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# Toward the Development of an Age Structure Teory for Family Dynamics I: General Frame

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## 1 Introduction

Few years ago, while introducing at the state of art of family demography (in the introduction of the well known book by Bongaarts e al.(1987) on this subject), Nathan Keyfitz was pointing out some of the theoretical difficulties involved with the development of a general mathematical theory of the family, by contrasting it with the traditional standard individualistic mathematical theory of population i.e. mainly the classical one-sex stable population theory and its several developments. In particular he was observing (1987, 4): "One direction of improvement of standard demographic theory has been to consider the marriage and reproduction of the two sexes jointly, but an authentic two-sex model is stubbornly non-linear, and none of the attempts to deal with it analytically can be regarded as fully satisfactory". Furthermore: "Recognizing families is even harder than recognizing the joint effects of the two sexes on marriage and reproduction, and no one ought to hope for closed form solutions to the analogue for the family of Lotkas renewal equation for individuals. Families do produce families in successive generations in an authentic renewal process, but even the formulation of this process is difficult". Anyway he was concluding with a hope, especially regarding the area of two-sex models: "Norman Ryder believes we can do better" (ibidem).

The aim of this work is to provide a first step in the direction shown by Keyfitz, looking for a full description of the dynamics of nuclear family formation and dissolution within the boundaries of a fully age-structured model, starting exactly where N. Keyfitz left, i.e. getting back again to two-sex age-structured pair formation models, which have experienced a number of important advances in the last few years and seem a quite natural starting point in the development of a more general family renewal model.

In particular two among these advances have supplied some of the motivations at the basis of the present work.

The first one of these is the developement of a quite general theory of homogeneous dynamical systems (Hadeler et al. 1988, Waldstaetter 1989; Hadeler 1989, 1993), which constitute the natural environment for the study of the dynamical properties of the processes of pair formation, due to the homogeneous structure of pairs formation functions. And the key observation with the goal of developing a mathematical theory of family is that the process of pair formation is the only one nonlinearity of the family renewal process.

The second one is the so called mixing theory (MT) recently developed by Busenberg, Castillo Chavez and coworkers, which is probably going to reveal a major contributions of mathematics to social sciences. Born outside from pure demography, its original environment being that of mathematical epidemiology, particularly the spread of AIDS, the MT is becoming a "key" tool for all problems concerning social interactions. Several contributions of the proponents of MT have especially been devoted to two-sex populations permitting to clarify the nature and the properties of the typical consistency restraints which act during the dynamics of sexual populations, and hence also for what concerns family dynamics.

The present paper is organized as follows: next section reviews existing materials about pair formation processes and the mathematical theory of family; in the third section we introduce all the fundamental assumptions of our formulation with the aim to make the pair formation machinery endowable to describe family dynamics; in the fourth section we collect all our ingredients to provide the development of our general model of nuclear family formation and dissolution; finally we breafly discuss the topics on which we are already working and consider some possible further developments.

# 2 From pair formation to a general family formation model

The structure of the typical age structured pair formation model (Fredrickson 1971, Hadeler 1989, Castillo-Chavez, Busenberg and Gerow 1991) usually rests on three PDEs defining respectively the evolution due to the aging process in the number $m(a;t)$ of single males aged $a$ at time $t$, of single females aged $a'$, $f(a';t)$ and finally in the number $W(a,a',x;t)$ of pairs compounded by two partners aged respectively $a,a'$ which were formed $x$ years before, of the form:

$$
\begin{aligned}
\partial_{a,t} m(a,t) &= -\mu_m(a,t)\, m(a,t) - \int_{\mathbb{R}_+} \rho(a,a',t)\, da' + \\
&\quad + \int_{\mathbb{R}_+^2} \left[\sigma(a,a',x,t) + \mu_f(a',t)\right] W(a,a',x,t)\, da'dx \\
\partial_{a',t} f(a',t) &= -\mu_f(a',t)\, f(a',t) - \int_{\mathbb{R}_+} \rho(a,a',t)\, da + \\
&\quad + \int_{\mathbb{R}_+^2} \left[\sigma(a,a',x,t) + \mu_m(a,t)\right] W(a,a',x,t)\, dadx \\
\partial_{a,a',x,t} W(a,a',x,t) &= -\left[\mu_m(a,t) + \mu_f(a',t) + \sigma(a,a',x,t)\right] W(a,a',x,t)
\end{aligned}
\tag{1}
$$

In the previous equations the $\mu_m$ and $\mu_f$ functions are age specific male and female mortality rates, $\sigma$ is the age specific dissolution rate while $\rho$, the marriage function, denotes the number of marriages between males aged $a$ and females aged $a'$. Furthermore:

$$
\partial_{x_1,\ldots,x_n} = \frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_n} \qquad\qquad \int_{\mathbb{R}_+^n} = \underbrace{\int_0^{+\infty} \cdots \int_0^{+\infty}}_{n \text{ times}}
$$

where $\partial_{x_1,\ldots,x_n}$ defines the classical Ross-McKendrick-Von Foerster "aging operator".

