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Theorems of the Alternative by way of Separation Theorems

A. Marchi

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Abstract

In this paper we deepen the study of the separation between a cone C and a linear subspace W, a study just outlined in a recent paper [2]. Our attention is concentrated on the inclusion of the face of the cone C, containing $W° = W \cap \text{Fr } C$, into separating hyperplanes in order to obtain a separation theorem between W and C and its geometric interpretation. Based on this theorem, we suggest a new approach to the proof of the alternative theorems; this approach allows us to prove the general theorem of the alternative, recently given in [1], for a wide class of linear systems.

1. Preliminary results

Consider a convex cone C in $\Re^m_+$ such that $C \neq \{0\}$ satisfying:

(1.1.a) $\text{int } C \subseteq C \subset \text{cl } C = \Re^m_+$

(1.1.b) $\text{cl } C + C = C$

and a linear subspace W of the m-dimensional space $\Re^m$. Let $M = \{1,2,..,m\}$, $e^{(i)}$ $i \in M$ be the unitary vectors of $\Re^m_+$, set $W^* = W - \text{cl } C$ the conic extension of W with respect to cone clC and denote by F the minimal face (with respect to the inclusion) of $\Re^m_+$ containing $W° = W \cap \text{cl}C$.

Set $I = \{i \mid e^{(i)} \in F\}$. It is easy to prove the following theorem which points out the role played by W* and F in the disjunction between W and C (the proof is given for sake of completeness):

Theorem 1.1: It results:

i) $W \cap C = \varnothing$ if and only if $W^* \cap C = \varnothing$,

ii) $W \cap \text{int } C = \varnothing$ if and only if $F = W^* \cap \text{Fr } C$,

iii) $W \cap C = \varnothing$ if and only if $F \cap C = \varnothing$.

Proof: i) $\Leftarrow$Directly since $W \subseteq W^*$.

$\Rightarrow$ Suppose that $W^* \cap C \neq \emptyset$, then there exists a $x^\circ = x' - h \in W^* \cap C$ such that $x' \in W$ and $h \in \text{cl } C$ so that from conditions (1.1.a,b), $x' = x^\circ + h \in W \cap C$ and this is absurd.

ii) $\Rightarrow$ Since F is the minimal face containing W° there exists an element $x' \in W$ with $x' = \sum_{i \in I} \lambda_i e^{(i)}$, $\lambda_i > 0$, so that $e^{(k)} = x' - \sum_{i \in I, i \neq k} \lambda_i/\lambda_k \ \ \lambda_i > 0$, for every $k \in I$ so $e^{(i)} \in W^*$. Since W^* is a convex cone it follows $F \subseteq W^* \cap \text{Fr } C$. Now we prove $F \supseteq W^* \cap \text{Fr } C$. We suppose that there exists an $x^\circ \in W^* \cap \text{Fr } C$ such that $x^\circ \notin F$. Then $x^\circ = x' - h = = \sum_{i \in M} \lambda_i e^{(i)}, \lambda_i \geq 0$ and $\lambda_k > 0$ $k \notin I$ where $x' \in W$ and $h \in \text{cl } C$. As a consequence $x' = x^\circ + h + \sum_{i \in M} \lambda_i e^{(i)} \in \text{cl } C \setminus F$, so that $x' \in W \cap \text{cl } C$ and $x' \notin F$. This is absurd since $W^\circ \subseteq F$.

$\Leftarrow$ Since $W^\circ \subseteq F \subset \text{Fr } C$ the thesis holds.

iii) $\Rightarrow$ From i) and ii) we have $F \cap C = (W^* \cap \text{Fr } C) \cap C = (W^* \cap C) \cap \text{Fr } C = \emptyset$.

$\Leftarrow$ Since $W^\circ \subseteq F$ then $W^\circ \cap C = \emptyset$, so that $(W \cap \text{cl } C) \cap C = W \cap C = \emptyset$.

