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Social Heterogeneities in Classical New Product Diffusion Models. I: "External" and "Internal" Models

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Abstract. Social heterogeneity is embedded within the deterministic models for new product diffusion. In particular a detailed analysis of the effects of heterogeneity on the "classical" models for external and internal diffusion is provided. Measures of the aggregation biases which appear when an heterogeneous situation is wrongly treated as homogeneous are given. Results from the recent epidemiological debate on the subject are also carefully discussed by aiming to clarify those advances which may result of critical interest for marketing science.

Keywords: heterogeneity, external diffusion, internal diffusion, epidemiological models, mixing patterns, aggregate innovation and imitation rates.

1. Introduction[1]. This paper aims to start a systematic discussion of the role of social heterogeneities within the classical deterministic mathematical models describing new products and/or information diffusion. As it is well known, the backbone of the deterministic theory of marketing of new products, the Bass model (Bass, 1969), superimposes the effects of two "classical" forces operating during the adoption process: those due to "innovation" and those due to imitative behaviours, well described since longtime ago in the classical papers by Fourt and Woodlock (1960) and Mansfield (1961). These two forces describe the action of the two basic communication mechanisms operating during the adoption process: the mass-media and word-of mouth, sometimes referred as the "external" and "internal" diffusion mechanisms (Mahajan et al. (1990,1993)).[2] Well recognised merits of these "classical" models for new product diffusion are, first, that of providing simple empirically testable predictions, and second, that of introducing parameters which give invaluable insight on the nature of the diffusion process itself: the intensity of diffusion by the external source and the imitation coefficient (denoted respectively as α and q in the sequel).

All the previously mentioned models postulate *homogeneous* situations: the social environment (i.e.: the population) in which the information spreads is homogeneous in that i)all the individuals are assumed to be homogeneously exposed to the action

[1] Even if this paper is the outcome of the joint discussion of the three authors, P. Manfredi is responsible for sections 3,4; A. Bonaccorsi is responsible for section two, and A. Secchi for the applications presented.

[2] Throughout the paper, following Mahajan et al. (1990), we talk about "external" and"internal" models.

of the public sources of information, ii)all the individuals mix homogeneously, i.e. are homogeneously exposed to internal communications.

This paper investigates what's happen when *social heterogeneity* exist in the population, i.e. when, on the contrary, individuals are exposed in an inhomogeneous manner to the action of the media and/or individuals do not mix homogeneously. Typical questions we pose are: are the individuals in the population exposed in the same way to the information source? Do these individuals share identical imitation patterns? Which are the actual patterns of interaction among and their relation with the social process of imitation? And, finally, which consequences arise when we wrongly treat as homogeneous an underlying heterogeneous situation? Classical new product diffusion models, being homogeneous, do not have answers to such questions, the relevance of which is indisputable. This may be realised from some limit case, such as the case of"isolated" groups, i.e. groups excluded from information in pure external diffusion world, or not mixing with the rest of the population in an internal diffusion world. In both cases these isolated individuals would not adopt the product, but a statistical apparatus of market surveys based on a wrong homogeneous model would hardly discover the problem in real time.

To our knowledge, up to the eighties the only serious tentative to embed social heterogeneities within the classical models for new product diffusion and/or models for the diffusion of news is the classical Bartholomew (1967) (see also Dimitri (1987)). Both these contributions, largely based on the epidemiological paper by Rushton and Mautner (1955), took place before the "heterogeneity revolution" in theoretical epidemiology. The heterogeneity revolution has started at the beginning of the eightiees, more or less closely tied to the need to understand the peculiar features of the spread of AIDS, and is still very active. Several results produced during this intensive debate, such as the development of a social mixing theory, proved to be of extreme fecundity in theoretical and applied epidemiology and constitute nowadays cornerstone results of social sciences.

Nonetheless the spread toward allied fields, such as that of the marketing of new products has been, quite limited, compared to the development of the concept of heterogeneity in other fields.

The main goal of this paper is that of investigating the effects of social heterogeneities on the two basic models for new product diffusion, i.e. the model for external diffusion and the model for internal diffusion, which are the constitutive ingredients of the Bass model (the effects of heterogeneities within the Bass model will be the object of a separated paper). Our basic assumption is that the "population" within which the diffusion occurs, is stratified in subgroups, which differ in the magnitudes of their α (in the external case) or q (the internal case) coefficients following some prescribed distribution of behaviour.

To this end the paper is subdivided in three parts. In the first we review the existing literature on heterogeneity within new product diffusion models. In the second the impact of heterogeneity within the model for external diffusion is considered: individuals are assumed to acquire the informations from the external source at different speeds. Both discrete and continuous heterogeneities are considered. The treatment is rather complete, and the results are, to our knowledge, new in marketing research. Finally, in the third part, we consider the role of heterogeneities within the epidemiological model which describes "pure imitation" processes. This part is quite articulated. First we consider the effects of heterogeneities in the speed at which individuals acquire internal information and some new results are given. Then we introduce to the main topic of the work, i.e. heterogeneity in the social interaction process (the "mixing problem"). This topic has represented a priority in the recent epidemiological research and has produced important theoretical results (most of which related to the dynamics of sexually transmitted diseases, especially HIV). As a first step we introduce those notions from mathematical epidemiology, i.e. the "heterogeneity and mixing toolbox", which are of first concern for marketing research. Then we show how the classical internal model for new product diffusion would be modifed taking into consideration heterogeneity and mixing. Some results are provided for some special mixing assumption. These results are obtained in a simple manner as they constitute special cases of more general result of mathematical epidemiology. For this reason we conclude the work by reviewing some more advanced epidemiological result concerning a more general model for new product diffusion, i.e. the internal model with removal. This seemed to us a good occasion to rediscover to marketing science the model with removal which despite its conceptual relevance never played a substantial role in marketing science. The conclusions stress the need for opening a serious field work on the problem.

2. Heterogeneities in models of innovation diffusion: a review. Since the notion of heterogeneity in diffusion models is not simple, it requires a careful review. The starting point is the absence of heterogeneity in the three classical models of Fourt and Woodlock (1960), Mansfield (1961), and Bass (1969). Our discussion starts with the later, more comprehensive model.

The Bass model assumes homogeneity in at least two different respects: with respect to characteristics of members of the population and with respect to the information diffusion mechanism. In the former sense, all members have the same probability to adopt the new product at the time defined by a stochastic allocation to various timing classes; in the latter sense all members are exposed homogeneously to external influence (advertising) and to internal influence (word-of-mouth of previous adopters). Let us call behavioral homogeneity the former, social homogeneity the latter. In both cases there have been attempts to incorporate heterogeneity in the basic model, but, as we shall see, in an unsatisfactory way.

In the former sense it has been correctly noted that the mathematical formulation of the Bass model requires the population to be homogeneous, in the sense that the distribution of adopting time is merely an aggregate property, while individuals are behaviorally homogeneous at the microlevel. The distribution of individuals across timing of adoption proposed in the early work of Rogers (1962) may for example be easily obtained from a Bass diffusion model (Mahajan, Muller and Srivastava, 1990) without any specification about intrinsic differences among members of the population.

The most challenging efforts to introduce heterogeneity have been developed in response to a related criticism to the epidemic models, namely the indeterminacy of its behavioral foundations. Why are some people more inclined to innovate than others? Which is the microfoundation of their adoption behavior? As it has been noted (Gatignon and Robertson, 1986; Grübler, 1992) epidemic models are empty of substantive theory regarding the adoption process of individuals. In response to this weakness of the basic model, there have been several attempts to develop microfounded models; it is in this context that heterogeneity is included into the basic framework. It is important to underline this point: heterogeneity is a major concern of the new generations of diffusion model builders, but only inasmuch as it is defined as behavioral heterogeneity, or heterogeneity of parameters that enter into the adoption decision process. In some sense, therefore, heterogeneity arises as an internal theoretical problem within an intrinsically epidemic approach. It is also considered one of the frontiers of research. Let us give a brief review of these developments, following the reviews of Sultan, Farley and Lehmann (1990), Mahajan, Muller and Bass (1990; 1995), Bass, Krishnan and Jain (1994) and Cestre (1996).

In general, heterogeneity is allowed in the population with regards to tastes and income, the initial perception of the quality of the new product or the expected benefit from the use, the price-performance trade-off, the perceived risk and risk propensity (Cestre, 1996; Le Louarn, 1997).