These equations are of course to be completed by suitable boundary conditions the role of which is that of introducing the processes of natality and pair formation. These conditions are usually of the form:

$$
\begin{aligned}
& m(0,t) = \int_{\mathbb{R}_+^3} \beta^m(a,a',x,t)\, W(a,a',x,t)\, dada'dx \\
& f(0,t) = \int_{\mathbb{R}_+^3} \beta^f(a,a',x,t)\, W(a,a',x,t)\, dada'dx \\
& W(a,0,x,t) = 0 \\
& W(0,a',x,t) = 0 \\
& W(a,a',0,t) = \rho(a,a',t)
\end{aligned}
$$

Finally, we have to add a set of initial conditions through the prescription of arbitrary initial age distributions:

$$
\begin{aligned}
& m(a,0) = m_0(a) \\
& f(a',0) = f_0(a') \\
& W(a,a',x,0) = W_0(a,a',x)
\end{aligned}
$$

For the basic pair formation model several results are available. In Hadeler 1989 and 1993 some special cases are investigated, in particular the role played by generalized harmonic mean as marriage function. Castillo Chavez and al. 1991 set the problem in the more general frame of their general mixing theory. Inaba 1993 and Prüss and Schappacher 1994 have studied the problem of well-posedness of solutions and found conditions under which model (1) exhibit persistent solutions. Recently Castillo Chavez and coworkers extended to two-sex aged structured demographies their remarkable "It takes two to tango" theorem.

Two-sex pair formation models assume the existence of a highly simplified world in which two kinds of "actors" exist: pairs and single individuals, with the singles "wandering around the world" looking for a partner. From this point of view they are criticizable in that they do not recognize at all that human reproduction mostly takes place through the intermediate role of family.

The present work is devoted to the explicit introduction of the role of families within the frame of two-sex models of human reproduction. We take as the basic unit of our analysis the *nuclear family* as compounded by, following the U.N. 1980 definition (Bongaarts 1983), a married couple, or one parent, and their nonmarried children, living in the same household which is the more classical unit in theoretical work in family demography (Hoehn 1987) beeng the notion naturally embodied in the concept and the analysis of the family life cycle. The notion of nuclear family together with that of *one-person family* constitutes a natural starting point for the study of family dynamics, beeing easily connectable to two-sex pair formation models. Our model is concerned only with demographically relevant facts. We will be so interested in the process of pair formation only for that part of it which concerns stable unions and we will always call them "marriages", independently on the fact they are formalized or not: we could explicitly rule out from our model unlegitimate births, but this is clearly unnecessary.

We will be concerned with *two generations families*: we rule out the possibility of larger families with three or more generations, by explicitly assuming that when a son/daughter get marry, he/she becomes imediately member of another family leaving so automatically his/her family of origin. This seems to be a quite reasonable assumption, in view of the simplifications which permits in the formulation of the problem, but can also be justified from a purely factual point of view since in many real world situations the importance of larger families is really limited: as reported for instance by Keyfitz (1987), families with three or more generations were only 2.3% in the 1981 canadian census. Furthermore, we point out that almost by definition the nuclear family is the "two-generation family".

Models as (1) seem to be quite straightforwardly extendable for the description of family dynamics. We just need now a set of functions $W_n(a, a', b_1, b_2, \ldots, b_m; c_1, c_2, \ldots$ defining, in the more general case, the number of families with $n$ children, i.e. compounded by parents of given ages $a, a'$ and by $m$ male children of the ages $b_1, b_2, \ldots, b_n$ and $n - m$ female children of ages $c_1, c_2, \ldots, c_{n-m}$ at time $t$. This approach would provide a full description of the age and family structure prevailing in the population. Even if this task does not seem difficult, it requires in fact nothing but an accurate bookkeping of the various flows regarding possible individual movements among the various kinds of families, it involves actually a tremendous amont of work due to the very large number of possible flows (and corresponding parameters). To make the problem a bit more transparent we will be concerned with an idealized world in which only small families with *at most one children per family* are formed: this assumption rather than being oversimplistic seems to us a reasonable compromise in that it permits to mantain a high degree of generality while keeping at the same time a reasonable degree of comprehensibility. Furthermore such "italian" assumption can anyway be justified on the basis of its pure demographic relevance, since the one child families is, or is probably going to be, a quite important demographic pattern in developed countries. Finally, it is to be noticed that such an assumption is not, from a dynamical point of view, as trivial as it could seem: the ultimate fate of the population is not necessarily extinction, at least as long as we permit to the rates of family formation and dissolution to be large enough.

Under previous assumption, the number of functions we need, for capturing all possible kind of family nuclei is exactly nine, namely:

a) single males/single females (2 family nuclei)

b) both parents without children (1)

c) both parents with children (2)

d) only one parent with children (4).

Before start with a detailed list of the main ingredients of our description let us state the more relevant modelling features, especially in comparison with similar work, present in our description.

All types of families are parameterized with their specific age, given by the lenght of the interval of time elapsed since the corresponding "event-origin" i.e. the demographic event which gave rise to the formation of that specific type of family; for instance, marriage dissolution in a family of two parents and one children is, in our model, the event origin for two new families of age zero: a single and a "parent with children" family. For a more homogeneous treatment this required to endowe also the single individuals, who constitute special families: one member families, with their own family age.

To picture the dynamics of the "family system" we need a set of demographic parameters, which are in principle of the same type utilized in the classical pair formation models. Among these we will have, as usual, the typical sets of mortality, fertility, and partnerships dissolution rates, to which we will also add, this is a novelty of our approach, a set of *children family-leaving rates*. A realistic description of the family evolution can not neglect the phenomenon of separate leaving as a consequence of the process, due to job, majority, study and so on, by which young people leave their family of origin. We want to stress that the phenomenon of "leaving the family" regards only young people not yet married and so has nothing to do with other forms of "leaving the family", as marriage dissolution. Separate living have become of interest for demographers quite late, since recent is the definitive acknowledgement in western demographies of important patterns alternative to family (Ongaro 1990, Goldsheider and Waite 1987 and references therein).

