have:

$$F(x_n) - F(x_0) = F'(x_0)(x_n - x_0) + o(\| x_n - x_0 \|) \tag{3.2}$$

with $lim_{x_n \to x_0} \frac{o(\|x_n - x_0\|)}{\|x_n - x_0\|} = 0$.
Consequently

$$\alpha_n(F(x_n) - F(x_0)) = F'(x_0)(\alpha_n(x_n - x_0)) + \alpha_n \ o(\| x_n - x_0 \|)$$

It results

$$\alpha_n \ o(\| x_n - x_0 \|) = \alpha_n \| x_n - x_0 \| \ \frac{o(\| x_n - x_0 \|}{\| x_n - x_0 \|} \to 0,$$

so that

$$\alpha_n(F(x_n) - F(x_0)) \to F'(x_0)d$$

On the other hand, the local efficiency of x_0 implies the existence of $\bar{n}$ such that $\forall n > \bar{n}$ it results $F(x_n) - F(x_0) \notin \Re^s_+ \backslash \{0\}$, so that $\alpha_n(F(x_n) - F(x_0)) \notin \Re^s_+ \setminus \{0\}$ and thus $lim_{x_n \to x_0} \alpha_n(F(x_n) - F(x_0) = F'(x_0)d \notin int\Re^s_+$. The thesis follows. □

The following simple example points out that (3.1) is not a sufficient optimality condition.

Example 3.1 Consider problem P where $F(x_1, x_2) = (x_1, -x_2^2)$, and $S = \{(x_1, x_2) : x_1 \geq 0\}$. At the point $(0,0)$, we have $F'(0,0)d = (d_1, 0)^T$ for every $d = (d_1, d_2)^T \in S'(0,0) = S$. Condition (3.1) is verified but $(0,0)$ is not a local Pareto point.

The following theorem states a sufficient optimality condition.

Theorem 3.2 *Consider problem P where F is differentiable at x_0. A sufficient condition for x_0 to be a local Pareto efficient point for P is*

$$F'(x_0)d \notin \Re^s_+, \forall d \in S'(x_0), d \neq 0 \tag{3.3}$$

Proof Assume that x_0 is not a local Pareto point for P. Then there exists a feasible sequence $\{x_n\}$ with $x_n \to x_0$ and $F(x_n) - F(x_0) \in \Re^s_+ \setminus \{0\}$. By Taylor's expansion (3.2), we have

$$\frac{F(x_n) - F(x_0)}{\| x_n - x_0 \|} = F'(x_0)(\frac{x_n - x_0}{\| x_n - x_0 \|}) + \frac{o(\| x_n - x_0 \|)}{\| x_n - x_0 \|}$$

Since $\frac{x_n - x_0}{\|x_n - x_0\|} \to d \in S^{'}(x_0)$ and $\frac{o(\|x_n - x_0\|)}{\|x_n - x_0\|} \to 0$, we have $F^{'}(x_0)d \in \Re^s_+$ and this contradicts (3.3). □

As we have pointed out in Example 3.1, condition (3.1) is a necessary but not in general sufficient optimality condition; nevertheless it becomes sufficient when the objective function is $\Re^s_+$-pseudoconcave as stated in the following theorem, whose proof is a direct consequence of Definition 2.3.

Theorem 3.3 *Consider problem P where F is differentiable at x_0. If S is star shaped at x_0 and F is $\Re^s_+$-pseudoconcave at x_0, then (3.1) is a sufficient condition for x_0 to be a Pareto point for P.*

A sufficient optimality condition which involves $\Re^s_+$-weakly pseudoconcavity is stated in the following theorem, whose proof follows immediately from Definition 2.2.

Theorem 3.4 *Consider problem P where F is differentiable at x_0. If S is star shaped at x_0 and F is $\Re^s_+$-weakly pseudoconcave at x_0, then (3.4) is a sufficient condition for x_0 to be a Pareto point for P:*

$$F^{'}(x_0)d \notin \Re^s_+ \setminus \{0\}, \forall d \in S^{'}(x_0), d \neq 0 \tag{3.4}$$

The following theorem specifies condition (3.4) when the feasible region S is a convex set.

Theorem 3.5 *Consider problem P where F is differentiable at x_0. If S is convex and F is $\Re^s_+$-weakly pseudoconcave at x_0, then (3.5) is a sufficient condition for x_0 to be a Pareto point for P:*

$$F^{'}(x_0)(x - x_0) \notin \Re^s_+ \setminus \{0\}, \forall x \in S \tag{3.5}$$

Proof Assume that x_0 is not a Pareto efficient point; then there exists $x \in S$ such that $F(x) - F(x_0) \in \Re^s_+ \setminus \{0\}$. Since F is $\Re^s_+$-weakly pseudoconcave at x_0 , we have $F^{'}(x_0)(x - x_0) \in \Re^s_+ \setminus \{0\}$ and this contradicts (3.5). □

Remark 3.1 The convexity of S implies that $\{d = \frac{x - x_0}{\|x - x_0\|}, x \in S\} \subset S^{'}(x_0)$; since such an inclusion is in general strict , (3.5) is a little more general than (3.4).

Corollary 3.1 *Consider problem P where F is differentiable at x_0. If S is convex and F is $\Re^s_+$-concave at x_0, then (3.5) is a sufficient condition for x_0 to be a Pareto point for P.*

Remark 3.2 Condition (3.5) is a necessary and sufficient optimality condition when S is convex and F is linear; but it is not necessary when F is $\Re^s_+$-concave as is shown in the following example.

Example 3.2 Consider problem P where $F(x) = (-x^2, -x^2 + 2x)$, $S = \Re$. The point $x_0 = 0$ is a Pareto point , but $F^{'}(x_0)(x - x_0) = (0, 2x)^T \in \Re^2_+$ for all $x > 0$.

We have just pointed out the role played by the Bouligand tangent cone $S^{'}(x_0)$ to the feasible region S at the point x_0 in establishing first order optimality conditions.
Taking into account the necessary condition and the sufficient one , it remains to analyze the directions $d \in S^{'}(x_0)$ for which $F^{'}(x_0)d$ belongs to the boundary of the Paretian cone $\Re^s_+$. We will refer to such directions as critical directions and we set

$$D(x_0) = \{d \in S^{'}(x_0) \setminus \{0\} : F^{'}(x_0)d \in Fr\Re^s_+\}$$

.