The following example points out that condition (1.1.b) is necessary in order that i) in Theorem 1.1 holds.

Example 1.1: Let us consider $C = \text{int } \Re^m_+ \cup \{(z,0,0) : z>0\}$ and $W = \{(1,1,0)\, t : t \in \Re\}$. We have $W \cap C = \emptyset$, $\text{cl } C + C \neq C$, $W^* \cap C \neq \emptyset$.

It is well known that if $W \cap \text{int } C = \emptyset$ then there exists a hyperplane Γ which separates the linear subspace W and $\Re^m_+$. More exactly:

(1.2) $\quad \exists \ \alpha \geq [0]$ [1] $: \langle \alpha \cdot w \rangle = 0 \ \ \forall w \in W, \langle \alpha \cdot c \rangle \geq 0, \forall \ c \in C.$

In general, the hyperplane $\Gamma = \{u \mid \langle \alpha \cdot u \rangle = 0\}$ is not unique, that is condition (1.2) can be verified for many $\alpha \geq [0]$. The problem to know when we can ensure the existence of $\alpha \geq [0]$ such that its i-th component is positive arises in important topics like, for example, in optimality conditions [see 2] or, as we will see in the following, in the alternative

[1] If $x \geqq z$ then $x_j \geqq z_j \ \forall j$, if $x > z$ then $x_j > z_j \ \forall j$, if $x \geq z$ then $x_j \geqq z_j \ \forall j$, $x \neq z$.

theorems. The answer to this problem is given in the following theorem [see 2,3] which points out the role of the conic extension W* and F.

Theorem 1.2: Suppose $W \cap \text{int}\, C = \emptyset$. Then i) and ii) hold.

i) For every $\alpha \geq [0]$ satisfying (1.2) it results $\alpha_i=0\ \forall i\in I$,

ii) There exists $\alpha \geq [0]$ satisfying (1.2) such that $\alpha_i>0\ \forall i\notin I$.

Remark: Theorem 1.2 implies that

1. Every hyperplane Γ_k satisfying (1.2) contains F, so that $F = \cap_k (\Gamma_k \cap \text{cl}C)$
2. There exists $\alpha\geq 0$ satisfying (1.2) with $m - |I|$ positive elements.

As a consequence of previous remark, since we don't know F a priori, we are interested in knowing the least upper bound of dim F; so that if $\dim F \leq m-k$, then there exists $\alpha\geq 0$ with at least k positive elements satisfying (1.2). With this logic in mind, we are going to fix some results. Consider the finite dimensional spaces $\Re^p_+$, $\Re^s_+$, $\Re_+^{ti}$, i=1,…h, with $m = p+s+\sum t_i$, i=1,..h, and denote with F_p, F_s and F_{ti}, i=1,..,h, a face of $\Re^p_+$, $\Re^s_+$ and $\Re_+^{ti}$, i=1,..h, respectively, so that a face F of $\Re^m_+ = \Re^p_+ \times \Re^s_+ \times \ldots \times \Re_+^{ti} \times \ldots$ is the Cartesian product $(F_p \times F_s \times \ldots \times F_{ti} \times \ldots)$. We will denote with $z= (z_p, z_s, z_1,\ldots,z_h)$ an element of $\Re^m_+$.

Let W be a linear subspace of the m-dimensional space $\Re^m$ and consider a paretian subcone $C = (\Re^p_+ \times \text{int}\Re^s_+ \times \Re_+^{t_1}\backslash\{0\} \times \ldots \times \Re_+^{t_h}\backslash\{0\})$. Obviously, C verifies condition (1.1). The following theorem holds.

Theorem 1.3: $W \cap C = \emptyset$ if and only if i) or ii) holds.

i) $\dim F_s \leq s-1$,

ii) $\dim F_s = s$, $\dim F_{ti} = 0$ for some $i\in\{1,2,\ldots,h\}$.