In addition, we will need a set of marriage functions. We will use functions rather than rates because the meaning of marriage rates in sexual populations in not evident. As recognized many times for pair formation models (see for instance Hadeler 1989), all the demographic processes involved in family dynamics are linear, with the exception of the process of mating-partnerships formation. So, while all the contributions due to mortality, fertility, divorce, etc. to family dynamics are representable by a linear mass action formalism (Metz and Diekmann 1986), just by applying the corresponding rates to the given population, the representation of the process of pair formation, the nature of which is (strongly) nonlinear, is more difficult and need more care. To avoid the introduction of such complications since the very beginning of our work, in which our goal is fundamentally that of the description of the basic relationships, we will simply use general marriage functions, following for instance Hadeler 1989. It is well known that the treatment of the pair formation process is, in general, the more delicate aspect of demographic models, and the more important source of difficulties. Recognizing, as we did, a fine family structure, with a lot of different compartments (the various types of families), implies, in our bookkeping, a very large number of marriage "flow terms" depending on the type of family from which the two given individuals mating at a certain time come from. So, for instance, the female spouse of a male individual who is getting marry still living in family with his parents, can be chosen in many ways: from the pool of female children still living their parents home, but also from the pool of single females, or from the pool of separated females who are living with their children. Our formulation explicitly recognizes the possibility of differential marriage behaviours due to the different family of origin of the partners.

As we will see, thank to our assumption of "small families" it will be possible to give, also in "aggregate" cases, a particularly simple representation of the marriage functions. In fact at any moment of time every existing family will always "put on the market" of available partners at most one partner of the various types (i.e. a parent,or a children or a single) permitting so to define the various marriage functions as functions, simply, of the number of families. But we will make precise this aspect later.

Among the various demographic determinants which affect the nuclear family composition as listed in Bongaarts 1983, the only one left out is adoption, which could anyway at least in principle very easily embodied.

The present formulation provides a systematic description of the process of family formation (through marriage of singles), expansion (birth of children) and contraction (due to home leaving of adult children and death of parents) and concentrates of course on those aspects typically of interest for classical mathematical demography, i.e. the family composition of the population and the typologies of such families with respect to the number, age and sex of children. In some way it can be considered complementary to the *theory of the family life cycle*, which traditionally neglects such aspects.

Our nuclear family model is, this is just a standard assumption, a *closed population model*. It must be nonetheless pointed out that such a model is quite appealing for the study of migration. In effect it really constitutes the minimal modelling structure adequate to the study of migratory phenomena, once we recognize that the fundamental unit of migrations is not the individual but, rather, the family (and so the single or the family, with all its children charge). And in fact, actually, the basic motivation to the present effort was provided by the acknowledgement of the difficulties which are met by the typical two-sex models when we try to extend them to the study of open populations.

Finally, to avoid any confusion: a pair formation model is, in some sense, something more general of a demographic model (even if the two processes are formally indistinguishable) in that pair formation phenomenon can in principle take into account of the entire process of partnerships among individuals, including very short or not stable unions, prostitutes frequentation, and so on, in sum relationships which will not affect the "demography" of our world because will not, in general, effect natality (this is why a pair formation model can be very useful for the study of facts as the transmission of infectious diseases, where "not demographically relevant" pairs can play a central role).

# 3 The model: general aspects

Let us consider first the *family functions* denoted by $w_{ij}$ and representing the numbers (to be precise: the densities) of the various types of nuclear families with at least two components. The suffix $ij$ permits to determine the composition of the family under the following rules: if $i = 2$ both parents are present while if $i = 1$ or $i = 0$ only the male or the female parent is present, respectively; the second index denotes the presence of a male ($j = 1$) or female ($j = 0$) children. Furthermore, we denote with $a$, $a'$, $b$, $c$ the ages of male parents, of female parents, and of male and female children respectively. All the $w_{ij}$ functions are assumed to depend on the ages of all the components of the family, on the "special" age $x$ of the family itself (computed since the event origin) and, of course, on the time ($t$). For example the function $w_{20} = w_{20}(a, a', c, x, t)$ denotes the number (density) of families aged $x$ at time $t$, whose members are a male parent aged $a$, a female parent aged $a'$ and a female children aged $c$. In any case the order of the variables follows the convention that parents come first (i.e. before children) and males come first (i.e. before females).

Two particular function are employed to indicate the number of one-dimensional families: $m = m(A, x, t)$ denotes the number of males aged $A$ which at time $t$ are single since $x$ years, whereas $f = f(A', x, t)$ denotes the number of females aged $A'$ that are single since $x$ years at time $t$. The following table lists all the family functions:

$$\begin{array}{lll} w_2 = w_2(a, a', x, t) & w_{21} = w_{21}(a, a', b, x, t) & w_{20} = w_{20}(a, a', c, x, t) \\ w_{11} = w_{11}(a, b, x, t) & w_{01} = w_{01}(a', b, x, t) & w_{10} = w_{10}(a, c, x, t) \\ w_{00} = w_{00}(a', c, x, t) & m = m(A, x, t) & f = f(A', x, t) \end{array}.$$

The following lists introduce to the basic demographic parameters of our model.

1) The **marriage functions** $\rho_j$, indicate the density of marriages between partners of given ages and coming from different families at time $t$. They will depend, in principle, on all the age variables which characterize each family: firstly we have the variables concerning the male partner group, secondly those of the female partner group. We will have several possibilities:

*Coupling* **parent-single**:

$$\begin{array}{ll} \rho_1 = \rho_1(a, b, x; A', y, t) & \rho_2 = \rho_2(A, x; a', b, y, t) \\ \rho_3 = \rho_3(a, c, x; A', y, t) & \rho_4 = \rho_4(A, x; a', c, y, t) \end{array}$$

where, for instance, the $\rho_1$ function denotes the density of "marriages" between a parent aged $a$ living with his son aged $b$ since $x$ years (when the mother left for some reason), with an $A'$ lady who was leaving alone since $y$ years.