Unfortunately, $S^{'}(x_0)$ is not an appropriate set in establishing second order optimality conditions, in the sense that the behaviour of the Hessian matrices of F along the critical directions does not allow to characterize the optimality of a point, as it happens in the scalar case [11, 14].
For such a reason, some authors [17, 19, 25] have introduced the following second order tangent set to S at x_0 in the direction d:

$$S^{"}(x_0, d) := \{w : \exists w_n \rightarrow w, \exists t_n \rightarrow 0^+, such\ that\ x_n = x_0 + t_n d + \frac{1}{2} t_n^2 w_n \in S\}$$

In [14] the following second order tangent set to S at x_0 in the direction d is introduced:

$$\begin{aligned} T_k^{"}(S, x_0, d) \; := \; & \{w : \exists \{x_n\} \subset S, x_n \rightarrow x_0, \exists \alpha_n \rightarrow +\infty, \exists \beta_n \rightarrow +\infty, \\ & with\ \frac{\beta_n}{\alpha_n} \rightarrow k, \alpha_n(x_n - x_0) \rightarrow d\ and\ \beta_n[\alpha_n(x_n - x_0) - d] \rightarrow w\} \end{aligned}$$

The relationship between $T_k^{"}(S, x_0, d)$ and $S^{"}(x_0, d)$ are stated in the following theorem [14].

Theorem 3.6 *For each $k > 0$, we have $T_k^{"}(S, x_0, d) = \frac{k}{2} S^{"}(x_0, d)$; as a particular case $T_2^{"}(S, x_0, d) = S^{"}(x_0, d)$.*

By means of $T_2^{"}(S, x_0, d)$ and $T_0^{"}(S, x_0, d)$, second order optimality conditions are obtained [14] in the scalar case. Now we will extend such an approach in order to obtain second order optimality conditions for a multiobjective problem.
With this aim we introduce the following notations:
- $I = \{1, ..., s\}$, $I_d = \{i \in I : F_i^{'}(x_0)d = 0, d \in D(x_0)\}$, $I_d^* = I \setminus I_d$;
- $F_{I_d}(x)$ is the subvector of $F(x)$ corresponding to the components $i \in I_d$;
- $F_{I_d}^{'}(x_0)$ is the submatrix of $F^{'}(x_0)$ corresponding to the gradients $F_i^{'}(x_0), i \in I_d$;
- $F_{I_d}^{"}(x_0)(w, w)$ is the vector whose components are $F_i^{"}(x_0)(w, w), i \in I_d$.

Theorem 3.7 *Consider problem P where F is a twice continuously differentiable function. Let x_0 be a local Pareto point of problem P. Then:*
i) $F'(x_0)d \notin int\Re^s_+, \forall d \in S'(x_0)$.
ii) For every $d \in D(x_0)$

$$F'_{I_d}(x_0)w + F''_{I_d}(x_0)(d,d) \notin int\Re^{|I_d|}_+, \quad \forall w \in T''_2(S,x_0,d)$$

iii) For every $d \in D(x_0)$

$$F'_{I_d}(x_0)z \notin int\Re^{|I_d|}_+, \quad \forall z \in T''_0(S,x_0,d)$$

Proof i) follows from Theorem 3.1.
ii) Let $d \in D(x_0)$ and $w \in T''_2(S,x_0,d)$; then there exist $\{x_n\} \subset S, \{\alpha_n\} \subset \Re_+, \{\beta_n\} \subset \Re_+, x_n \to x_0, \alpha_n \to +\infty, \beta_n \to +\infty,$ *such that* $\frac{\beta_n}{\alpha_n} \to 2, \alpha_n(x_n - x_0) \to d$ *and* $\beta_n[\alpha_n(x_n - x_0) - d] \to w$.
Let us note that $d \in D(x_0)$ implies $F'_{I^*_d}(x_0)d \in int\Re^{|I^*_d|}_+$ and, consequently there exists $\bar{n}$ such that $F_{I^*_d}(x_n) - F_{I^*_d}(x_0) \in int\Re^{|I^*_d|}_+, \forall n > \bar{n}$.
Since x_0 is a local Pareto point, necessarily we have, for every $n > \bar{n}$, that:

$$F_{I_d}(x_n) - F_{I_d}(x_0) \notin \Re^{|I_d|}_+ \setminus \{0\} \tag{3.6}$$

Set $v_n = \alpha_n(x_n - x_0) - d$. By Taylor's expansion, we have:

$$F_{I_d}(x_n) - F_{I_d}(x_0) = \frac{1}{\alpha_n}F'_{I_d}(x_0)v_n + \frac{1}{2\alpha_n^2}F''_{I_d}(x_0)(v_n+d, v_n+d) + o(\| x_n - x_0 \|^2) \tag{3.7}$$

with $lim_{x_n \to x_0} \frac{o(\|x_n - x_0\|^2)}{\|x_n - x_0\|^2} = 0$.
Let us note that $\beta_n\alpha_n \;\; o(\| x_n - x_0 \|^2) \to 0$.
In fact
$\beta_n\alpha_n \;\; o(\| x_n - x_0 \|^2) = \beta_n\alpha_n \| x_n - x_0 \|^2 \;\; \frac{o(\|x_n - x_0\|^2)}{\|x_n - x_0\|^2} = \frac{\beta_n}{\alpha_n}\alpha_n^2 \| x_n - x_0 \|^2$ $\frac{o(\|x_n - x_0\|^2)}{\|x_n - x_0\|^2} \to 2 \| d \|^2 \, 0 = 0$
The thesis follows multiplying (3.7) for $\beta_n\alpha_n$, taking the limit for $n \to +\infty$ and taking into account (3.6).
iii) Let $d \in D(x_0)$ and $z \in T''_0(S,x_0,d)$, then there exist $\{x_n\} \subset S, \{\alpha_n\} \subset \Re_+, \{\beta_n\} \subset \Re_+, x_n \to x_0, \alpha_n \to +\infty, \beta_n \to +\infty,$ *such that* $\frac{\beta_n}{\alpha_n} \to 0, \alpha_n(x_n - x_0) \to d$ *and* $\beta_n[\alpha_n(x_n - x_0) - d] \to z$.
For such a sequence $\{x_n\}$, (3.6) and (3.7) hold.
It is easy to prove once again that $\beta_n\alpha_n \;\; o(\| x_n - x_0 \|^2) \to 0$, so that, multiplying (3.7) for $\beta_n\alpha_n$, taking the limit for $n \to +\infty$ and taking into account (3.6), the thesis follows.
This complete the proof. □

The following theorem states a sufficient second order optimality condition for problem P.