Proof: Set $F = (F_p \times F_s \times \ldots \times F_{ti} \times \ldots) = W^* \cap \text{cl}C$. If $\dim F_s \leq s-1$, the thesis follows. From iii) of Theorem 1.1 $W \cap C = \emptyset$ if and only if $W \cap F = \emptyset$. If $\dim F_s = s$ for every hyperplane separating W and C, since F_s is a convex cone, there exists a $z \in F$ such that $z_s\in \text{int}\,\Re^s_+$. As a consequence $z \notin C$ that is $F \cap C = \emptyset$ if and only if $z_i = 0$ for some $i\in\{1,2,\ldots,h\}$. The thesis follows.

2. Linear separation between W and C

Taking into account the previous results, we propose in this section a separation theorem which allows us to further specify the sign of the vector of the coefficients of a hyperplane separating a linear subspace and a cone.

Let W be a linear subspace of the m-dimensional space $\Re^m$ and $C = (\Re^p_+ \times \text{int}\Re^s_+ \times \Re_+^{t_1}\backslash\{0\} \times \ldots \times \Re_+^{t_h}\backslash\{0\})$

Theorem 2.1: $W \cap C = \varnothing$ if and only if there exists a hyperplane $\Gamma = \{(u,v,w_1,\ldots,w_h)$: $\langle\alpha\cdot u\rangle + \langle\beta\cdot v\rangle + \langle\gamma_1\cdot w_1\rangle + \ldots + \langle\gamma_h\cdot w_h\rangle = 0\}$ separating W and C with $(\alpha,\beta,\gamma_1,\ldots,\gamma_h) \geqq [0]$

and at least one of the following relations holds:

i) $\beta \geq [0]$ or

ii) $\beta = [0]$ and a $\gamma_i > [0]$ for some $i \in \{1,2,\ldots,h\}$.

Proof: Directly from Theorems 1.2 and 1.3.

With the aim to give a geometric interpretation of the previous results (Theorems 1.3 and 2.1) we consider some particular cases of C. For instance,

$W \cap \text{int}\,\Re^s_+ = \varnothing$ if and only $F \cap \text{int}\,\Re^s_+ = \varnothing$. This happens when $F = F_s$ for every F_s such that $\dim F_s \leq s-1$. This means that W may lean to a generic face of $\Re^s_+$; as a consequence of Theorem 2.1 there exists a hyperplane separating W and C with $\alpha \geq [0]$.

$W \cap \Re^t_+\backslash\{0\} = \varnothing$ if and only $F \cap \Re^t_+\backslash\{0\} = \varnothing$. This happens when $F = \{0\}$, i.e. $\dim F_t = 0$. This means that W may lean to no generic face of $\Re^t_+$ as a consequence of Theorem 2.1 there exists a hyperplane separating W and C with $\alpha > [0]$.

$W \cap (\Re^p_+ \times \Re^t_+\backslash\{0\})$ if and only $F \cap (\Re^p_+ \times \Re^t_+\backslash\{0\}) = \varnothing$. This happens when $F = (F_p \times \{0\})$ for every F_p, This means that W may lean to a generic face of $\Re^p_+$ but to no generic face of $\Re^t_+$, in fact $(F_p \times \{0\}) \notin (\Re^p_+ \times \Re^t_+\backslash\{0\})$ while $(\{0\} \times F_t) \in (\Re^p_+ \times \Re^t_+\backslash\{0\})$ as a consequence of Theorem 2.1 there exists a hyperplane separating W and C with $\alpha \geqq [0]$ and $\beta > [0]$.

With similar arguments we can state the following particular cases:

$W \cap (\text{int}\, \Re^s_+ \times \Re^t_+ \setminus \{0\}) = \varnothing \Leftrightarrow F = (F_s \times F_t)\ \forall F_s, F_t : \dim F_s \leq s\text{-}1$ or $F = (\Re^s_+ \times \{0\})$,

$W \cap (\Re^p_+ \times \text{int}\, \Re^s_+ \times \Re^t_+ \setminus \{0\}) = \varnothing \Leftrightarrow F = (F_p \times F_s \times F_t)\ \forall F_p, F_s, F_t : \dim F_s \leq s\text{-}1$ or $F = (F_p \times \Re^s_+ \times \{0\})$.