*Coupling* **single-single**: we have only one marriage function:

$$\rho_5 = \rho_5\left(A, x; A', y, t\right)$$

*Coupling* **parent-children**:

$$\begin{array}{ll}
\rho_6 = \rho_6\left(a, b, x; \bar{c}, \bar{a}, \bar{a}', y, t\right) & \rho_{12} = \rho_{12}\left(a, c, x; \bar{c}, \bar{a}, y, t\right) \\
\rho_7 = \rho_7\left(a, c, x; \bar{c}, \bar{a}, \bar{a}', y, t\right) & \rho_{13} = \rho_{13}\left(a, c, x; \bar{c}, \bar{a}', y, t\right) \\
\rho_8 = \rho_8\left(b, a, a', x; \bar{a}', \bar{b}, y, t\right) & \rho_{14} = \rho_{14}\left(b, a, x; \bar{a}', \bar{b}, y, t\right) \\
\rho_9 = \rho_9\left(b, a, a', x; \bar{a}', \bar{c}, y, t\right) & \rho_{15} = \rho_{15}\left(b, a', x; \bar{a}', \bar{b}, y, t\right) \\
\rho_{10} = \rho_{10}\left(a, b, x; \bar{c}, \bar{a}, y, t\right) & \rho_{16} = \rho_{16}\left(b, a, x; \bar{a}', \bar{c}, y, t\right) \\
\rho_{11} = \rho_{11}\left(a, b, x; \bar{c}, \bar{a}', y, t\right) & \rho_{17} = \rho_{17}\left(b, a', x; \bar{a}', \bar{c}, y, t\right)
\end{array}$$

*Coupling* **single-children**:

$$\begin{array}{ll}
\rho_{18} = \rho_{18}\left(A, x; \bar{c}, \bar{a}, \bar{a}', y, t\right) & \rho_{21} = \rho_{21}\left(A, x; \bar{c}, \bar{a}', y, t\right) \\
\rho_{19} = \rho_{19}\left(b, a, a', x; \overline{A}', y, t\right) & \rho_{22} = \rho_{22}\left(b, a, x; \overline{A}', y, t\right) \\
\rho_{20} = \rho_{20}\left(A, x; \bar{c}, \bar{a}, y, t\right) & \rho_{23} = \rho_{23}\left(b, a', x; \overline{A}', y, t\right)
\end{array}$$

*Coupling* **children-children**:

$$\begin{array}{ll}
\rho_{24} = \rho_{24}\left(b, a, a', x; \bar{c}, \bar{a}, \bar{a}', y, t\right) & \rho_{29} = \rho_{29}\left(b, a, x; \bar{c}, \bar{a}', y, t\right) \\
\rho_{25} = \rho_{25}\left(b, a, a', x; \bar{c}, \bar{a}, y, t\right) & \rho_{30} = \rho_{30}\left(b, a', x; \bar{c}, \bar{a}, \bar{a}', y, t\right) \\
\rho_{26} = \rho_{26}\left(b, a, a', x; \bar{c}, \bar{a}', y, t\right) & \rho_{31} = \rho_{31}\left(b, a', x; \bar{c}, \bar{a}, y, t\right) \\
\rho_{27} = \rho_{27}\left(b, a, x; \bar{c}, \bar{a}, \bar{a}', y, t\right) & \rho_{32} = \rho_{32}\left(b, a', x; \bar{c}, \bar{a}', y, t\right) \\
\rho_{28} = \rho_{28}\left(b, a, x; \bar{c}, \bar{a}, y, t\right) &
\end{array}$$

The marriage functions $\rho$ have to satisfy the well known "marriage axioms" (McFarland 1975, Keilman 1985) among which we remember the known requirements of: a) positivity, b) availability, c) monotonicity, d) simmetry, e) first degree homogeneity, f) consistency g) competition, h) substitution, that will not discussed in detail here since they will the central objects of separate works.

For what concerns the **family-leaving rates** usual assumptions on the variables hold (Proprietà?)

$$\begin{array}{lll}
\nu_{21} = \nu_{21}\left(a, a', b, x, t\right) & \nu_{11} = \nu_{11}\left(a, b, x, t\right) & \nu_{01} = \nu_{01}\left(a', b, x, t\right) \\
\nu_{20} = \nu_{20}\left(a, a', c, x, t\right) & \nu_{10} = \nu_{10}\left(a, c, x, t\right) & \nu_{00} = \nu_{00}\left(a', c, x, t\right)
\end{array}$$

the interpretation of which is straightforward: for instance $\nu_{21}\left(a, a', b, x, t\right)$ is the instantaneous rate of leaving the family, with parents aged $(a, a')$ for a son aged $b$ leaving in his family since $x$ years ($x$ is not, of course, not necessarily equal to $b$).