Theorem 3.8 *Consider problem P where F is a twice continuously differentiable function. If:*
i) $F'(x_0)d \notin int\Re^s_+, \forall d \in S'(x_0)$.
ii) For every $d \in D(x_0)$

$$F'_{I_d}(x_0)w + F''_{I_d}(x_0)(d,d) \notin \Re^{|I_d|}_+, \quad \forall w \in T''_2(S, x_0, d)$$

iii) For every $d \in D(x_0)$

$$F'_{I_d}(x_0)z \notin \Re^{|I_d|}_+, \quad \forall z \in T''_0(S, x_0, d)$$

then x_0 is a local Pareto point for problem P.

Proof Assume x_0 is not a local Pareto point for problem P. Then there exists $\{x_n\} \subset S, x_n \to x_0$ such that

$$F(x_n) - F(x_0) \in \Re^s_+ \setminus \{0\} \tag{3.8}$$

Let $\alpha_n \subset \Re_+, \alpha_n \to +\infty$ such that $\alpha_n(x_n - x_0) \to d \in S'(x_0)$. Taking into account i) and (3.8), we have $d \in D(x_0)$. Performing a Taylor's expansion on the function F_{I_d} with respect to the sequence $\{x_n\}$ and setting $v_n = \alpha_n(x_n - x_0) - d$ we get (3.7).
There are two possible cases:
a) the sequence $\{\alpha_n v_n\}$ is a convergent sequence with $\alpha_n v_n \to \frac{1}{2}w$.
b) the sequence $\{\alpha_n v_n\}$ is unbounded that is $\alpha_n \parallel v_n \parallel \to +\infty$.
case a)
Let us note that $\alpha_n^2 o(\parallel x_n - x_0 \parallel^2) \to 0$ so that , multiplying (3.7) for $2\alpha_n^2$ and taking the limit for $n \to +\infty$ we obtain:

$$F'_{I_d}(x_0)w + F''_{I_d}(x_0)(d,d) \tag{3.9}$$

Since $2\alpha_n v_n = 2\alpha_n[\alpha_n(x_n - x_0) - d]$, choosing $\beta_n = 2\alpha_n$, it results $w \in T''_2(S, x_0, d)$. From (3.8), we have: $F'_{I_d}(x_0)w + F''_{I_d}(x_0)(d,d) \in \Re^{|I_d|}_+$ and this contradicts ii).
case b)
Multiplying (3.7) for $\frac{v_n}{\|v_n\|}$ we obtain

$$\frac{\alpha_n}{\parallel v_n \parallel}(F_{I_d}(x_n) - F_{I_d}(x_0) = F'_{I_d}(x_0)\frac{v_n}{\parallel v_n \parallel} + \frac{1}{2\alpha_n v_n}F''_{I_d}(x_0)(d + v_n, d + v_n) +$$
$$+\frac{\alpha_n}{\parallel v_n \parallel}o(\parallel x_n - x_0 \parallel^2)$$

Since $\alpha_n^2 \parallel x_n - x_0 \parallel^2 \to \parallel d \parallel^2$, $\alpha_n \parallel v_n \parallel \to +\infty$, and $\frac{o(\|x_n - x_0\|^2)}{\|x_n - x_0\|^2} \to 0$, it results

$$\frac{\alpha_n}{\parallel v_n \parallel}o(\parallel x_n - x_0 \parallel^2) = \frac{\alpha_n^2 \parallel x_n - x_0 \parallel^2}{\alpha_n \parallel v_n \parallel}\frac{o(\parallel x_n - x_0 \parallel^2)}{\parallel x_n - x_0 \parallel^2} \to 0$$

On the other hand $\frac{v_n}{\|v_n\|} \to z \in T_0^{''}(S, x_0, d)$ since, choosing $\beta_n = \frac{1}{\|v_n\|}$, we have $\frac{\beta_n}{\alpha_n} = \frac{1}{\alpha_n \|v_n\|} \to 0$. Taking the limit in (3.10) for $n \to +\infty$ and taking into account (3.8), we have $F'_{I_d}(x_0)z \in \Re_+^{|I_d|}$, $z \in T_0^{''}(S, x_0, d)$ and this contradicts iii).
The proof is complete. □

In the next sections we will specify the above results when the feasible region is an open set or when it is defined by constraint functions.

4 Optimality conditions for unconstrained problems

Consider problem P where now S is an open set; in such a case the tangent cone $S'(x_0)$ is the whole space $\Re^n$ so that the necessary optimality condition (3.1) becomes

$$F'(x_0)d \notin int\Re_+^s, \forall d \neq 0 \tag{4.1}$$

while the sufficient optimality condition (3.2) becomes

$$F'(x_0)d \notin \Re_+^s, \forall d \neq 0 \tag{4.2}$$

Remark 4.1 In the scalar case (s=1), (4.1) reduces to the classical condition $F'(x_0) = 0$, while (4.2) is inconsistent (unlike the vector case).

The optimality conditions (4.1) and (4.2) can be expressed by means of multipliers, as it is stated in the following theorem.

Theorem 4.1 *Consider the differentiable problem P where S is an open set.*
i) If x_0 is a local Pareto efficient point then

$$\exists \alpha \in \Re_+^s \setminus \{0\} : \alpha^T F'(x_0) = 0 \tag{4.3}$$

ii) If (4.3) holds with $\alpha \in int\Re_+^s$ and $KerF'(x_0) = \{0\}$, then x_0 is a local Pareto efficient point.