3. Alternative Theorems

By means of Separation Theorem 2.1, we are able to propose a new approach to the proof of the Alternative Theorems. First of all, we will use this approach for a generalization of Motzkin's Theorem of the Alternative.

Let A_i is a matrix of order $m_i \times n$, $i \in \{0,1,2,\ldots,h+2\}$ and $x \in \Re^n$.

Theorem 3.1: The linear homogeneous system S:

$$\begin{cases} A_0\, x = [0] \\ A_1\, x \geqq [0] \\ A_2\, x \geq [0] \\ A_{j+2}\, x > [0] \quad j \in \{1,\ldots,h\} \end{cases}$$

has no solution if and only if system S':

$$\begin{cases} y^0 A_0 + y^1 A_1 + y^2 A_2 + \ldots.. y^{j+2} A_{j+2} = [0] \\ y^0 \in \Re^{m_0}, y^1 \geqq [0], y^2 \geqq [0], y^{j+2} \geqq [0]\ j \in \{1,\ldots,h\} \end{cases}$$

and at least one of the following relations holds:

i) $y^2 \geq [0]$ or

ii) $y^2 = [0]$ and a $y^{j+2} > [0]$, for some $j \in \{1,2,\ldots,h\}$

has solution.

Proof: Let us substitute $A_0\, x = [0]$ with $A_0\, x \geqq [0]$ and $- A_0\, x \geqq [0]$ and set $W = \{ [A_0 : -A_0 : A_1 : \ldots : A_{j+2} : \ldots]^T x \mid x \in \Re^n \}$ $j \in \{1,2,\ldots,h\}$ and $C = \Re^{2m_0+m_1}_+ \times \text{int}\Re^{m_2}_+ \times \ldots \times \Re^{m_i}_+ \setminus \{0\} \times \ldots$, $i \in \{3,..,h+2\}$. System S is impossible if and only if $W \cap C = \varnothing$. From Theorem 2.1 $W \cap C = \varnothing$ if and only if there exists an hyperplane $\Gamma = \{(u,v,w_1,\ldots,w_h) : \langle \alpha \cdot u \rangle + \langle \beta \cdot v \rangle + \langle \gamma_1 \cdot w_1 \rangle + \ldots + \langle \gamma_h \cdot w_h \rangle = 0\}$ separating W and C with $(\alpha,\beta,\gamma_1,..,\gamma_h)\geqq[0]$ and at least one of the following relations holds: i) $\beta\geq[0]$ or ii) $\beta=[0]$ and a $\gamma_j>[0]$, for some $j\in\{1,2,\ldots,h\}$. Since $W\subset\Gamma$ [1] for all $x\in\Re^n$, set $u=[A_0\vdots -A_0\vdots A_1]^T x$, $v=A_2x$, $w_j=A_{j+2}x$, $j\in\{1,2,\ldots,h\}$ we have: $[(\alpha^0-\alpha^1)A_0+\alpha^2 A_1+\beta A_2+\ldots+\gamma_j A_{j+2}+\ldots]x=0$, so set $y^0=(\alpha^0-\alpha^1)$, $y^1=\alpha^2$, $y^2=\beta$ and $y^{j+2}=\gamma_j$ $j\in\{1,2,\ldots,h\}$ we have a solution of system S' and viceversa.

Remark: It has to be underlined that the solution of the system S' becomes $y^1=\alpha$, $y^2=\beta$ and $y^{j+2}=\gamma_j$, $j\in\{1,..,h\}$ when system S does not contain equations of the type $A_0 x=0$. From a geometrical point of view, as a consequence of this result we have that every vector $(\alpha,\beta,\gamma_1,\ldots,,\gamma_h)$ of the coefficients of the hyperplanes separating W and C is a solution of system S' and viceversa.