For which regards the **separation rates:**

$$\begin{array}{lll} \sigma_2^m = \sigma_2^m\left(a, a', x, t\right) & \sigma_2^f = \sigma_2^f\left(a, a', x, t\right) & \sigma_{21}^m = \sigma_{21}^m\left(a, a', b, x, t\right) \\ \sigma_{21}^f = \sigma_{21}^f\left(a, a', b, x, t\right) & \sigma_{20}^m = \sigma_{20}^m\left(a, a', c, x, t\right) & \sigma_{20}^f = \sigma_{20}^f\left(a, a', c, x, t\right) \end{array}.$$

The justification for the introduction of such forms is obvious. First, we have assumed that the separation rates are family specific (there is empirical evidence for this). Second we have assumed they depend on all the "ages" included in the model, in particular on those of the family and of children: there is strong evidence for this too. Third we have introduced both male and female separation rates to recognize how the family would subdivide in the presence of children. So obviously $\sigma_2^f = \sigma_2^m$, since there are no children in such families. Viceversa the distinction between $\sigma_{21}^m$ and $\sigma_{21}^f$, and $\sigma_{20}^m, \sigma_{20}^f$ is useful because, for instance, the "fission" of a $w_{21}$ family due to marriage dissolution, will give rise to two distinct possibilities

a) the son remains with his father: in this case we assume that the mother left

b) the son remains with his mother: in this case we assume that the father left.

So, $\sigma_{21}^f$ will denote the proportion, in the total number of $w_{21}$ family-type dissolution per unit time, given by $(\sigma_{21}^m + \sigma_{21}^f)w_{21}$, which are due to "mother leaving the family", so giving rise to, at the same time, $\sigma_{21}^f w_{21}$ new "single females" families and to $\sigma_{21}^f w_{21}$ new families of the $w_{11}$ type (father with his son). In the sequel sometimes wee will put $\sigma_{20} = \sigma_{20}^f + \sigma_{20}^m$ and $\sigma_{21} = \sigma_{21}^f + \sigma_{21}^m$.

The **mortality rates** are assumed to depend only on the age of the individuals (and of course on sex and time) but we will explicitly rule out, at least at beginning, the possibility of differential mortality due to the belonging to a particular type of family. So:

$$\mu^m = \mu^m\left(\tau, t\right) \quad \mu^f = \mu^f\left(\tau, t\right)$$

The male and female **fertility rates** are assumed to be , as usual in the literature, of the form:

$$\beta^m = \beta^m\left(a, a', x, t\right) \qquad \beta^f = \beta^f\left(a, a', x, t\right)$$

where:

$$\beta\left(a, a', x, t\right) = \beta^m\left(a, a', x, t\right) + \beta^f\left(a, a', x, t\right)$$

is the total fertility rate of couples of the $(a, a', x, t)$ type.

We will of course assume that all the demographic parameters just introduced are demographically "well behaved", leaving all detailed assumption to future works.

# 4 The model: dynamics

Let us now proceed to the formulation of the dynamics of each family. As a first step we will show how the typical machinery of Ross-McKendrick-VonFoerster balance PDEs, traditionally used in the description of the dynamics of sexual populations, can actually be extended to the description of the renewal process of (nuclear) family demography.We will distinguish three hierarchical levels of analysis, depending on the amount of demographical detail embedded in our description. The first level, the more general, will consider the fully age-structured problem in which all the functions previously introduced are assumed to depend on all the variables. The second level, the intermediate one, neglects the age of families and so considers, as in the typical demographic models, only the age of individuals. The transition from one level to the other is obtained by performing suitable integrations ove the entire span of the variable to be neglected, in this case the age of families, based on the assumption that the various parameters do not depend on such variable. We will consider, finally, an aggregate level, in which the age structure is omitted, due to the assumption of constant coefficients (or, at most, time dependent). The latter problem, the "aggregate one" will have an ODE form, while the two former problems will be described by systems of partial integro-differential equations. The fundamental difference between the two age structured formulations is that all the processes of family formations, in particular the nonlinear terms due to the marriage functions are embodied in the boundary terms in the more general case (ie: the first level). The integration over the age of families moves such terms from the boundaries and carries them within the equations (the second level).

$$R_j(\tau_1, \tau_2, t) = \int_{\mathbb{R}_+^{k+m}} \rho_j(\tau_1, \delta_1, \ldots, \delta_k; \tau_2, \gamma_1, \ldots, \gamma_m, t)\, d\delta_1 \ldots d\delta_k d\gamma_1 \ldots d\gamma_m$$

### 4.0.1 Families without children

The dynamics of a couple without children is described by:

$$\partial_{a,a',x,t} w_2(a, a', x, t) = -\Lambda_2(a, a', x, t)\, w_2(a, a', x, t) \qquad (2)$$

where

$$\Lambda_2(a,a',x,t) = \mu^m(a,t) + \mu^f(a',t) + \sigma_2(a,a',x,t) + \beta^m(a,a',x,t) + \beta^f(a,a',x,t)$$

with the boundary conditions

$w_2(a,a',x,t) = 0$ if $\min\{a,a'\} \le x$ (in particular $w_2 = 0$ if $a$ or $a'$ are zero);

$$\begin{aligned} w_2(\tau_1,\tau_2,0,t) &= \int_{\mathbb{R}^2_+} [\mu^m(b,t) + v_{21}(\tau_1,\tau_2,b,x,t)]\, w_{21}(a,a',b,x,t)\, dbdx + \\ &+ \int_{\mathbb{R}^2_+} \left[\mu^f(c,t) + v_{20}(\tau_1,\tau_2,c,x,t)\right] w_{20}(a,a',c,x,t)\, dcdx + \\ &+ \sum_{j\in J_2} R_j(\tau_1,\tau_2,t) \end{aligned}$$

where $J_2 = \{5, 18, 19, \ldots, 31, 32\}$.