Proof Set $V = \{v = F'(x_0)d, d \in \Re^n\}$.
i) (4.1) is equivalent to state $V \cap int\Re_+^s = \emptyset$; the thesis follows from i) of Theorem 2.1.
ii) Taking into account ii) of Theorem 2.1, condition (4.3) with $\alpha \in int\Re_+^s$,is equivalent to $V \cap \Re_+^s = \{0\}$. On the other hand $KerF'(x_0) = \{0\}$, so that $(V \setminus \{0\}) \cap \Re_+^s = \emptyset$ and the thesis follows from (4.2). □

Remark 4.2 In the scalar case, condition (4.3) is equivalent to state that x_0 is a critical point; for such a reason we will refer to points verifying (4.3) as critical points of a vector function.

Remark 4.3 Let us note that $KerF'(x_0) = \{0\}$ if and only if $s \geq n$ and $rankF'(x_0) = n$.

When the objective function verifies some generalized concavity assumptions, we can specify the previous optimality conditions. Taking into account Theorems 3.3, 3.4, 3.5 and Corollary 3.1, we have the following theorem.

Theorem 4.2 *Consider the differentiable problem P where S is an open set.*
i) If F is $\Re^s_+ -$ pseudoconcave at $x_0 \in S$ and $\exists \alpha \in \Re^s_+ \setminus \{0\} : \alpha^T F'(x_0) = 0$, then x_0 is a Pareto efficient point.
ii) If F is $\Re^s_+ -$ weakly pseudoconcave at $x_0 \in S$ and $\exists\ \alpha \in int\Re^s_+ : \alpha^T F'(x_0) = 0$, then x_0 is a Pareto efficient point.
iii) If F is a linear multiobjective function, then x_0 is a Pareto efficient point if and only if $\exists\ \alpha \in int\Re^s_+ : \alpha^T F'(x_0) = 0$.

Now we specify the necessary second order optimality condition and the sufficient one to an unconstrained problem. First of all, let us note that $T_2''(S, x_0, d) = T_0''(S, x_0, d) = \Re^n$. In fact, let $w \in \Re^n$; obviously it is possible to find a sequence of positive scalars $\alpha_n \to +\infty$ such that the sequence $x_n = x_0 + \frac{d}{\alpha_n} + \frac{w}{2\alpha_n^2}$ is feasible, so that $w \in T_2''(S, x_0, d)$. On the other hand it is possible to find $\alpha_n \to +\infty$, $\beta_n \to +\infty$ with $\frac{\beta_n}{\alpha_n} \to 0$ (for instance choosing $\beta_n = \sqrt{\alpha_n}$), such that the sequence $x_n = x_0 + \frac{d}{\alpha_n} + \frac{w}{\alpha_n \beta_n}$ is feasible and $z \in T_0''(S, x_0, d)$.
As a consequence, the necessary second order optimality condition expressed by Theorem 3.7 can be specified in the following way.

Theorem 4.3 *Consider Problem P where F is a twice continuously differentiable function and S is an open set. If x_0 is a local Pareto point then i) and ii) hold.*
i) $F'(x_0)d \notin int\Re^s_+ \ \forall d \in \Re^n$.
ii) For every $d \in D(x_0)$, it results

$$F'_{I_d}(x_0)w + F''_{I_d}(x_0)(d, d) \notin int\Re_+^{|I_d|} \ \forall w \in \Re^n \tag{4.4}$$

Proof i) and ii) follow from i) and ii) of Theorem 3.7. It remains to prove that $F'_{I_d}(x_0)z \notin int\Re_+^{|I_d|} \ \forall z \in \Re^n$ is always satisfied. If not, there exists z^* such that $F'_{I_d}(x_0)z^* \in int\Re_+^{|I_d|}$. Consider the vector $d^* = td + z^*$, $t > 0$. It results

$$F'(x_0)d^* = t \begin{pmatrix} F'_{I_d}(x_0)d \\ F'_{I_d^*}(x_0)d \end{pmatrix} + \begin{pmatrix} F'_{I_d}(x_0)z^* \\ F'_{I_d^*}(x_0)z^* \end{pmatrix} = \begin{pmatrix} F'_{I_d}(x_0)z^* \\ tF'_{I_d^*}(x_0)d + F'_{I_d^*}(x_0)z^* \end{pmatrix}$$

Since $F^{'}_{I^*_d}(x_0)d \in int\Re_+^{|I^*|}$, for t large enough it results $F^{'}(x_0)d \in int\Re^s_+$ and this contradicts i). □

Corollary 4.1 *Consider the twice differentiable problem P where S is an open set. If x_0 is a local Pareto point then i) and ii) hold.*
i) $F^{'}(x_0)d \notin int\Re^s_+ \ \forall d \in \Re^n$.
ii) For every $d \in D(x_0)$,

$$F^{"}_{I_d}(x_0)(d,d) \notin int\Re_+^{|I_d|}. \tag{4.5}$$

Proof It is sufficient to note that ii) of Theorem 4.2 holds for $w = 0$. □

The following example points out that the necessary second order optimality condition expressed by Corollary 4.1 is stronger than the one given in Theorem 4.3 in the sense that ii) of Corollary 4.1 does not imply necessarily ii) of Theorem 4.3.

Example 4.1 Consider problem P where $F(x_1, x_2) = (x_1 - x_2 + x_1^2, -x_1 + x_2)$, $S = \Re^2$ and the point $x_0 = (0, 0)$. It is easy to verify that x_0 is not a local Pareto point; nevertheless, for $d = (1, 1) \in D(x_0)$, i) and ii) of Corollary 4.1 are satisfied while ii) of Theorem 4.3 is not verified for $w = (0, 1)$.

Now, we specify Theorem 3.8.

Theorem 4.4 *Consider Problem P where F is a twice continuously differentiable function and S is an open set. If:*
i) $F^{'}(x_0)d \notin int\Re^s_+ \ \forall d \in \Re^n$.
ii) For every $d \in D(x_0)$,

$$F^{'}_{I_d}(x_0)w + F^{"}_{I_d}(x_0)(d,d) \notin \Re_+^{|I_d|} \ \ \forall w \in \Re^n \tag{4.6}$$

then x_0 is a local Pareto point.