Kalish (1985) develops the idea that potential adopters differ with respect to tastes and income. Each consumer has his own valuation of the intrinsic value of the new product, which is expressed in monetary terms, and is willing to adopt it only if this value is larger then actual market price. Given some uncertainty regarding the true quality of the new product, the reservation price can be considered as a function of the expected value of a distribution of quality outcomes. Therefore only those consumers whose risk-adjusted valuation exceeds the product's price will be potential buyers.

A model which tries to develop formally the idea of heterogeneity has been proposed by Chatterjee and Eliashberg (1985). Consumers are allowed to be heterogeneous with respect to a large number of parameters: risk aversion, initial perception of product quality, relative weight of price and quality in the evaluation of product, information sensitivity.

Heterogeneity is also used to cope with another major shortcomings of the original model. Deterministic diffusion models require the size of the market, or number of potential adopters, to be fixed during all the diffusion process. This means that the market potential is fixed at the initial time and remains at that level during the entire process. This assumption has been relaxed in several dynamic diffusion model (e.g. Mahajan and Peterson (1978)), in which the size of the market grows over time. A related avenue is to make the size of the market dependent on price. This is an important strategy for incorporating heterogeneity in a parsimonious way. It is assumed that individuals have a distribution of reservation prices. The reservation price is the maximum price an individual is willing to pay for a product, given his budget constraint and preferences for other products. A natural way to obtain a dynamic demand model is to assume that price is decreasing over time (for example due to experience curves), so that an increasing number of individuals find the market price becomes smaller than the reservation price. The parameters of the distribution determines the rate of growth of the potential market (Dolan and Jeuland, 1981; Feichtinger, 1982; Kalish, 1985).

Heterogeneity is also accounted for in models of diffusion that explicitly incorporate stochastic elements. As an example, agents may be associated to different probabilities to be exposed to advertising, to be influenced by advertising and word-of-mouth, to purchase or to repeat the purchase (Tapiero, 1983; Boker, 1987, Wheat and Morrison, 1990; Mahajan and Peterson, 1978). In all these cases, heterogeneity is again obtained by representing agents as drawn from an appropriate distribution.

One of the striking features of this stream of literature is the complete independence of behavioral heterogeneity from social heterogeneity. Individuals that differ by income, tastes, quality perception or risk attitudes are otherwise entirely similar in their social behavior: they do not differ in the speed they acquire information from public sources, nor in the intensity with which they communicate interpersonally regarding the new product. Works that claim they have reached a truly microfounded theory of diffusion exhibit a remarkable incompleteness in the characterization of human behavior.

The problem of social heterogeneity is perceived as a separate weakness of the Bass model, but again the responses available in the literature are rather weak. As an example, in the model by Mahajan, Muller and Kerin (1984), it is taken into consideration the possibility that word-of-mouth has a negative, instead of positive, sign (i.e. dissatisfied customers). The population is therefore divided into groups according to the stage of the adoption process they are in (non aware, aware, adopter or non-adopter) and to the sign of the information they produce for other members of the population. The flows between these groups determine the aggregate dynamics of diffusion. However, heterogeneity is admitted only for senders of messages, not for receivers. All members of the population have the same probability of receiving any given piece of information, so that the aggregate probability is proportional to the relative size of the message sender groups.

Our statement of the state of the discipline is that the concern with heterogeneity is a genuine one in diffusion literature, but it receives an inadequate treatment. Heterogeneity is assumed in the distribution of economic variables affecting individual adoption decisions, but the emphasis is then on aggregation into market-level properties. As it is clear from the surveys of Mahajan, Muller and Bass (1990) and Mahajan and Muller (1993), heterogeneity is not really on the research agenda.[3]

To our knowledge, the only paper that explicitly uses the epidemiological concept of heterogeneity in marketing by means of a deterministic model is Putsis, Balasubramanian, Kaplan and Sen (1997). They study mixing behavior in several European countries and analyze the rate of contacts between individuals both intra- and cross-country and find that mixing is indeed an important consideration in new product diffusion.

An earlier paper introducing heterogeneity can be found also in the literature on

[3] A partially different situation can be found in several streams of stochastical product diffusion models, such as the avalanche model proposed by Steyer (Le Nagard and Steyer, 1995) or the information contagion model of Arthur and Lane (1993), in which social heterogeneity plays a role.

technology diffusion. Gore and Lavaraj (1987) proposed a model of diffusion via internal communication of a breeding innovation within a spatially stratified population. Diffusion among the population of rural villages depends on the number of previous adopters in villages themselves and the number of adopters in town, the centre of innovation. The structure of intermixing between members of villages and citizens is not specified; villages are located at a distance of 2-15 kilometers from the town and communication is not easy. The authors show that estimates obtained using two separate models of diffusion improve slightly the fit statistics with respect to an aggregate logistic model. What is interesting, however, is that the logistic model applied to all data (i.e. without considering the heterogeneity) severely overestimates the saturation level (a level of 10,07% of all families in villages was estimated, while the model with heterogeneity delivers a value of only 5,9%). Consequently, the rate at which diffusion takes place is overestimated when homogeneity is wrongly assumed.

2.1. The empirical relevance of the heterogeneity hypothesis. Interestingly, while heterogeneity seems to play a minor role in the mainstream diffusion theory, and there are only limited and very recent attempts to take it into account, its empirical importance is out of discussion in consumer behavior. A rather impressionistic tour into the large, qualitatively heterogeneous literature on personal influence and opinion leadership on consumer behavior shows a remarkably consistent pattern.

Since the early '60s several researchers underlined how large is the amount of social communication that goes around products (Dichter, 1966). Word-of-mouth is generally considered a reliable and trustworthy type of information, whose influence on consumer behavior is high because it allows two-way communication, is associated to strong social support and is often backed up by social-group pressure. Arndt (1967) carried out a survey among members of a large community (a campus for married students) and found that product-related word-of-mouth closely follows the pattern of social relations, so that those individuals that are more central in the social network are also those that generate more communication about the new product. This is an important finding, since it shows that product-related conversation follows the (relatively) invariant channels of communication that individuals open and keep active in their social life.

The amount of product-related conversation may indeed be huge for some categories of product: in a study of attendance to movies Mahajan et al. (1984) found that more than 60% of subjects engaged in movie-related conversation with friends. Interestingly, "friends were the major source of information", much more than advertising or reviews in newspapers.

According to a survey on external search behavior, consumers perceive personal information sources to be more important than non-personal sources because the former provide two-way communication, and bring at the moment of communication more knowledge, generating more attention and interest (Pinson and Roberto, 1988).

One of the most robust findings is that individuals sharply differ in the amount of information they search for and, moreover, in the intensity of product-related conversation. The concept of opinion leader captures the characteristics of those individuals that engage in more information diffusion. A survey of several studies (e.g, Robertson and Myers, 1969; see the surveys in Mullen and Johnson, 1990; Loudon and Della Bitta, 1993) shows that opinion leaders have the same social and class position as non-leaders, but enjoy better reputation and social status within the group. They have greater interest in the area of influence, are more exposed to mass media and try new products relatively early (although they are not innovators but rather early adopters). They normally develop much more social communication than non-leaders, are more sociable and more loyal to group values and norms. Because of these characteristics opinion leaders' influence is accepted within the relevant group as trustworthy and relevant for other individuals' consumer decisions. The proportion of opinion leaders out of the population is of course an empirical matter, but market surveys seem to converge, approximately, around 10%: the so-called Influential Americans are one of every ten adults in the United States, while the "must-know men" discovered by the Yankelovich research company, to whom many individuals refer for problems of mechanics, electronics and car purchase, are one-quarter of the adult male population (Loudon and Della Bitta, 1993).

An interesting question that comes about is whether opinion leaders have a general influence that cut across several product categories or are rather "specialist" in one product. There are conflicting findings on this question. Specialized opinion leaders seem to prevail: in studies of product areas, it is generally found that few of the respondents are opinion leaders for many products, with overlapping strongly related to similarities in the interest raised by products (King and Summers, 1970).

General opinion leaders are also present, however. Feick and Price (1987) have developed the label of market maven (where maven is a Yiddish word for a neighborhood expert) to characterize those individuals that enjoy shopping and collecting information about products, become aware of new products earlier and engage in product-related conversation with many others over a large range of product categories. The interesting finding is that market mavens do not need to be actual users of all products they are asked about by other individuals. In general, market mavens are women.