Now we can use the proposed approach for proving the general Theorem of the alternative given in [1]. Let a real matrix A of order $m\times n$ and a column-vectors $b\in\Re^m$ and $x\in\Re^n$ be partitioned in the form:

$$A=\begin{bmatrix}A_{11} & A_{12} & \cdots & A_{1q}\\ A_{21} & A_{22} & \cdots & A_{2q}\\ \vdots & \vdots & \ddots & \vdots\\ A_{p1} & A_{p2} & \cdots & A_{pq}\end{bmatrix},\quad x=\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_q\end{bmatrix} \text{ e } b=\begin{bmatrix}b_1\\ b_2\\ \vdots\\ b_p\end{bmatrix}$$

with A_{ij} of order $m_i\times n_j$, $b^i\in\Re^{mi}$ and $x_j\in\Re^{nj}$, $i=1,2,..,p$, $j=1,2,..,q$. We consider the standard form of a linear non homogeneous system S given in [3]:

$$\text{System S}\quad\begin{cases}A_1 x=b^1\\ A_2 x\leqq b^2\\ A_3 x<b^3\\ A_i x\leq b^i \;\; i\in\{4,\ldots,p\}\\ x^1\in\Re^{n_1}\\ x^2\geqq[0]\\ x^3>[0]\\ x^j\geq[0]\;\; j\in\{4,\ldots,q\}\end{cases}$$

where $A_i=[A_{i1}\ A_{i2}\ \ldots\ A_{iq}]$ is of order $m_i\times n$, $i=1,2,..,p$.

We will prove that the linear system in alternative is the following one:

System S' $\begin{cases} y^T A^1 = [0],\ y^T A^j \geqq [0]\ j \in \{2,...,q\},\ -y^T b \geqq 0 \\ y^1 \in \Re^{m_1},\ y^i \geqq [0]\ i \in \{2,...,p\} \end{cases}$

and at least one of the following relations holds:

i) $y^T A^3 \geq [0]$ or

ii) $y^3 \geq [0]$ or

iii) $y^T b < 0$ or

iv) $y^T A^3 = [0]$, $y^3 = [0]$, $y^T b = 0$ and $y^T A^j > [0]$ for some $j \in \{4,5\ldots,q\}$ or $y^i > [0]$ for some $i \in \{4,5\ldots,p\}$

where $y^T = [\, y^1\ y^2\ \ldots\ y^p] \in \Re^m$ and $A^i = [\, A_{1i}\ A_{2i}\ \ldots\ A_{pi}\,]^T$, $i = 1,2,\ldots,q$.

This result is established in [1], by adopting in a suitable way the Motzkin's theorem of the alternative. In this section, we point out that this result follows directly from Theorem 2.1.With this aim we set $R = \begin{bmatrix} A_1 & -b_1 \\ -A_1 & b_1 \\ -A_2 & b_2 \\ I_2 & 0 \end{bmatrix}$, $S = \begin{bmatrix} -A_3 & b_3 \\ I_3 & 0 \\ 0 & 1 \end{bmatrix}$,

$T_k = [\, -A_k\ \ b^k\,]^T$, $k = i-3$, with $i \in \{4,5\ldots,p\}$, $V_k = [\, I_k\ \ 0\,]^T$, $k = p+j-6$, $j \in \{4,5\ldots,q\}$ and $z = \begin{bmatrix} x \\ t \end{bmatrix}$ where $I_2 = [0\ \ I_{n2}\ \ 0\ ..\ 0]^T$, $I_3 = [0\ \ 0\ \ I_{n3}\ \ldots\ 0]^T$ and $I_j = [0\ \ 0\ \ 0..I_{nj}..]^T$, $j \in \{4,5\ldots,q\}$,

where I_{nj} is the identity matrix of order n_j. We may rewrite the linear system S in this way:

$$\begin{cases} R\,z \geqq [0] \\ S\,z > [0] \\ T_k \geq [0]\ k \in \{1,\ldots,p+q-6\} \end{cases}$$