Proof Assume that x_0 is not a local Pareto point; then there exists a feasible sequence $\{x_n\}$ with $x_n \to x_0$ such that $F(x_n) - F(x_0) \in \Re^s_+ \setminus \{0\}$. It is easy to verify that $\lim_{x_n \to x_0} \frac{x_n - x_0}{\|x_n - x_0\|} = d \in D(x_0)$. By Taylor's expansion we have

$$F(x_n) - F(x_0) = F^{'}(x_0)(x_n - x_0) + \frac{1}{2}F^{"}(x_0)(x_n - x_0, x_n - x_0) + o(\| x_n - x_0 \|^2). \tag{4.7}$$

with $\lim_{x_n \to x_0} \frac{o(\|x_n - x_0\|^2)}{\|x_n - x_0\|^2} = 0$.
Taking into account iii) of Theorem 2.1, condition (4.6) implies the existence of a multiplier $\alpha_d \in \Re_+^{|I_d|}$ such that $\alpha_d^T F^{'}_{I_d}(x_0) = 0$ and

$$\alpha_d^T F^{"}_{I_d}(x_0)(d,d) < 0 \tag{4.8}$$

Let $\bar{\alpha} = (0, \alpha_d) \in \Re^s_+ \setminus \{0\}$ where 0 is the null vector of $\mid I^*_d \mid$ components. Multiplying (4.7) for $\frac{\bar{\alpha}^T}{\|x_n - x_0\|^2}$, we obtain

$$\bar{\alpha}^T \frac{F(x_n) - F(x_0)}{\| x_n - x_0 \|^2} = \frac{1}{2}\alpha_d^T F^{"}_{I_d}(x_0)(\frac{x_n - x_0}{\| x_n - x_0 \|}, \frac{x_n - x_0}{\| x_n - x_0 \|}) + \bar{\alpha}^T \frac{o(\| x_n - x_0 \|^2)}{\| x_n - x_0 \|^2}$$

so that

$$\lim_{x_n \to x_0} \bar{\alpha}^T \frac{F(x_n) - F(x_0)}{\| x_n - x_0 \|} = \frac{1}{2}\alpha_d^T F^{"}_{I_d}(x_0)(d, d)$$

Since $F(x_n) - F(x_0) \in \Re^s_+ \setminus \{0\}$, we have $\frac{1}{2}\alpha_d^T F^{"}_{I_d}(d, d) \geq 0$ and this contradicts (4.8). □

Remark 4.4 We have seen in Theorem 4.1 that the first order optimality conditions can be expressed in a dual form by means of multipliers. Also the second order optimality conditions (4.4), (4.5), (4.6) can be stated in a dual form but now, unlike the first order case, the multipliers are depending from the choosen direction $d \in D(x_0)$, as pointed out in [2, 1].
In section 6 we will deep this aspect in order to obtain second order optimality condition involving fixed multipliers.

5 Optimality conditions for constrained problems

In this section we will consider a multiobjective problem where the feasible region S is expressed by means of constrained functions, that is the problem

$$P^* : max\ F(x), x \in S = \{x \in X : G(x) = (g_1(x), ..., g_m(x)) \in \Re^m_+\}.$$

For sake of simplicity, corresponding to a feasible point x_0 , we will assume, without loss of generality, that x_0 is binding to all the constraints. Set:
- $C^0 = \{d : G^{'}(x_0)d \in int\Re^m_+\}$;
- $C = \{d : G^{'}(x_0)d \in \Re^m_+\}$.
-$H^{'}(x_0) = \begin{pmatrix} F^{'}(x_0) \\ G^{'}(x_0) \end{pmatrix}$
-$\Omega = \{\omega = H^{'}(x_0)d, d \in \Re^n\}$;
It is well known that $C^0 \subseteq S^{'}(x_0) \subseteq C$.
The following theorem states a first order necessary optimality condition which can be considered the natural extension in vector optimization of the classical Fritz-John conditions.

Theorem 5.1 *Consider problem P^* where F and G are continuously differentiable functions. The following equivalent conditions (5.1), (5.2) are necessary for x_0 to be a local Pareto point:*

$$H^{'}(x_0)d \notin int\Re^s_+ \times int\Re^m_+, \ \forall d \in \Re^n \tag{5.1}$$

$$\exists \alpha \in \Re^s_+, \exists \beta \in \Re^m_+, (\alpha, \beta) \neq 0 : \alpha^T F'(x_0) + \beta^T G'(x_0) = 0 \tag{5.2}$$

Proof Taking into account that $C^0 \subseteq S'(x_0)$, from (3.1) we have that (5.1) holds $\forall d \in C^0$. On the other hand, (5.1) is trivially verified $\forall d \notin C^0$. The thesis follow from i) of Theorem 2.2, taking into account that (5.1) is equivalent to state that $\Omega \cap (int\Re^s_+ \times int\Re^m_+) = \emptyset$. □

As in the scalar case, it can happen that $\alpha = 0$ in (5.2); requiring a constraint qualification it is possible to prove the following theorem which extends to vector optimization the classical Kuhn-Tucker conditions.

Theorem 5.2 *Consider problem P^* where F and G are continuously differentiable functions and assume that $C = S'(x_0)$. If x_0 is a local Pareto point, then*

$$\exists \alpha \in \Re^s_+ \setminus \{0\}, \exists \beta \in \Re^m_+ : \alpha^T F'(x_0) + \beta^T G'(x_0) = 0 \tag{5.3}$$

Proof The Abadie constraint qualification $C = S'(x_0)$ implies that $\Omega \cap (int\Re^s_+ \times \Re^m_+) = \emptyset$. From ii) of Theorem 2.2 we get (5.3). □

Remark 5.1 The Abadie constrint qualification $C = S'(x_0)$ ensure $\alpha \neq 0$ in (5.2), but some of these multipliers may be equal to 0. In order to get positive multipliers, Maeda in [21] suggests several constraint qualifications.

As in the scalar case, under suitable assumptions of generalized concavity, (5.3) becomes a sufficient optimality condition.

Theorem 5.3 *Consider problem P^* where F and G are continuously differentiable functions. If F is $\Re^s_+$−pseudoconcave at x_0, G is $\Re^m_+$−quasiconcave at x_0 and (5.3) holds, then x_0 is a Pareto point for problem P.*

Proof Assume that x_0 is not a Pareto point; then there exists $\bar{x}$ such that $F(\bar{x}) - F(x_0) \in \Re^s_+ \setminus \{0\}$ and $G(\bar{x}) = G(\bar{x}) - G(x_0) \in \Re^m_+$. The pseudoconcavity of the objective function F and the quasiconcavity of G imply $F'(x_0)(\bar{x} - x_0) \in int\Re^s_+$, $G'(x_0)(\bar{x} - x_0) \in \Re^m_+$, so that
$\alpha^T F'(x_0)(\bar{x} - x_0) + \beta^T G'(x_0)(\bar{x} - x_0) > 0$ and this contradicts (5.3). □

Now we will state some other first order sufficient optimality conditions.