The notion of active consumer has recently been proposed to try to link more explicitly a realistic representation of consumer behavior to the hard core of economic theory. The basic idea is that individuals enjoy in searching for goods, and particularly in experimenting novelty and surprise in consumption. In this perspective, search is not considered a cost (a negative element in the utility function), bur rather an important part of the utility itself of agents. This idea, fully developed in Bianchi (1998) is consistent with various institutional and neo-Austrian theories of consumption. It has clearly a linkage with theories of social communication. As the editor explicitly says, "the social dimension of the consumer- which includes the communication side of individual choices- must recover a more refined and articulated place in the motivational structure of consumer decision procedures" (Bianchi, 1998, 8).

2.2. Sales forecast and use of early data. The limitations of homogeneous models can be shown with respect to an important application of Bass-type models, namely forecasting of sales. Several studies have found that robust estimates require data covering the peak of the noncumulative adoption curve (Heeler and Hustad, 1980). Which is to say- forecasting is lost. Therefore, how to accurately estimate the parameters of a diffusion model from early sales data is considered a crucial problem in the literature. One of the difficulties lies in the fact that early sales data may be not sufficient to provide stable and robust estimates of the parameters.

The analogical approach (Blattberg and Golanty, 1978; Choffray and Lilien, 1986; Easingwood, 1989) suggests to utilize the parameters of diffusion models of other, assumed similar, products to forecast sales of the new product. A drawback of this approach is that new products always incorporate some new features that make judgments of similarity unreliable. A large number of case histories have been collected in which early estimates of diffusion rates have been in fact wildly disconfirmed by actual experience. A more sophisticated approach involves the use of Bayesian procedures (Sultan, Farley and Lehmann, 1990). As an example, Lilien, Rao and Kalish (1981) used Bayesian methods to produce forecasts in a repeat purchase diffusion model. A modified version of their model was proposed by Rao and Yamada (1988), who found that parameters of the diffusion model can be estimated using potential adopters' perceptions of key attributes of families of similar new products (ethical drugs). In both cases, the value of the market size N is critical for deriving optimal policies for firms, in terms of marketing effort to be allocated during the diffusion process. Thus an interesting question is what happens if N is a collection of heterogeneous potential adopters. Here Bayesian updating would not solve the forecasting uncertainty, since the problem would not be the updating of parameters of a correct model, but rather the identification of the true model- i.e. with or without heterogeneity. Relying on early sales data would be dramatically misleading.

Our treatment makes often use of the notion of aggregation bias in the estimation of diffusion parameters relying on early sales data to display the effects of heterogeneity. Even if we are not interested in the problem of optimal estimation procedures, we believe we are offering a broader perspective on how to build reasonable market models.

3. Heterogeneities in innovation due to external information. The basic model for the diffusion of innovation through external information (Fourt and Woodlock, 1960) is subsumed by the differential equation (ODE):

$$\dot{Y}(t) = \alpha(m - Y(t)) \qquad \alpha, m > 0 \tag{1}$$

plus the initial condition $Y(0) = 0$. In (1) $Y(t)$ denotes the cumulative number of adopters ("A" individuals: those who already received the new[4]) at time t, m the market size, ie the size of the population of potential adopters (assumed constant), so that $X(t) = m - Y(t)$ denotes the number of individuals who have not yet been informed

[4] As all Bass-type models, the "external" model offers an explanation of the social process of diffusion of an information, neglecting the decision process leading to the actual decision to adopt. Hence, at best, the representation of the decision process is the simplest: a constant fraction of those who received the information will actually adopt the product (i.e.: individuals are homogeneous on their decision of adoption).

("N" individuals), and α the *innovation* rate (assumed constant), which reflects the intensity of the source spreading the information. As it is known (1) implies a progressive saturation of the market through a concave dynamics. The formulation (1) is justified as follows: as a consequence of the acquisition of the new (=adoption), individuals transfer at constant rate α from the "N" compartment to the "A" compartment. This transfer of material obeys the pair of differential equations:

$$\dot{X}(t) = -\alpha X(t) \;\; ; \;\; \dot{Y}(t) = \alpha X(t)$$

where $\dot{X} + \dot{Y} = 0$, implying that the total population $m = X + Y$ is constant over time. Hence $Y = m - X$ which definitively leads to (1).

3.1. Discrete heterogeneity. Let us now explicitly assume that the population is *heterogenous* from the point of view of information acquisition, i.e. that the individuals of the adopting population are not exposed in the same ways to the information source and, as a consequence, are characterised by different time scales in their adoption process. Formally we will suppose that the population is stratified in n different groups depending on the intensity of their innovation rate α_i. The adoption process is then described by the following system of n independent equations:

$$(2) \qquad \dot{Y}_i(t) = \alpha_i(m_i - Y_i(t)) \quad i = 1, 2, .., n$$

The size m_i of the i-th group is assumed to be constant. Intra-groups migrations of individuals are ruled out by assumption. To fix the ideas let us assume $\alpha_1 < \alpha_2 < .. < \alpha_n$.

The qualitative behaviour of each equations (2) is identical to that of the basic model (1). What's happen to the overall population? By adding up all the (2), we obtain the aggregate equation:

$$(3) \qquad \dot{Y}(t) = \left(\sum_i \alpha_i \frac{m_i - Y_i(t)}{m - Y(t)}\right)(m - Y(t)) = \alpha(t)(m - Y(t))$$

where $\alpha(t)$ is the *aggregate innovation rate*, defined as:

$$(4) \qquad \alpha(t) = \sum_i \frac{\alpha_i(m_i - Y_i(t))}{m - Y(t)} = \sum_i \alpha_i \frac{X_i(t)}{X(t)} = \mu_\alpha(t)$$

Equation (4) shows that the aggregate innovation rate observed during the dynamics, being the weighted average[5] of the innovation rates of the various groups weighted with the noninformed fractions $w_i = X_i/X$, is not constant. Obviously[6]:

[5] This implies that the dynamics of $\alpha(t)$ will always be bounded in the set (α_1, α_n).

[6] It is to be noticed that the process of external diffusion is "the other side" of a death process (with constant hazard rate): heterogeneity may therefore be studied much in the same way of heterogeneity in the mortality process (Vaupel et al. 1979).

(5) $$\dot{X}(t) = -\alpha(t)X(t)$$

It is easy to see that:

$$\dot{Y}(t) > 0 \quad \forall t \quad ; \quad \ddot{Y}(t) = -\ddot{X}(t) = (-1)X(t)\sum_i \alpha_i^2 \frac{X_i(t)}{X(t)} < 0 \quad \forall t;$$

confirming that the actual qualitative dynamics of the aggregate model is identical to that of the homogeneous one. Hence any relevant effect of heterogeneity is purely quantitative and wholly explained by $\alpha(t)$. To see how the dynamics of $\alpha(t)$ influences the overall dynamics of $Y(t)$ remark that $\alpha(0) = m^{-1}\sum_i \alpha_i m_i$; morever:

- The aggregate innovation rate strictly decreases over time:

  (6) $$\frac{d}{dt}(\alpha(t)) = (-1)Var_\alpha(t) < 0 \quad \forall t$$

  where:

  (7) $$Var_\alpha(t) = \left(\sum_i \alpha_i^2 \frac{X_i(t)}{X(t)} - (\alpha(t))^2\right)$$

  The result (6) follows straightforwardly by taking the time derivative of $\alpha(t)$, and using $\dot{X}_i(t) = -\alpha_i X_i(t)$; $\dot{X}(t) = -\alpha(t)X(t)$.
- The asymptotic aggregate innovation rate coincides with the innovation rates of the slowest group (the true "laggards" of the process):

  (8) $$\alpha(\infty) = \sum_i \alpha_i w_i(\infty) = \alpha_1$$

The result (8) is obvious: in the long term the fastest adopters have been eliminated and only the slowest "survive". In sum: the aggregate innovation rate decreases monotonically over time from from its initial value $\alpha(0) = \sum_i \alpha_i \frac{m_i}{m}$ to its long term value $\alpha_1 = \min_i(\alpha_i)$. Consequently the time scale of the market saturation process will coincide with the time scale typical of the slowest group (proof of (8) and other details on the dynamics of $\alpha(t)$ and the weights $w_i(t)$ are postponed in the appendix).