If we suppose that $\beta, \sigma$ and $v$ do not depend on the age of the family, integrating (2) with respect to this variable, we obtain the following PDE problem involving the number of "families of $a$, $a'$ type" denoted by $W_2$:

$$\begin{aligned} \partial_{a,a',t} W_2(a,a',t) &= -\Lambda_2(a,a',t)\, W_2(a,a',t) + \sum_{j\in J_2} R_j(a,a',t) + \\ &+ \int_{\mathbb{R}_+} [\mu^m(b,t) + v_{21}(\tau_1,\tau_2,b,t)]\, W_{21}(a,a',b,t)\, db + \\ &+ \int_{\mathbb{R}_+} \left[\mu^f(c,t) + v_{20}(\tau_1,\tau_2,c,t)\right] W_{20}(a,a',c,t)\, dc \end{aligned} \tag{3}$$

It is to be noticed that the effect of the integration is that of removing the second boundary condition by directly inserting it into the dynamical equation.

If we finally suppose that $\beta, \sigma$ and $v$ depend only on $t$ we obtain, by integrating (3) on the ages of individuals, the ODE problem

$$\begin{aligned} \dot{W}_2^\star(t) &= -\Lambda_2(t)\, W_2^\star(t) + \left(\mu^f(t) + v_{20}(t)\right) W_{20}^\star(t) + \\ &+ \left(\mu^m(t) + v_{21}(t)\right) W_{21}^\star(t) + \sum_{j\in J_2} R_j^\star(t). \end{aligned} \tag{4}$$

Let us say few words just to comment a bit such rather horrible-looking equations. Despite their lenght (but this is the family bookkeping) their meaning is obvious. For example the latter ODE equation tells us that the evolution per unit time in the total number of couples without children (and independently on their age) is due to the (positive) inflow of new marriages or to the dissolution of larger families, due to the death of one or both of the two parents, or to the leaving of one of the children, and to the outflow given by separation or death of one of the two parents, or to the birth of a children.

### 4.1 Couples with a children

The dynamics for a couple with a male children is:

$$\partial_{a,a',b,x,t} w_{21}(a,a',b,x,t) = -\Lambda_{21}(a,a',b,x,t)\, w_{21}(a,a',b,x,t) - \sum_{j\in J_{21}} R_j(b,a,a',x,t)$$

where

$$\Lambda_{21}(a,a',b,x,t) = \mu^m(a,t) + \mu^m(b,t) + \mu^f(a',t) + \sigma_{21}(a,a',b,x,t) + \nu_{21}(a,a',b,x,t)$$

$$J_{21} = \{8,9,19,24,25,26\}$$

and with the boundary conditions

$$\begin{aligned} &w_{21}(a,a',b,x,t) = 0 \text{ if } \min\{a,a',b\} < x \text{ (and thus } w_{21} = 0 \text{ if } a \text{ or } a' \text{ are zero);}\\ &w_{21}(\tau_1,\tau_2,b,0,t) = \textstyle\sum_{j\in J'_{21}} R_j(\tau_1,b,\tau_2,t), \quad J'_{21} = \{1,2,6,8,10,11,14,15\}\\ &w_{21}(\tau_1,\tau_2,0,0,t) = \int_{\mathbb{R}_+} \beta^m(\tau_1,\tau_2,x,t)\, w_2(\tau_1,\tau_2,x,t)\, dx. \end{aligned}$$

If we suppose that $\beta,\sigma$ and $v$ do not depend on $x$, integrating with respect to $x$, we obtain

$$\partial_{a,a',b,t} W_{21}(a,a',b,t) = -\Lambda_{21}(a,a',b,t)\, W_{21}(a,a',b,t) + \sum_{j\in ??} R_j(a,a',b,t)$$

with the initial condition $W_{21}(a,a',0,t) = \beta^m(a,a',t)\, W_2(a,a',t)$.

If we suppose that $\beta,\sigma$ and $v$ depend only on $t$ we obtain the ODE problem

$$\dot{W}^\star_{21}(t) = -\widetilde{\Lambda}_{21}(t)\, W^\star_{21}(t) + \beta^m(t)\, W^\star_2(t) + \sum_{j\in J} R^\star_j(t).$$

Reasoning in the same way, we easily obtain the equations which characterize the dynamic of a couple with a female children:

$$\partial_{a,a',c,x,t} w_{20}(a,a',c,x,t) = -\Lambda_{20}(a,a',c,x,t)\, w_{20}(a,a',c,x,t) - \sum_{j\in J_{20}} R_j(a,a',c,x,t)$$

where

$$J_{20} = \{6, 7, 18, 24, 27, 30\}$$

and with the boundary conditions

$$\begin{array}{ll} w_{20}(a, a', c, x, t) & = 0 \text{ if } \min\{a, a'\} \leq x \text{ (and thus } w_{20} = 0 \text{ if } a \text{ or } a' \text{ are zero);} \\ w_{20}(\tau_1, \tau_2, c, 0, t) & = \sum_{j \in J'_{20}} R_j(\tau_1, c, \tau_2, t), \quad J'_{20} = \{3, 4, 7, 9, 12, 13, 16, 17\} \\ w_{20}(\tau_1, \tau_2, 0, 0, t) & = \int_{\mathbb{R}_+} \beta^f(\tau_1, \tau_2, x, t) \, w_2(\tau_1, \tau_2, x, t) \, dx \end{array}$$

If we suppose that $\beta, \sigma$ and $v$ do not depend on $x$, integrating with respect to $x$, we obtain

$$\partial_{a,a',c,t} W_{20}(a, a', c, t) = -\Lambda_{20}(a, a', c, t) \, W_{20}(a, a', c, t) + \sum_{j \in J''_{20}} R_j(a, a', c, t)$$

with the initial condition $W_{20}(a, a', 0, t) = \beta^f(a, a', t) \, W_2(a, a', t)$.