Consider the linear subspace $W = \{\, [R : S : T_k]^T z \mid z \in \Re^{n+1} \,\}$ and $C = \Re_+^r \times \text{int}\, \Re_+^s \times \ldots \times \Re_+^{mi} \setminus \{0\} \ldots \times \Re_+^{nj} \setminus \{0\} \times \ldots$, $i \in \{4,5\ldots,p\}$, $j \in \{4,5\ldots,q\}$, where $r = 2m_1 + m_2 + n_2$, $s = m_3 + n_3 + 1$.

Theorem 2.1: System S is impossible if and only if System S' has solution.

Proof: S is impossible if and only if $C \cap W = \varnothing$. For Theorem 2.1 there exists a $\Gamma = \{(u,v,w_1,\dots,w_h): <\alpha \cdot u> + <\beta \cdot v> + <\gamma_1 \cdot w_1> + \dots \dots + <\gamma_{p+q-6} \cdot w_{p+q-6}> = 0\}$ separating W and C with $(\alpha,\beta,\gamma_1,\dots,\gamma_{p+q-6}) \geq [0]$ and at least one of the following relations holds: i) $\beta \geq [0]$ or ii) $\beta=[0]$ and $\gamma_i > [0]$ for some $i \in \{1,2,\dots,p+q-6\}$. This hyperplane is such that $[<\alpha \cdot R> + <\beta \cdot S> + \dots + <\gamma_k \cdot T_k> + \dots]z = 0$, $k=1,\dots,p+q-6$. From the definition of the matrices P, S and T_i we have the following system :

$$
\begin{array}{llllll}
(\alpha_1-\alpha_2)A_{11} - \alpha_3 A_{21} & -\beta_1 A_{31} & & -\gamma_1 A_{41} \dots - \gamma_{p-3} A_{p1} & & = [0] \\
(\alpha_1-\alpha_2)A_{12} - \alpha_3 A_{22} + \alpha_4 & -\beta_1 A_{32} & & -\gamma_1 A_{42} \dots - \gamma_{p-3} A_{p2} & & = [0] \\
(\alpha_1-\alpha_2)A_{13} - \alpha_3 A_{23} & -\beta_1 A_{33} + \beta_2 & & -\gamma_1 A_{43} \dots - \gamma_{p-3} A_{p3} & & = [0] \\
(\alpha_1-\alpha_2)A_{14} - \alpha_3 A_{24} & -\beta_1 A_{34} & & -\gamma_1 A_{44} \dots - \gamma_{p-3} A_{p4} & + \gamma_{p-2} & = [0] \\
\dots & \dots & & \dots & \dots & \\
(\alpha_1-\alpha_2)A_{1q} - \alpha_3 A_{2q} & -\beta_1 A_{3q} & & -\gamma_1 A_{4q} \dots - \gamma_{p-3} A_{pq} & +\gamma_{p+q-6} & = [0] \\
-(\alpha_1-\alpha_2)b_1 + \alpha_3 b_2 & +\beta_1 b_3 & +\beta_3 & +\gamma_1 b_4 \dots + \gamma_{p-3} b_q & & = 0.
\end{array}
$$

with $(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\gamma_1,\dots,\gamma_{p+q-6}) \geq [0]$ and at least one of the following relations holds: I) $\beta \geq [0]$ or ii) $\beta=[0]$ and $\gamma_i > [0]$ for some $i \in \{1,2,\dots,p+q-6\}$. Set $y_1 = \alpha_2 - \alpha_1$ (y_1 is sign unrestricted), $y_2 = \alpha_3$, $y_3 = \beta_1$, $y_i = \gamma_{i-3}$, $i=4,\dots p$, we obtain a solution of problem S' and viceversa.

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