Theorem 5.4 *Consider problem P^* where F and G are continuously differentiable functions. The following equivalent conditions (5.4), (5.5) are sufficient for x_0 to be a local Pareto point:*

$$H'(x_0)d \notin (\Re^s_+ \times \Re^m_+), \ \ \forall d \in \Re^n \tag{5.4}$$

$$\exists\alpha \in int\Re^s_+, \exists\beta \in int\Re^m_+ : \alpha^T F^{'}(x_0) + \beta^T G^{'}(x_0) = 0, \;\; KerH^{'}(x_0) = \{0\} \tag{5.5}$$

Proof The equivalence between (5.4) and (5.5) follows from ii) of Theorem 2.2. Condition (5.4) implies $F^{'}(x_0)d \notin \Re^s_+$ $\forall d \in C \supseteq S^{'}(x_0)$. The thesis follows from Theorem 3.2. □

Now we will specify second order optimality conditions, established in Section 3, to problem P^*. The following theorem holds:

Theorem 5.5 *Consider problem P^*, where F and G are twice continuously differentiable functions. If x_0 is a local Pareto point then i) and ii) hold.*
i) $H^{'}(x_0)d \notin int\Re^s_+ \times int\Re^m_+$, $\forall d \in \Re^n$
ii) For every critical direction $d \in D(x_0)$ we have

$$\begin{pmatrix} F^{'}_{I_d}(x_0)w \\ G^{'}_{J_d}(x_0)w \end{pmatrix} + \begin{pmatrix} F^{"}_{I_d}(x_0)(d,d) \\ G^{"}_{J_d}(x_0)(d,d) \end{pmatrix} \notin int\Re^{|I_d|}_+ \times int\Re^{|J_d|}_+, \;\; \forall w \in \Re^n \tag{5.6}$$

where $J_d = \{j \in \{1, ..., m\} : G^{'}_j(x_0)d = 0\}$.

Proof i) It follows from Theorem 5.1
ii) Assume that there exist $d \in D(x_0)$, $w \in \Re^n$ such that (5.7), (5.8) hold

$$F^{'}_{I_d}(x_0)w + F^{"}_{I_d}(x_0)(d,d) \in int\Re^{|I_d|}_+ \tag{5.7}$$

$$G^{'}_{J_d}(x_0)w + G^{"}_{J_d}(x_0)(d,d) \in int\Re^{|J_d|}_+ \tag{5.8}$$

Consider the sequence $x_n = x_0 + \frac{\sqrt{2}}{n}d + \frac{1}{n^2}w$. First of all we will prove that a subsequence of x_n is feasible.
By means of Taylor'expansion it is easy to prove that

$$\lim_{n\to+\infty} n^2(G_{J_d}(x_n) - G_{J_d}(x_0)) = G^{'}_{J_d}(x_0)w + G^{"}_{J_d}(x_0)(d,d) \in int\Re^{|J_d|}_+.$$

Consequently, since $G(x_0) = 0$, there exists n_1, such that $G_{J_d}(x_n) \in int\Re^{|J_d|}_+$ $\forall n > n_1$. On the other hand, $d \in D(x_0)$ implies $G^{'}(x_0)d \in \Re^m_+$, so that $G^{'}_{J^*_d}(x_0)d \in int\Re^{|J^*_d|}_+$, where $J^*_d = \{j \in \{1, ..., m\} : j \notin J_d\}$. By Taylor's expansion we have

$$\lim_{n\to+\infty} n(G_{J^*_d}(x_n) - G_{J^*_d}(x_0)) = G^{'}_{J*_d}(x_0)d \in int\Re^{|J^*_d|}_+$$

and thus there exists n_2, such that $G_{J^*_d}(x_n) \in int\Re^{|J^*_d|}_+$ $\forall n > n_2$. Setting $\bar{n} = max\{n_1, n_2\}$, the sequence $\{x_n\}$ is feasible $\forall n > \bar{n}$.
Performing a Taylor's expansion on the function F_{I_d} at x_0, we have

$$\lim_{n\to+\infty} n^2(F_{I_d}(x_n) - F_{I_d}(x_0)) = F^{'}_{I_d}(x_0)w + F^{"}_{I_d}(x_0)(d,d)$$

so that from (5.7) $\exists n_3 : \forall n > n_3 \; F_{I_d}(x_n) - F_{I_d}(x_0) \in int\Re_+^{|I_d|}$. On the other hand $F'_{I_d^*}(x_0)d \in int\Re_+^{|I_d^*|}$ and thus $\exists n_4 : \forall n > n_4 \; F_{I_d^*}(x_n) - F_{I_d^*}(x_0) \in int\Re_+^{|I_d^*|}$. Choosing $n^* = max\{n_3, n_4, \bar{n}\}$, we have $F(x_n) - F(x_0) \in int\Re_+^s, \; \forall n > n^*$ and this contradicts the optimality of x_0. □

Theorem 5.6 *Consider problem P^*, where F and G are twice continuously differentiable functions. If i) and ii) hold, then x_0 is a local Pareto point.*
i) $H'(x_0)d \notin int\Re_+^s \times int\Re_+^m, \; \forall d \in \Re^n$
ii) For every critical direction $d \in D(x_0)$ it results

$$\begin{pmatrix} F'_{I_d}(x_0)w \\ G'_{J_d}(x_0)w \end{pmatrix} + \begin{pmatrix} F''_{I_d}(x_0)(d,d) \\ G''_{J_d}(x_0)(d,d) \end{pmatrix} \notin \Re_+^{|I_d|} \times \Re_+^{|J_d|}, \; \forall w \in \Re^n \tag{5.9}$$