If we suppose that $\beta, \sigma$ and $v$ depend only on $t$ we obtain the ODE problem

$$\dot{W}^\star_{20}(t) = \beta^f(t) \, W^\star_2(t) + \sum_{j \in ??} R^\star_j(t) - \widetilde{\Lambda}_{20}(t) \, W^\star_{20}(t).$$

## 4.2 Family with male parent and male children

The dynamics is:

$$\partial_{a,b,x,t} w_{11}(a, b, x, t) = -\Lambda_{11}(a, b, x, t) \, w_{11}(a, b, x, t) - \sum_{j \in J_{11}} R_j(a, b, x, t)$$

where

$$\Lambda_{11}(a, b, x, t) = \mu^m(a, t) + \mu^m(b, t) + v_{11}(a, b, x, t)$$
$$J_{11} = \{1, 6, 10, 11, 14, 16, 22, 27, 28, 29\}$$

and with the boundary conditions

$$\begin{array}{l} w_{11}(a, b, x, t) = 0 \text{ if } \min\{a, b\} \leq x \text{ ( in particular, } w_{11} = 0 \text{ if } a \text{ or } b \text{ are zero);} \\ w_{11}(\tau_1, b, 0, t) = \int_{\mathbb{R}^2_+} \left[\mu^f(a', t) + \sigma^f_{21}(\tau_1, a', b, x, t)\right] w_{21}(\tau_1, a', b, x, t) \, da' dx. \end{array}$$

If we suppose that $\beta, \sigma$ and $v$ do not depend on $x$, integrating with respect to $x$, we obtain

$$\begin{array}{ll} \partial_{a,b,t} W_{11}(a, b, t) & = \int_{\mathbb{R}_+} \left[\mu^f(a', t) + \sigma^f_{21}(a, a', b, t)\right] W_{21} da' + \\ & -\Lambda_{11}(a, b, t) \, W_{11}(a, b, t) - \sum_{j \in J_{11}} R_j(a, b, t). \end{array}$$

If we suppose that $\beta, \sigma$ and $v$ depend only on $t$ we obtain the ODE problem

$$\dot{W}_{11}^{\star}(t) = \left[\mu^f(t) + \sigma_{21}^f(t)\right] W_{21}^{\star}(t) - \widetilde{\Lambda}_{11}(t)\, W_{11}^{\star}(t) - \sum_{j\in J_{11}} R_j^{\star}(t)\,.$$

## 4.3 Family with male parent and female children

The dynamics is:

$$\partial_{a,c,x,t} w_{10}(a,c,x,t) = -\Lambda_{10}(a,c,x,t)\, w_{10}(a,c,x,t) - \sum_{j\in J_{10}} R_j(a,c,x,t)$$

where

$$\Lambda_{10}(a,c,x,t) = \mu^m(a,t) + \mu^f(c,t) + \nu_{10}(a,c,x,t)$$
$$J_{10} = \{3,7,10,12,13,20,25,28,31\}$$

and with the boundary conditions

$$w_{10}(a,c,x,t) = 0 \text{ if } \min\{a,c\} \leq x \text{ (in particular, } w_{10} = 0 \text{ if } a \text{ or } c \text{ are zero);}$$
$$w_{10}(\tau_1,c,0,t) = \int_{\mathbb{R}_+^2} \left[\mu^f(a',t) + \sigma_{20}^f(\tau_1,a',c,x,t)\right] w_{20}(\tau_1,a',c,x,t)\, da'dx.$$

If we suppose that $\beta, \sigma$ and $v$ do not depend on $x$, integrating with respect to $x$, we obtain

$$\begin{aligned}\partial_{a,c,t} W_{10}(a,c,t) &= \textstyle\int_{\mathbb{R}_+} \left[\mu^f(a',t) + \sigma_{20}^f(a,a',c,t)\right] W_{20}(a,a',c,t)\, da' + \\ &\quad -\Lambda_{10}(a,c,t)\, W_{10}(a,c,t) - \textstyle\sum_{j\in J_{10}} R_j(a,c,t)\,.\end{aligned}$$

If we suppose that $\beta, \sigma$ and $v$ depend only on $t$ we obtain the ODE problem

$$\dot{W}_{10}^{\star}(t) = \left[\mu^f(t) + \sigma_{20}^f(t)\right] W_{20}^{\star}(t) - \widetilde{\Lambda}_{10}(t)\, W_{10}^{\star}(t) - \sum_{j\in J_{10}} R_j^{\star}(t)\,.$$

## 4.4 Family with female parent and male children

The dynamics is:

$$\partial_{a',b,x,t} w_{01}(a',b,x,t) = -\Lambda_{01}(a',b,x,t)\, w_{01}(a',b,x,t) - \sum_{j\in J_{01}} R_j(a',b,x,t)$$

where

$$\Lambda_{01}(a',b,x,t) = \mu^f(a',t) + \mu^m(b,t) + \nu_{01}(a',b,x,t)$$

$$J_{01} = \{2, 8, 14, 15, 17, 23, 30, 31, 32\}$$

with the boundary conditions

$$w_{01}(a', b, x, t) = 0 \text{ if } \min\{a', b\} \leq x \text{ (in particular, } w_{01} = 0 \text{ if } a \text{ or } b \text{ are zero);}$$
$$w_{01}(\tau, b, 0, t) = \int_{\mathbb{R}_+^2} [\mu^m(a,t) + \sigma_{21}^m(a, \tau, b, x, t)]\, w_{21}(a, \tau, b, x, t)\, da dx.$$