What are the most relevant phenomenological consequences of these facts? As we have seen, in the present case, heterogeneity has essentially quantitative effects. A practical way to illustrate them is through the notion of *aggregation bias* in the time scales of market saturation. Let us suppose we are unaware of the heterogeneous structure of the population and try to estimate from sales data the wrong homogeneous model (1), even in the simplest case, with m known. Independently on the lenght of the available data set we would unavoidably overestimate the true innovation rate, thereby systematically underestimating the true time scale of the adoption process, i.e. the true market saturation time, a quite undesidered effect for management. The explanation is simply that in presence of heterogeneity the initial dynamics of the innovation process

Saturated shares		75%	90%	95%	99%
$\alpha = 0,0278$ (Homog.model)	CV=0	49	83	107	165
$\alpha_1 = 0,027$ $\alpha_2 = 0,0358$	CV=0,094	50	83,3	109	168
$\alpha_1 = 0,025$ $\alpha_2 = 0,0538$	CV=0,309	52	88	116	180
$\alpha_1 = 0,023$ $\alpha_2 = 0,0718$	CV=0,525	56	96	126	196
$\alpha_1 = 0,021$ $\alpha_2 = 0,0898$	CV=0,740	61	105	138	214
$\alpha_1 = 0,019$ $\alpha_2 = 0,1078$	CV=0,955	67	116	153	237

TABLE 1

Times needed to saturate prescribed maket-shares in an "external" model with two groups; w1=0.9 ; w2 =0.1; m=10.000.000.

will exceedingly reflect the role of the groups with faster adoption time scale.[7] Table 1, reporting the lenght of times needed to saturate prescribed market shares in a set of heterogeneous populations with a small fastly-adopting group coexisting with a large slowly-adopting group, is an instance of these effects.

3.2. Continuous heterogeneities. If the innovation rate is a continuous variable u taking values in some prescribed set $U \in R^+$, the relevant formulation becomes:

$$\dot{Y}_u(t) = u(m_u - Y_u(t)) \qquad u \in U \tag{9}$$

where m_u (with: $\int_0^\infty m_u du = m$) is the absolute (non-normalised) density of individuals with innovation rate u at time zero, and: $f(u) = m_u/m = \frac{X_u(0)}{X(0)}$ is the corresponding relative density. The aggregate equation is:

$$\dot{Y}(t) = \frac{d}{dt}\int_0^\infty Y_u(t)du = \int_0^\infty \dot{Y}_u(t)du = \alpha(t)(m - Y(t)) \tag{10}$$

where:

$$\alpha(t) = \int_0^\infty u\frac{X_u(t)}{X(t)}du = \int_0^\infty u\frac{f_u e^{-ut}}{\int_0^\infty f_u e^{-ut}du}du \tag{11}$$

and:

$$\dot{X}_u(t) = -uX_u(t) \quad u \in U \quad ; \quad \dot{X}(t) = -\alpha(t)X(t) \tag{12}$$

Results completely similar to those of the previous subsection hold. In particular:

$$\alpha(0) = \int_0^\infty u f_u du \;\; ; \;\; \dot{\alpha}(t) = (-1)\left(\int_0^\infty u^2\frac{X_u(t)}{X(t)}du - \alpha^2(t)\right) = (-1)Var_U(t)$$

[7] Our reasoning has been based on the comparison between a prescribed heterogeneous situation and the underlying homogeneous model, in which heterogeneity is absent. The problem of the graduation of the effects of different heterogeneous situations will be considered elsewhere. It is nonetheless easy to acknowledge that, other things being equal, more heterogeneous situations, measured for instance by a larger initial variance, imply a faster initial decay of the aggregate innovation rate and a longer saturation time. These aspects will be deepened in further developments (but see the example of next subsection).

A remarkable example: the gamma case. Let us assume that the initial distribution in the levels of the innovation rate is Gamma-type: $f_u = \frac{X_u(0)}{X(0)} \sim Ga(\xi, \eta)$, where $(\xi, \eta) > 0$, so that $\alpha(0) = \int_0^\infty uGa(\xi, \eta)du = \xi/\eta$. As:

$$\int_0^\infty f_u e^{-ut} du = \frac{\eta^\xi}{(\eta + t)^\xi}; \quad \int_0^\infty u f_u e^{-ut} du = \frac{\xi \eta^\xi}{(\eta + t)^{\xi+1}}$$

we find:

$$\alpha(t) = \left(\frac{\eta^\xi}{(\eta + t)^\xi} \right)^{-1} \frac{\xi \eta^\xi}{(\eta + t)^{\xi+1}} = \frac{\xi}{\eta + t} \tag{13}$$

The result (13) leads to the following linear ODE for $X(t)$:

$$\dot{X}(t) = -\alpha(t) X(t) = -\left(\frac{\xi}{\eta + t} \right) X(t)$$

the solution of which gives the overall market dynamics. By solving we find:

$$Y(t) = m - X(t) = m \left(1 - \frac{\eta^\xi}{(\eta + t)^\xi} \right) \tag{14}$$

The result (14) (which satisfies all the expected requisites of innovation curves) shows that the aggregate dynamics is not anymore exponential, as it was in the homogeneous model.

Graduating the effects of heterogeneity. For a U random variable having a density $Ga(\xi, \eta)$, its mean and variance are: $\mu(U) = \xi/\eta$; $Var(U) = \xi/\eta^2$. Hence:

$$Var(U) = \eta^{-1} \mu(U) \quad or: \quad \eta^{-1} = \frac{Var(U)}{\mu(U)}$$

showing that η^{-1} represents a measure of dispersion (even if not adimensional). This permits to write:

$$\alpha(t) = \frac{\xi}{\eta + t} = \frac{\xi}{\eta} \frac{\eta}{\eta + t} = \alpha(0)(1 + \eta^{-1} t)^{-1} \tag{15}$$

The last relation suggests the appropriate ordering of the effects of a whole family of heterogeneous models in which a background homogeneous model defines the origin of the scale. By keeping fixed the ratio $\xi/\eta = \mu = \alpha(0)$, and by considering diminishing levels of η, i.e. increasing levels of $B = \eta^{-1}$ we are automatically considering initial distributions characterised by the same mean, i.e. by the same initial aggregate innovation rate, and by increasing levels of heterogeneity, as measured by an increasing ratio variance/mean. As:

$$\frac{d\alpha(t)}{dB} = \alpha(0) \frac{-t}{(1 + Bt)^2} < 0$$

we see that the aggregate innovation rate is, at any time, a decreasing function of the level of heterogeneity. In other terms: a larger initial heterogeneity is responsible, other things being equal, of a faster time decay in the aggregate innovation rate.

Time lenghts needed to saturate prescribed market shares. In the homogeneous case the time needed to absorb the first $(1-q)\%$ of the market is the solution of the equation: $m(1-e^{-\alpha t}) = m(1-q)$ given by:

$$T_{Hom}^{1-q} = \left(\frac{-1}{\alpha}\right) \log q = T_{Hom} \tag{16}$$

When heterogeneities follow the Gamma distribution we have the corresponding equation:

$$\frac{\eta^{\xi}}{(\eta+t)^{\xi}} = q \tag{17}$$

which may be solved analytically for t (on the contrary of the discrete case). To compare the heterogeneous case with the homogenous one let us consider a heterogeneous situation with $\alpha(0) = \xi/\eta = \alpha$ (the innovation rate α of the corresponding homogeneous population). From (17) we find:

$$(-1)\,\eta \log \frac{\eta}{\eta+t} = -\frac{1}{\alpha(0)} \log q \quad \to \quad (-1)\,\eta \log \frac{\eta}{\eta+t} = T_{Hom}$$

By solving for t the last equation we get the following relation between the time unit in the heterogeneous model, T_{Het}, and its homogeneous counterpart T_{Hom}:

$$T_{Het} = \beta \left(\frac{1}{e^{-T_{Hom}/\beta}} - 1 \right) \tag{18}$$

It is not difficult to show that $T_{Het}(\beta)$ is a monotonically increasing function of the degree of initial heterogeneity, as measured by β^{-1}, which reduces to T_{Hom} in the limit case in which heterogeneity goes to zero (other things being equal).

4. Heterogeneity and mixing within the internal adoption process. The basic model for "internal" ("word of mouth") diffusion of innovation (dating back to Mansfield, 1961) is subsumed by the following logistic ODE:

$$\dot{Y}(t) = \frac{q}{m} Y(t)(m - Y(t)) = rY(t)(m - Y(t)) \quad r = \frac{q}{m} \tag{19}$$

plus the initial condition $Y(0) = 1$ (needed to start the transmission process). The q coefficient ($q > 0$) has been defined by Bass as the *imitation coefficient.* In view of our subsequent developments the most useful justification of (19) is based on the following epidemiologic argument: at every time t further spread of information totally depends on the rate of encounters between "A" (let us call them now "infected") and "N" ("susceptible" to the acquisition of the information) individuals. Let's make the following assumptions: A)each individual in the population encounters C individuals p.u.t (independently on the fact he already adopted or not). C is the *meeting rate*, or the *rate of social activity,* defining the total rate of "social partners" encountered per unit time, B)the pattern of encounters is perfectly at random; C)there exists a constant probability β that a meeting beetween an "A" individual and an "N" individual gives rise to a new infection, i.e. to a new "A".