Proof Assume that x_0 is not a local Pareto point; then there exists a feasible sequence $\{x_n\}$ with $x_n \to x_0$ such that $F(x_n) - F(x_0) \in \Re_+^s \setminus \{0\}$. It is easy to verify that $\lim_{x_n \to x_0} \frac{x_n - x_0}{\|x_n - x_0\|} = d \in D(x_0)$. By Taylor's expansion we have

$$F(x_n) - F(x_0) = F'(x_0)(x_n - x_0) + \frac{1}{2}F''(x_0)(x_n - x_0, x_n - x_0) + o(\| x_n - x_0 \|^2). \tag{5.10}$$

$$G(x_n) - G(x_0) = G'(x_0)(x_n - x_0) + \frac{1}{2}G''(x_0)(x_n - x_0, x_n - x_0) + o(\| x_n - x_0 \|^2). \tag{5.11}$$

with $\lim_{x_n \to x_0} \frac{o(\|x_n - x_0\|^2)}{\|x_n - x_0\|^2} = 0$.
Taking into account iii) of Theorem 2.1, condition (5.9) implies the existence of a multiplier $\alpha_d \in \Re_+^{|I_d|}$ and a multiplier $\beta_d \in \Re_+^{|J_d|}$, with $(\alpha_d, \beta_d) \neq 0$, such that $\alpha_d^T F'_{I_d}(x_0) + \beta_d^T G'_{J_d}(x_0) = 0$ and

$$\alpha_d^T F''_{I_d}(x_0)(d,d) + \beta_d^T G''_{J_d}(x_0)(d,d) < 0 \tag{5.12}$$

Let $\bar{\alpha} = (0, \alpha_d) \in \Re_+^s \setminus \{0\}$ where 0 is the null vector of $| I_d^* |$ components, and let $\bar{\beta} = (0, \beta_d) \in \Re_+^m \setminus \{0\}$ where 0 is the null vector of $| J_d^* |$ components. Multiplying (5.10) for $\frac{\bar{\alpha}^T}{\|x_n - x_0\|^2}$, we obtain

$$\bar{\alpha}^T \frac{F(x_n) - F(x_0)}{\| x_n - x_0 \|^2} = \frac{1}{2}\alpha_d^T F''_{I_d}(x_0)(\frac{x_n - x_0}{\| x_n - x_0 \|}, \frac{x_n - x_0}{\| x_n - x_0 \|}) + \alpha^T \frac{o(\| x_n - x_0 \|^2)}{\| x_n - x_0 \|^2}$$

so that

$$\lim_{x_n \to x_0} \bar{\alpha}^T \frac{F(x_n) - F(x_0)}{\| x_n - x_0 \|} = \frac{1}{2}\alpha_d^T F''_{I_d}(x_0)(d,d)$$

Since $F(x_n) - F(x_0) \in \Re_+^s \setminus \{0\}$, we have

$$\frac{1}{2}\alpha_d^T F''_{I_d}(d,d) \geq 0 \tag{5.13}$$

Multiplying (5.11) for $\frac{\bar{\beta}^T}{\|x_n - x_0\|^2}$, we obtain

$$\bar{\beta}^T \frac{G(x_n) - G(x_0)}{\| x_n - x_0 \|^2} = \frac{1}{2}\beta_d^T G^{''}_{I_d}(x_0)(\frac{x_n - x_0}{\| x_n - x_0 \|}, \frac{x_n - x_0}{\| x_n - x_0 \|}) + \beta^T \frac{o(\| x_n - x_0 \|^2)}{\| x_n - x_0 \|^2}$$

so that

$$\lim_{x_n \to x_0} \bar{\beta}^T \frac{G(x_n) - G(x_0)}{\| x_n - x_0 \|} = \frac{1}{2}\beta_d^T G^{''}_{J_d}(x_0)(d, d)$$

The feasibility of $\{x_n\}$ implies $G(x_n) - G(x_0) \in \Re^m_+$, so that

$$\frac{1}{2}\beta_d^T G^{''}_{J_d}(d, d) \geq 0 \tag{5.14}$$

From (5.13) and (5.14) we obtain

$$\alpha_d^T F^{''}_{I_d}(x_0)(d, d) + \beta_d^T G^{''}_{J_d}(x_0)(d, d) \geq 0 \tag{5.15}$$

and this contradicts (5.12). □

Remark 5.2 In the scalar case ($s = 1$), Theorems 5.6 and 5.7 reduce to the ones given by Ben-Tal in [2].

Now we will point out that condition ii) of Theorem 5.5 and condition ii) of Theorem 5.6 can be expressed by means of the disjunction of two suitable sets; such a disjunction will be useful in order to obtain, in the next section, second order optimality conditions involving fixed multipliers.
With this aim set :
$L = \{\ell = H^{'}(x_0)d, d \in \Re^n\}$
$Q = \{q = H^{''}(x_0)(d, d), d \in D(x_0)\}$
The following Theorem holds:

Theorem 5.7 *Consider problem P^* where F and G are continuously differentiable functions and let x_0 be a feasible point.*
i) Condition (5.6) is equivalent to state that

$$(L + Q) \cap (int\Re^s_+ \times int\Re^m_+) = \emptyset \tag{5.16}$$

ii) Condition (5.9) is equivalent to state that

$$(L + Q) \cap (\Re^s_+ \times \Re^m_+) = \emptyset \tag{5.17}$$

Proof i) Obviously (5.6) implies (5.16). Viceversa, suppose that there exist $w \in \Re^n$, $d \in D(x_0)$ such that (5.6) does not hold. Since $D(x_0) \subseteq S^{'}(x_0) \subseteq C$, it results $G^{'}(x_0)d \in \Re^m_+$, so that $G^{'}_{J_d^*}(x_0)d \in int\Re^{|J_d^*|}_+$; furthermore $d \in D(x_0)$ implies $F^{'}_{I_d^*}(x_0)d \in int\Re^{|I_d^*|}_+$. Then it is possible to find $\bar{t} > 0$ such that