The previous assumptions describe a population of (homogenous) individuals mixing homogeneously. The absolute rate of change in the number of infectious individuals (whose who received the information), i.e. the number of new infections per unit time, can be computed as follows. The single infected individual meets C individuals p.u.t. As individuals mix homogeneously, a fraction $S(t) = X(t)/(X(t) + Y(t))$ of these C encounters will take place with susceptibles ($0 \leq S(t) \leq 1$ is the *susceptible fraction*). A fraction β of these encounters will end in new infections. The quantity $\beta CS(t)$ hence defines the load of new infections caused by a single infected individual per unit time. By adding up the actions of all the infected individuals we find the total load of new infections p.u.t:

$$B(t) = \sum_{i=1}^{Y(t)} \beta CS(t) = Y(t)\beta CS(t) = \beta C\frac{X(t)Y(t)}{X(t)+Y(t)} \tag{20}$$

By defining $\beta C = q$, and using: $X(t) + Y(t) = m$ (population size is constant), the equation (19) is obtained. In the epidemiological jargon model (19) is called an SI ("*susceptible-infective*") model. The epidemiological foundation followed here leads to define the *imitation coefficient* q as the product between two constitutive parameters, the meeting rate C and the probability of infection per single encounter β. Let's remark that the imitation coefficient is the rate of growth of the initial exponential phase of the dynamics of (19).

4.1. A preliminary: heterogeneity in the acquisition of information. Let us consider first a simplified situation in which individuals are still assumed to mix homogeneously, as in the basic model (19), but differ in the speed at which their acquire the information from encounters. This situation is quite similar to that of the previous section: epidemiologically we would say that individuals have different degrees of susceptibility. This leads, assuming a discrete pattern of heterogeneity (extension to continuous heterogeneity is straightforward.), to the following extension of (19)[8]:

$$\dot{Y_i}(t) = r_i(m_i - Y_i(t))Y(t) \quad i = 1, 2, ..., n \tag{21}$$

where: $m_i - Y_i(t) = X_i(t), r_i = q_i/m, Y(t) = \sum_i Y_i(t)$. Obviously:

$$\dot{X_i}(t) = -r_i(m_i - Y_i(t))Y(t) = -r_iX_i(t))Y(t) \quad i = 1, 2, ..., n$$

The equations (21) are, compared to external diffusion, more difficult to treat analytically. Nonetheless the main qualitative feature of the homogeneous model (19), i.e. asymptotic saturation to m, is preserved. Moreover, considerations quite similar to

[8] In this case the overall susceptible fraction at time t is: $(X_1 + .. + X_n)/m$. Hence each single infective individual leads to $\beta C(X_1 + ... + X_n)/m$ new infections p.u.t., etc.

those developed for the model for external diffusion hold. Let us assume for simplicity $q_1 < ... < q_n$. The aggregate dynamics is described by:

$$\dot{Y}(t) = Y(t)\sum_i r_i(m_i - Y_i(t)) = \frac{q(t)}{m}X(t)Y(t) \tag{22}$$

where the aggregate (time-varying) imitation coefficient $q(t)$ is given by:

$$q(t) = \sum_i q_i\frac{X_i(t)}{X(t)} = \sum_i q_i w_i = \mu_q(t) \tag{23}$$

Hence, similarly to the external case, the aggregate imitation coefficient is the average of the q_i weighted with the susceptibility weights. In particular: $q(0) = \sum_i q_i w_i(0) \cong \sum_i q_i \frac{m_i}{m}$. The following facts, which straightforwardly extend results from the case of external diffusion, hold:

- The rate of growth of the initial exponential phase is: $r(0)X(0) \cong q(0)$, which is exactly by the initial innovation rate. Moreover:

$$\left(\frac{d}{dt}q(t)\right) = (-1)Y(t)Var_q(t) < 0 \tag{24}$$

  i.e. the aggregate imitation coefficient is strictly decreasing over time.
- The time scale needed to the saturation of the whole market is the time-scale of the slower group.

Hence the same type of aggregation bias observed for external diffusion is observed in model (21) as well.

4.2. Heterogeneities in social activity: the mixing problem. The introduction of heterogeneities in the rates of social activity makes the internal mechanism much more complex and interesting of the external one: a new problem, the so called mixing problem, largely investigated in the recent epidemiological literature, arises. Heterogeneity and the mixing problem ("who mixes with whom?") have represented a core problem of mathematical epidemiology in the last 15 years. A key motivation to the birth of a general theory of heterogeneity and mixing (HMT since now on) in epidemiology has been the need to provide flexible tools for the mathematical analysis of HIV and other sexually transmitted diseases (STD), where a central aspect of the dynamics is "*...the marked heterogeneity in degrees of sexual activity within the overall population*" (Anderson and May 1991, 228). A huge literature has been developed since then, of which we may quote here only a very small, although highly representative, subset. Even if an overall HMT in theoretical epidemiology is still far from being complete, several quite general problems are nowadays quite well understood.

Altough most of the efforts produced up to now have been connected with the analysis of STD, the HM approach is completely general and it may be applied to practically any type of social interaction. In recent times several tentatives (Castillo-Chavez et al. (1995), Edmunds et al. (1997)) have been made aimed to extend the basic framework for STD to other areas, ranging from nonsexually transmitted diseases to general contact patterns in biology.

Along this direction we will try to show how the recent epidemiological developments fit into the area of marketing models. We will first introduce the mixing theory tool-box, by showing how the HM approach provides a convenient framework for the investigation of the effects of heterogeneity within internal diffusion models, in that it clarifies which are the crucial parameters in the representation of social interaction processes: *activity levels* plus *mixing functions.*

During this phase we will systematically exploit background materials derived from the recent epidemiological literature (for instance Anderson and May (1991), or Jacquez et al. (1989,1995)).

Some highly stylized prototype models of mixing patterns will then be presented. We will subsequently show how the basic simple model (19) for internal diffusion modifies when heterogeneity and mixing are considered and discuss the effects of some special mixing assumption. We finally review a noteworthy result, taken from the recent epidemiological literature, cooncerning the model with removal.

4.3. Mixing frameworks; mixing parameters. To formulate the internal model with heterogeneity in social interaction we will not merely assume that the individuals of the involved population are subdivided in groups depending on their different imitation coefficients. This is not a fruitful approach. Rather, as suggested by the recent epidemiological approaches (and implicit in the epidemiological foundation of (19)), we will assume that individuals are stratified on the basis of their rates of *social activity.* More precisely we assume that social activity within the population is characterised by a prescribed (discrete) distribution of *activity levels*:

$$C = \begin{matrix} C_1 & C_2 & \dots & C_k \\ m_1 & m_2 & \dots & m_k \end{matrix} \tag{25}$$

where C_i is the number of "social" partners [9] p.u.t. of individuals belonging to group i and $\sum m_i = m$ is the total population. Extension to the continuous case is straightforward. Once this heterogeneity has been recognised, the definition of the rate of change over time in the number of adopters requires the specification, at the very least, of the *functions of social interaction* or *mixing functions* (Jacquez et al. 1989, Blythe et al. 1991) p_{ij}, denoting the fraction of his C_i social partner p.u.t that the generic individual of group i has with individuals belonging to group j. Provided all the groups are socially active, the mixing functions satisfy the following properties:[10]

$$i) p_{ij} \geq 0 \quad ; \quad ii) \sum_{j=1}^{n} p_{ij} = 1 \quad ; \quad iii) C_i m_i p_{ij} = C_j m_j p_{ji}$$

The "probabilistic" properties i) and ii) quickly follow from the process of allocation of the social activity of the various groups. For the $i-th$ group we have the obvious

[9] It is assumed that these social relations are of a type which is adequate for the transmission of information. In the theory of STD this quantity typically represents the mean number of (new or total) sexual partners p.u.t. of an individual belonging to group i.

[10] The mixing properties are sometimes called the mixing axioms in the more theoretical treatment of the subject (Blythe et al., 1991, Busenberg and Castillo-Chavez, 1991).

relation: $C_ip_{i1} + C_ip_{i2} + ... + C_ip_{ik} = C_i$. Property iii) has a more substantive nature: it represents a *law of conservation of social activity.* The physical meaning of iii): $C_im_ip_{ij} = C_jm_jp_{ji}$ is that the social activity (i.e. the number of relations) of individual type "i" with individuals type "j" cannot be different from the total number of relations of "j" individuals with "i" individuals. As these relations involve quantities which are not necessarily constant, they represent *concrete restraint operating at every time during the social process.* Let us assume that for some reason the number m_j of individuals in group j should diminish, being fixed the numbers in the other groups: this necessarily implies that all the restraints involving group j are not anymore satisfied and that, therefore, something has to change to preserve the restaints satisfied. To stress the dynamical nature of the mixing axioms it is useful to write therefore: $C_i(t)m_i(t)p_{ij}(t) = C_j(t)m_j(t)p_{ji}(t)$.

REMARK 1. *In compact terms the overall process of social interaction may be represented by the so called mixing matrix: $P = [p_{ij}]$ (which is obviously a markov matrix).*

Heterogeneities in social activity do not exhaust the range of possible heterogeneities in our framework. A further possible source of heterogeneity is represented by the probability of "infection" per single encounter which should be written as β_{ij} to reflect possible effects of heterogeneity from the sides of both partners of a social relations. For simplicity we will only consider heterogeneity in social activity and β_{ij} will always be taken as constant.

REMARK 2. *The approach to social mixing presented here, in which the unique source of heterogeneity considered is the level of social activity, measured by an index of speed in social circulation, is as simplest as possible. More involved treatments are possible, for instance introducing explicit dependencies of mixing parameters on socio-economic or demographic variables and so on (Castillo-Chavez et al. 1995).*

4.4. Special mixing structures. The analysis of special mixing cases is of great help in defining the boundaries of the problem. Noteworthy examples of mixing structures employed in the recent epidemiological literature (Jacquez et al.(1989), Uche and Anderson (1996)), are the following:

1. *Restricted mixing (perfect assortativeness or "like with like"):* 100% of the social activity of the individuals of a given group is confined within their own group. The corresponding form of the mixing matrix is the identity matrix I.
2. *Proportionate mixing (P.M.):* the concept of PM (for instance Nold 1980), extends to heterogeneous situations the notion of homogeneous mixing with random selection of social partners. To understand this notion let us first consider the special case in which all individuals in all groups have the same number of relations C per unit time. In this case, provided individuals choose their relations at random, the mixing functions would have the form: $p_{ij} = N_j/N = p_j$ as the probability for a "i" individual to encounter a "j" individual would only depend on the relative frequency of "j" individuals. In the more general case in which individuals have different C_i the relative frequencies need to be weighted with the rates of social activity. Let us consider a (large)

community compounded by two groups of identical size; individuals of the first group have a low rate of social activity, while individuals of the second group have a large rate of social activity. This means that the group of "very social" individuals contribute more than the other group to the the overall social activity of the community. In other words: if an individual choose a relation at random, he will have a larger probability to meet an individual from the "very social" rather than from the "unsocial" group. This leads to the following definition:

DEFINITION 1. *(proportionate mixing) Given the distribution of activity levels $C = \{C_i; m_i\}$, the corresponding proportionate mixing function is:*

$$p^*_{ij} = \frac{C_j m_j}{\sum_{j=1}^{n} C_j m_j} \quad i = 1, 2, ..., n \tag{26}$$

$C_i m_i$ is clearly a measure of the social activity by "i" individuals, while $\sum C_i m_i$ is a measure of the total social activity taking place in the overall community.

1. *Preferred mixing*: in this type of mixing (Jacquez et al. 1989, 1992) an arbitrary fraction h_i of each group's contacts are reserved for within-group contacts; the remaining contacts of each group (given by: $(1 - h_i)C_i N_i$) are subject to the proportionate mixing rule. Hence:

$$p_{ij} = h_i \delta_{ij} + (1 - h_i) \frac{(1 - h_j)C_j N_j}{\sum_{j=1}^{K}(1 - h_j)C_j N_j} \tag{27}$$

   where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$.
2. *Perfectly disassortative mixing*: a mixing pattern is perfectly disassortative when all the social activity of a group is allocated within a unique different group. The ensuing mixing matrix is therefore a non-identity zero–one matrix. The matrix:

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

   is the simplest example of a fully disassortative pattern.

4.5. The model for internal diffusion embedding heterogeneity and mixing. The explicit use of the mixing parameters C_i, p_{ij}, β_{ij} lead to the following general formulation of the internal model:

$$\dot{Y}_i = C_i(m_i - Y_i) \sum_j \beta_{ij} p_{ij} \frac{Y_j}{m_j} \quad i = 1, 2, ..., n \tag{28}$$

As for (19), the expression (28) is justified by computing the load of new "infections" caused p.u.t by a single infected individual belonging to group j (for instance Jacquez et al. 1995). This individual encounters C_j (new) individuals p.u.t.; of these C_j encounters a fraction p_{ji} is allocated within group i, i.e. $C_j p_{ji}$ relations are with type "i" individuals. Therefore, by assuming that the mixing process between "i" and "j" individual is perfectly at random, our single "j" infected individual will meet p.u.t $C_j p_{ji}(X_i/m_i)$ susceptible individuals in group i, causing $\beta C_j p_{ji}(X_i/m_i)$ new infections. By cumulating the previous quantity for all "j" individuals we find the total load of new infections caused p.u.t by the infected individuals in group j within group i: $B_{i,j}(t) = Y_j \beta C_j p_{ji} \frac{X_i}{m_i}$ To find the total number of new infections p.u.t in group "i" we have to add up all the quantities $B_{i,j}(t)$ with respect to all groups of possible infected "partners":

$$B_i(t) = \sum_j B_{i,j}(t) = \sum_j \beta C_j p_{ji} Y_j \frac{X_i}{m_i} \tag{29}$$

The case of a β variable across groups does not change the point. We simply have:

$$B_i(t) = \sum_j B_{i,j}(t) = \sum_j \beta_{ij} C_j p_{ji} Y_j \frac{X_i}{T_i} \tag{30}$$

By using the third mixing property $m_i C_i p_{ij} = m_j C_j p_{ji}$,the (30) may also be expressed as follows (leading to (28)):

$$B_i(t) = X_i \sum_j \beta_{ij} \frac{m_i C_i p_{ij}}{m_j} \frac{Y_j}{m_i} = C_i X_i \sum_j \beta_{ij} p_{ij} \frac{Y_j}{m_j} \tag{31}$$

The (30) or (31) define the total rate of change per unit time in the number of infected individuals (whose who have received the information) in group i. Remembering that: $X_i = m_i - Y_i$ the equation (28) is definitively obtained.

REMARK 3. *(imitation rates in presence of heterogeneity) The contribution to the rate of change of the number of infecteds of group i in (28) due to encounters with individuals belonging to group j: $\beta_{ij} C_i p_{ij} \frac{Y_i}{m_j}(m_i - Y_i)$ suggests that the rate of imitation of "j" individuals by "i" individuals is given by the quantity:*

$$q_{ij} = \beta_{ij} C_i p_{ij} \tag{32}$$

In theoretical epidemiology the model (28) is a heterogeneous SI model with constant population. Its qualitative properties are well understood (Lajmanovich and Yorke, 1976). Two main qualitative situations are of concern: i)all the groups are connected (i.e. the case of an imprimitive mixing matrix),[11] ii)isolated groups exist (a primitive mixing matrix). In the former case, independently on the initial conditions, the whole community will be reached by the information in the long term, while in the latter isolated groups will never be informed unless an informed individual reaches the group at some stage (formally: if the initial condition $Y_i(0)$ of an isolated group is zero, than the epidemics will never take-off). Hence in a community characterised by restricted mixing, or by the presence of isolated groups, special care should be reserved by planners to reach isolated groups.

[11] The connection may be direct or indirect, due to the intermediate interaction with other groups.

On the other hand it is hard to provide a synthetic view of the overall quantitative behaviour of the system (28) for arbitrary mixing patterns. By resorting to special assumptions on the mixing patterns, valuable information is nonetheless obtained. We report here two important results concerning respectively the case of restricted and proportionate mixing. A more detailed classification is the target of future work (Manfredi et al. 1998).