$F'_{I_d^*}(x_0)(\bar{t}d+w) + F''_{I_d^*}(x_0)(d,d) = \bar{t}F'_{I_d^*}(x_0)d + (F'_{I_d^*}(x_0)w + F''_{I_d^*}(x_0)(d,d)) \in int\Re_+^{|I_d^*|}$
$G'_{J_d^*}(x_0)(\bar{t}d+w) + G''_{J_d^*}(x_0)(d,d) = \bar{t}G'_{J_d^*}(x_0)d + (G'_{J_d^*}(x_0)w + G''_{J_d^*}(x_0)(d,d)) \in int\Re_+^{|J_d^*|}$
On the other hand
$F'_{I_d}(x_0)(\bar{t}d+w) + F''_{I_d}(x_0)(d,d) = F'_{I_d}(x_0)w + F''_{I_d}(x_0)(d,d) \in int\Re_+^{|I_d|}$
$G'_{J_d}(x_0)(\bar{t}d+w) + G''_{J_d}(x_0)(d,d) = G'_{J_d}(x_0)w + G''_{J_d}(x_0)(d,d) \in int\Re_+^{|J_d|}$
Consequently the point $(\bar{t}d+w, d) \in (L+Q)\cap(int\Re_+^s \times int\Re_+^m)$ which is absurd.
ii) The proof is similar to the one given in i). □

Taking into account Theorems 5.5, 5.6 and 5.7 and setting $\bar{Q} = Q \cup \{0\}$, we have the following result:

Corollary 5.1 *Consider problem P^* where F and G are twice continuously differentiable functions.*
i) If x_0 is a local Pareto point then

$$(L+\bar{Q})\cap(int\Re_+^s \times int\Re_+^m) = \emptyset \tag{5.18}$$

ii) If

$$(L+\bar{Q})\cap(\Re_+^s \times \Re_+^m) = \{0\} \tag{5.19}$$

then x_0 is a local Pareto point.

6 Second order optimality conditions expressed by means of multipliers

We have seen in Corollary 5.1 that second order optimality conditions can be expressed by means of the disjunction between $L+\bar{Q}$ and a suitable cone. This suggests the study of their separation or, equivalently, the possibility of finding optimality conditions involving multipliers.
Let us note that, in general, an hyperplane which separates $L+\bar{Q}$ and $int\Re_+^s \times int\Re_+^m$ does not exist, as it is shown in the following example which is obtained adapting to the vector case the one proposed by Ben-Tal [2] in the scalar case.

Remark 6.1 Consider problem P^* where
$F(x_1,x_2,x_3) = (-2x_1x_2 - \frac{1}{2}x_3^2, -2x_1x_3 - \frac{1}{2}x_2^2, -2x_2x_3 - \frac{1}{2}x_1^2)$ and
$G(x_1,x_2,x_3) = (x_1^4, x_2^4, x_3^4)$.
Following the same line proposed by Ben-Tal [2] in Example 2.1, it is possible to prove that $x_0 = (0,0,0)$ is a local Pareto point and that there do not exist

fixed multipliers. In order to have separation between $L + \bar{Q}$ and a suitable cone, we need of a regularity condition.

In [16] the following two different kinds of regularity conditions are introduced:
weak regularity Assume that $(L+\bar{Q}) \cap (int\Re^s_+ \times int\Re^m_+) = \emptyset$. We will say that a condition is a weak second order regularity condition if it implies:

$$Co(L + \bar{Q}) \cap (int\Re^s_+ \times int\Re^m_+) = \emptyset$$

where $Co(L + \bar{Q})$ denotes the convex hull of $L + \bar{Q}$
strong regularity Assume that $(L+\bar{Q}) \cap (int\Re^s_+ \times int\Re^m_+) = \emptyset$. We will say that a condition is a strong second order regularity condition if it implies:

$$Q \subseteq L - (\Re^s_+ \times \Re^m_+)$$

Some second order regularity conditions are given in the following Theorems [16]:

Theorem 6.1 *Consider problem P^* where F and G are twice continuously differentiable functions. Then the following conditions are weak second order regularity conditions:*
i) $L + \bar{Q}$ is a convex cone;
ii) $\bar{Q}$ is a convex cone;
iii) $dimL = n - 1$.

Theorem 6.2 *Consider problem P^* where F and G are twice continuously differentiable functions. Then the following conditions are strong second order regularity conditions:*
i) $Q \subseteq L$;
ii) $Q = \emptyset$;
iii) $dimL = n$;
iv) $dimL = s + m - 1$.

Remark 6.2 When $s = 1$, the linearly independence of the gradients of the constraints binding at x_0 is equivalent to iv) of Theorem 6.2. Another second order strong regularity condition, which generalizes to the vector case the McCormick constraint qualification, can be found in [5, 16].

The following theorem points out the role played by weak and strong second order regularity in stating necessary second order optimality conditions involving fixed multipliers.

Theorem 6.3 *Consider problem P^* where F and G are twice continuously differentiable functions and let x_0 be a local Pareto point.*

i) If a second order weak regularity condition holds then:

$$\exists\alpha \in \Re^s_+, \exists\beta \in \Re^m_+, (\alpha,\beta) \neq 0, such\ that\ \alpha^T F'(x_0) + \beta^T G'(x_0) = 0$$
$$and\ \alpha^T F''(x_0)(d,d) + \beta^T G''(x_0)(d,d) \leq 0\ \ \forall d \in D(x_0)$$

ii) If a second order strong regularity condition holds then:

$$\forall\alpha \in \Re^s_+, \forall\beta \in \Re^m_+, (\alpha,\beta) \neq 0, such\ that\ \alpha^T F'(x_0) + \beta^T G'(x_0) = 0$$
$$it\ results\ \alpha^T F''(x_0)(d,d) + \beta^T G''(x_0)(d,d) \leq 0\ \ \forall d \in D(x_0)$$

Remark 6.3 When $s = 1$, ii) of Theorem 6.3 reduces to the well known classic second order optimality condition.

At last the following theorem gives a sufficient second order optimality condition expressed by means of fixed multipliers.

Theorem 6.4 *Consider problem P^* where F and G are twice continuously differentiable functions. If x_0 verifies conditions i) and ii), then x_0 is a local Pareto point for P^* .*
i) $\exists\alpha \in int\Re^s_+, \exists\beta \in int\Re^m_+, such\ \ that\ \ \alpha^T F'(x_0) + \beta^T G'(x_0) = 0$
ii) $\alpha^T F''(x_0)(d,d) + \beta^T G''(x_0)(d,d) < 0\ \ \forall d \in D(x_0)$.

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