The case of restricted mixing. As abovementioned, when restricted mixing is the rule the overall behaviour of the market wholly depends on initial conditions: the information will only spread within those groups which at some stage are reached by some initial spreader. Although this case is mathematically very simple (the various groups follow independent time paths) it may give rise to surprising facts such as the appearance of distinct adoption waves when the imitation rates of the various groups differ sufficiently in magnitude. This is illustrated in fig. 1 which reports the aggregate dynamics of a three groups community experiencing restricted mixing.

The case of proportionate mixing. When the mixing patterns are of the PM type, (28) reduces to (under $\beta_{ij} = \beta$):

$$\dot{Y}_i = \beta^* C_i (m_i - Y_i) \sum_j C_j Y_j \qquad (33)$$

where: $\beta^* = \beta / \sum_j C_j m_j$. The aggregate dynamics is described by the equation:

$$\dot{Y} = r^*(t) Y(t)(m - Y(t)) \qquad r^*(t) = \beta^* C_X(t) C_Y(t) \qquad (34)$$

in which the aggregate imitation rate depends on the product between two weighted averages of the levels of social activity, namely:

$$C_X(t) = \sum_i C_i \frac{X_i}{X} \;\; ; \;\; C_Y(t) = \sum_j C_j \frac{Y_j}{Y}$$

To characterise the behaviour of (34)[12] let us consider again: i)the rate of growth of the exponential early phase; ii)the time scale relevant for the saturation of the market. For what concerns this latter the stability analysis of the saturation equilibrium $(m_1, .., m_n)$ shows that, as expected, the saturation time of the overall market depends on the time to stability of the less socially active group with contact rate C_1. Viceversa, the initial exponential phase is characterised by the Jacobian matrix $J(0)$ which characterizes the stability properties of the $(0, ...0)$ equilibrium of model (33). Since:

$$J(0) = \beta^* \begin{pmatrix} m_1 C_1^2 & m_1 C_1 C_2 & \dots & m_1 C_1 C_n \\ m_2 C_2 C_1 & m_2 C_2^2 & \dots & m_2 C_2 C_n \\ \vdots & \vdots & \ddots & \vdots \\ m_n C_n C_1 & m_n C_n C_2 & \cdots & m_n C_n^2 \end{pmatrix}$$

[12] It is easy to show that, similarly to other findings of this work:

$$\frac{dr^*(t)}{dt} = (-1)\beta^* \left(\sum_j C_j Y_j\right) \left(C_X(t) Var_C^Y(t) + C_Y(t) Var_C^X(t)\right) < 0$$

we have (notice that the determinants of all the principal minors of orders $2, 3, ..., n$ vanish) the characteristic equation:

$$\lambda^n - \left(\beta^* \sum_j C_j^2 m_j\right) \lambda^{n-1} = 0$$

It follows that the (dominant) eigenvalue, which lead the initial phase growth is given by:

$$\lambda_0 = \beta^* \sum_j C_j^2 m_j = \beta \frac{\sum_j C_j^2 m_j}{\sum_j C_j m_j} = \beta \mu_C \left(1 + \frac{\sigma_C^2}{\mu_C^2}\right) \tag{35}$$

where μ_C and σ_C^2 are respectively the mean and variance of the distribution of the levels of social activity and σ_C^2/μ_C^2 the square of the variation coefficient, which is the appropriate measure of heterogeneity in the present situation (notice that for $\sigma_C^2 = 0$ the homogeneous result is recovered). The rate (35) is the rate of growth which prevails after the initial transient phase (provided no saturation effects take place). In fact at the very beginning:

$$\frac{\dot{Y}(0)}{Y(0)} = r^*(0)(m - Y(0)) \cong \beta^* C_X(0) C_Y(0)(m - Y(0)) = \beta^* C_Y(0)$$

which essentially reflects the initial distribution of infectives.

The result (35) which has been rederived here in an original manner for the basic SI model typical of new product diffusion theory, is in effects a cornerstone in theoretical epidemiology, where it has been found to hold even in more complex situations (Anderson and May 1991). It shows that distributions of social activity characterised by identical mean but larger heterogeneity will lead to faster initial growth.

In sum, remembering that a higher heterogeneity will in general also imply, coeteris paribus, a longer saturation time compared to situations with lower heterogeneity we may say that: *market situations characterised by larger levels of heterogeneity tend to (other things being equal) take-off more quickly and to saturate more slowly.* This noteworthy effect is illustrated in fig. 2, which reports the aggregate dynamics of a set of increasing heterogeneity situations (under PM) grounded against their homogenous counterpart. The next subsection reviews a more popular epidemiological version of the last result (Anderson and May 1991) which will add further insight to our understanding.

4.5.1. Internal diffusion: the case of spreaders who cease to spread. The possibility of permanent removal from the state of spreader in models of diffusion of news was considered by Bartholomew (1970, 223), borrowing from epidemiology the popular model of general epidemics (Kermack and McKendrick (1927)): "*People may cease to be spreader for a variety of reasons: they may forget, loose interest, or gain the impression that every body knows*". The introduction of permanent removal at constant rate modifies the homogeneous SI model into the following SIR model:

$$\dot{X} = -\frac{q}{m} XY \quad ; \quad \dot{Y} = \frac{q}{m} XY - vY \quad ; \quad \dot{Z} = vY \tag{36}$$

where X=susceptibles (who have not yet received the information), Y=infectives (who have been informed), Z=removed (who cease to spread); v is the removal rate. Typical initial conditions are $X(0) = m - 1; Y(0) = 1; Z(0) = 0$.

The introduction of permanent removal makes predictions about market dynamics more complex, compared to the "classical" models of innovation. The dynamic of these latter, summarised in the Bass model, is qualitatively simple, unavoidably ending with the asymptotic saturation of the whole population. This is a necessary consequence of homogeneity and of the fact that the spreading source and word-of-mouth spreaders never cease to spread (i.e.: they circulate for an infinitely long time compared to the diffusion time scales). In the model with removal the following remarkable facts hold:

- The epidemics/new will not necessarily spread. This will actually happen only when the removal rate is not too large compared to the imitation rate. This is intuitive: if the removal rate is larger, compared to the imitation rate, spreaders will get exhausted before being able to "infect" a relevant part of the susceptible population. This fact is expressed via a remarkable *threshold theorem* (dating back to Kermack and McKendrick (1927)), leading to a central epidemiological parameter, the *basic reproduction ratio (BRR)* R_0. The BRR expresses the number of secondary infections caused by a single infected individual during his whole infective period (i.e.:before being removed) in a wholly susceptible population. Hence R_0 naturally acts as a threshold parameter: when $R_0 < 1$ we expect the disease dies out, while when $R_0 > 1$ the disease will invade the population. For model (36):

  (37) $$R_0 = \frac{\beta C}{v}$$

  All relevant facts concerning (36) are driven by R_0. In particular the rate of growth r_0 of the initial exponential phase of the epidemics is related to R_0 by:

  (38) $$r_0 = v(R_0 - 1) = \beta C - v$$

- Even if $R_0 > 1$ (so that the disease invades the host population), the disease will not be able to "reach" the whole population. In the long term the disease dies out without having infected the whole population. The fraction eventually infected, which may be significantly smaller than unity, is an increasing function of R_0.

4.5.2. Proportionate mixing in the heterogeneous model with removal. The classification of the overall effects of heterogeneitiy on (36) is not an easy matter. Complete results are nonetheless available (Anderson and May 1991) for the case of proportionate mixing. The heterogeneous version of (36) is:

(39) $$\dot{X}_i = -B_i(t) \quad ; \quad \dot{Y}_i = B_i(t) - vY_i \quad ; \quad \dot{Z}_i = fvY_i$$

Let PM be assumed: $B_i(t) = \beta^* C_i X_i \sum_j C_j Y_j$. The two following results by Anderson and May (1990, 1991) nowadays constitute a classical theorem of mathematical epidemiology:

- For the model (39) it holds: $R_0 = \beta C^*/v$; $r_0 = v(R_0 - 1)$ where:

$$C^* = \mu_C(1 + \sigma_C^2/\mu_C^2) \tag{40}$$

The (40) extend our previous findings: "*This result simply reflects the disproportionate role played by individuals in the more (socially) active groups, who are both more likely to acquire the infection and more likely to spread it*" (Anderson and May 1990, 287). This variance effect has proven to be valuable in the study of the transmission dynamics of STD, where the patterns in the distribution of levels of sexual activity usually generate very large variances, compared to their mean. In terms of diffusion of news, we may observing that heterogeneity exagerates the role of the more socially active groups (the "superspreeders") who are both more likely to acquire the new and to retransmit it.

- The asymptotic fraction of the population which experiences the disease in the long term, is a strictly decreasing function of the degree in heterogeneity in social activity: "*This is essentially because, other things being equal, the epidemics tend to burn itself out among those in the highly active classes, thus driving the effective value of the BRR below unity before a large fraction of those in the low activity classes have been infected*" (Anderson and May, 1990).

The substantive meaning of the last result is crucial for new products diffusion: when within the population there are very socially active groups, the information tends to spread very quickly among them. But if removal exists, these individuals will also, soon or later, cease to spread the new, leaving the charge to continue to spread the information to the less active groups, those characterised by lower degrees of social activity. This fact will reduce the *actual reproduction ratio*, and finally, in the long term, will lead to a smaller final prevalence. In other terms: *markets characterised by the same initial aggregate imitation coefficient may lead to completely different final prevalences, depending on the magnitude of their heterogeneity.*[13]

5. Beyond models: conclusions and data issues. The fact that a wrong homogenous treatment of an underlying heterogeneous situation may give rise to dramatically wrong predictions, i.e. that "heterogeneity matters", raises perhaps the main questions of the present paper: *how do individuals mix in the actual processes of social diffusion of innovations? which are the social interactions patterns relevant for the diffusion of new products*"?

[13] The results presented here do merely constitute instances of the consequences that heterogeneity may generate. They are in fact derived under special assumption on mixing patterns and hence do not exhaust the problem of classifying the effects of heterogeneities in epidemiological models. At present the understanding of the properties of SIR models (39) in presence of general mixing patterns is far from being complete (Jacquez et al. 1988, Jacquez et al. 1995; Jacquez and Simon1992). Nonetheless several important problems have been quite fairly well understood, such as the theoretical foundation and computation of the basic reproduction ratio in presence of general heterogeneity patterns (Diekmann et al. 1990, Heesterbeek and Dietz,1996 and refs.), and the problem of the general representation of mixing patterns (Busenberg and Castillo-Chavez (1991a,b)), Blythe et al. (1991, 1995), Castillo-Chavez et al. (1995)).

This central question, which is just a part of the more general question of how do individuals actually socially mix, has been largely neglected in classical new product diffusion models, essentially based on a traditional homogeneous apparatus. The heterogeneity and mixing approach provides a powerful and flexible theoretical framework. Its empirical applications are non-trivial: a hunger of data and not irrelevant statistical problems arise, particularly in comparison with the original Bass model and of many of its modified versions which require which require the estimation of only three parameters. In contrast, our model requires the estimation of many independent parameters, depending on the assumptions on the number of heterogeneous groups of potential adopters. One possible line of defense is that if the underlying phenomenon is truly heterogeneous, the modelization strategy must take it into account, even at the cost of loosing effectiveness in the estimation.

Since the beginning of the heterogeneity revolutions theoretical and applied epidemiologists have tried to start to anwer the analogous questions, but remaining mainly confined within the domain of sexual interactions (Shu-Fang et al. 1994). Hence on the empirical side there are no ready anwers for fields different from the original one. We believe it is now time to seriously fill this gap.

A. Dynamics of the sum of independent exponentially evolving populations. Let us consider a population compounded by n non interacting subgroups exponentially decaying at different constant rates $\alpha_1 < .. < \alpha_n$:

(41) $$\dot{X}_i\,(t) = -\alpha_i X_i(t) \qquad \alpha_i > 0 \quad i = 1, .., n$$

The dynamics of the total population $X(t)$ is given by the ODE:

(42) $$\dot{X}(t) = -\left[\sum_{i=1}^{n} \alpha_i w_i(t)\right] X(t) = \alpha(t) X(t)$$

where the w_i define the survivors weights $w_i = X_i(t)/X(t)$. We have:

(43) $$\dot{w}_i(t) = w_i(t)\left[\frac{\dot{X}_i\,(t)}{X_i(t)} - \frac{\dot{X}(t)}{X(t)}\right] = -w_i(t)\left[\alpha_i - \sum_{j=1}^{n} \alpha_j w_j\right]$$

The system (43) formally defines a Lotka-Volterra n-species cooperative system on the unit simplex. The dynamics of (43) is easily ascertained: the system will converge to the long term globally stable equilibrium $(1, 0, .., 0)$[14] where the only nonzero element

[14] Use the Liapunov function:

$$V(t) = (w_1 - 1)^2 + \sum\nolimits_{j=2}^{n} w_j^2$$

pertains to the fastest population (the one characterised by the highest growth rate.i.e. by the smallest decay rate). This "competitive exclusion" result totally rests on the existence of a group which is fastest compared to others. In practical terms: the fastest population will in be, in percentage terms, more and more important as times goes on. More insight is obtained by writing: $\alpha_j = \alpha_1 + \varepsilon_j$; where $\varepsilon_j > 0$, $\varepsilon_2 < .. < \varepsilon_n$ We find:

$$\frac{\dot{w}_1(t)}{w_1(t)} = \sum_j \varepsilon_j w_j > 0 \quad ; \quad \frac{\dot{w}_j(t)}{w_j(t)} = \frac{\dot{w}_1(t)}{w_1(t)} - \varepsilon_j \ ; \quad \frac{\dot{w}_n(t)}{w_n(t)} = -\sum_j \varepsilon_j w_j < 0$$

showing that the fastest group always gains weight during the dynamics, the slowest one always looses weight, while the other groups may experience an initial phase in which they gain weight. In the case $n = 2$, the ODE for w is a simple logistic equation, showing that the weight of the faster population, which is initially determined by the initial size of the two subgroups, will saturate to unity with an s-shaped dynamics.

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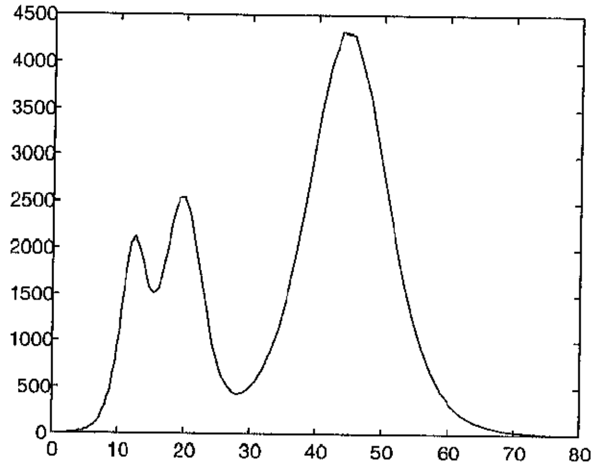

FIG. 1. *Separate adoption waves in the total number of adopters in a restricted mixing model with three groups ($m = 100.000$; $q_1 = 0.25$; $w_1 = 0.7$; $q_2 = 0.5$; $w_2 = 0.2$; $q_3 = 0.75$; $w_3 = 0.1$)*

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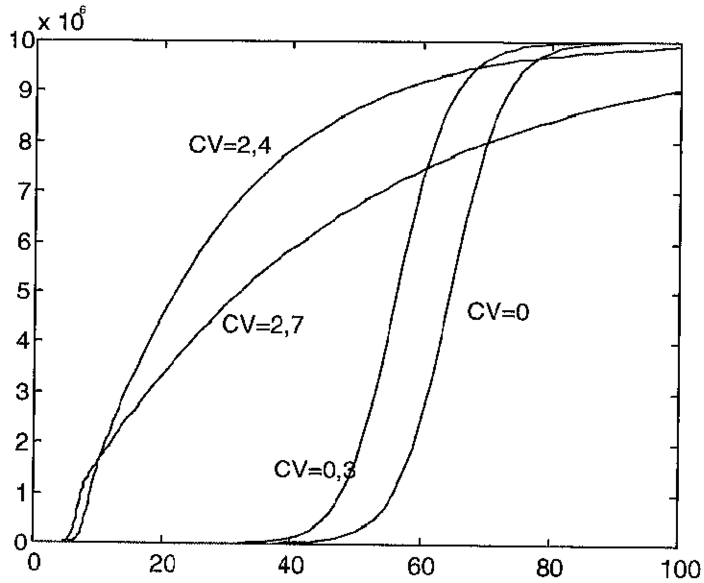

FIG. 2. *Internal diffusion: aggregate dynamics under the proportionate mixing assumption (CV=coefficient of variation).*