If we suppose that $\beta, \sigma$ and $v$ do not depend on $x$, integrating with respect to $x$, we obtain

$$\begin{aligned}\partial_{a',b,t} W_{01}(a', b, t) &= \int_{\mathbb{R}_+} [\mu^m(a,t) + \sigma_{21}^m(a, a', b, t)]\, W_{21}(a, a', b, t)\, da + \\ &\quad - \Lambda_{01}(a', b, t)\, W_{01}(a', b, t) - \sum_{j \in J_{01}} R_j(a', b, t).\end{aligned}$$

If we suppose that $\beta, \sigma$ and $v$ depend only on $t$ we obtain the ODE problem

$$\dot{W}_{01}^{\star}(t) = [\mu^m(t) + \sigma_{21}^m(t)]\, W_{21}^{\star}(t) - \widetilde{\Lambda}_{01}(t)\, W_{01}^{\star}(t) - \sum_{j \in J_{01}} R_j^{\star}(t).$$

## 4.5 Family with female parent and female children

The dynamics is:

$$\partial_{a',c,x,t} w_{00}(a', c, x, t) = -\Lambda_{00}(a', c, x, t)\, w_{00}(a', c, x, t) - \sum_{j \in J_{00}} R_j(a', c, x, t)$$

where

$$\Lambda_{00}(a', c, x, t) = \mu^f(a', t) + \mu^f(c, t) + \nu_{00}(a', c, x, t)$$
$$J_{00} = \{4, 9, 11, 13, 16, 17, 21, 26, 29, 32\}$$

with the boundary conditions

$$w_{00}(a', c, x, t) = 0 \text{ if } \min\{a', c\} \leq x \text{ (in particular, } w_{00} = 0 \text{ if } a' \text{ or } c \text{ are zero);}$$
$$w_{00}(\tau, c, 0, t) = \int_{\mathbb{R}_+^2} [\mu^m(a,t) + \sigma_{20}^m(a, \tau, c, x, t)]\, w_{20}(a, \tau, c, x, t)\, da dx.$$

If we suppose that $\beta, \sigma$ and $v$ do not depend on $x$, integrating with respect to $x$, we obtain

$$\begin{aligned}\partial_{a',c,t} W_{00}(a', c, t) &= \int_{\mathbb{R}_+} [\mu^m(a,t) + \sigma_{20}^m(a, a', c, t)]\, W_{20}(a, a', c, x, t)\, da + \\ &\quad - \left[\mu^f(c,t) + \mu^f(a',t) + \nu_{00}(a', c, t)\right] W_{00}(a', c, t) - \sum_{j \in J_{00}} R_j(a', c, t).\end{aligned}$$

If we suppose that $\beta, \sigma$ and $v$ depend only on $t$ we obtain the ODE problem

$$\dot{W}_{00}^{\star}(t) = [\mu^m(t) + \sigma_{20}^m(t)] W_{20}^{\star}(t) - \widetilde{\Lambda}_{00}(t) W_{00}^{\star} - \sum_{j \in J_{00}} R_j^{\star}(t).$$

## 4.6 Male single

The dynamics is:

$$\partial_{A,x,t} m(A,x,t) = -\mu^m(A,t) m(A,x,t) - \sum_{j \in J_m} R_j(A,x,t)$$

where

$$J_m = \{2, 4, 5, 18, 20, 21\}$$

with the boundary conditions

$$\begin{aligned}
& m(A,x,t) = 0 \text{ if } A \leq x; \\
m(\tau,0,t) &= \int_{\mathbb{R}_+^2} [\mu^m(b,t) + v_{11}(a,b,x,t)] w_{11}(\tau,b,x,t)\, dxdb + \\
&+ \int_{\mathbb{R}_+^2} [\mu^m(a,t) + \nu_{11}(a,\tau,x,t)] w_{11}(a,\tau,x,t)\, dxda + \\
&+ \int_{\mathbb{R}_+^2} \left[\mu^f(a',t) + \nu_{01}(a',\tau,x,t)\right] w_{01}(a',\tau,x,t)\, dxda' + \\
&+ \int_{\mathbb{R}_+^2} \left[\mu^f(c,t) + \nu_{10}(\tau,c,x,t)\right] w_{10}(\tau,c,x,t)\, dxdc + \\
&+ \int_{\mathbb{R}_+^3} \nu_{21}(a,a',\tau,x,t) w_{21}(a,a',\tau,x,t)\, dada'dx + \\
&+ \int_{\mathbb{R}_+^2} \left[\sigma_2(\tau,a',x,t) + \mu^f(a',t)\right] w_2(\tau,a',x,t)\, dxda' + \\
&+ \int_{\mathbb{R}_+^3} \sigma_{21}^m(\tau,a',b,x,t) w_{21}(\tau,a',b,x,t)\, dbda'dx + \\
&+ \int_{\mathbb{R}_+^3} \sigma_{20}^m(\tau,a',c,x,t) w_{20}(\tau,a',c,x,t)\, dcda'dx.
\end{aligned}$$

If we suppose that $\beta, \sigma$ and $v$ do not depend on $x$, integrating with respect to $x$, we obtain

$$\begin{aligned}
\partial_{A,t} M(A,t) &= \int_{\mathbb{R}_+} [\mu^m(b,t) + v_{11}(a,b,t)] W_{11}(a,b,t)\, db + \\
&+ \int_{\mathbb{R}_+} [\mu^m(a,t) + v_{11}(a,b,t)] W_{11}(a,b,t)\, da +
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}_+}\left[\mu^f(a',t)+v_{01}(a',b,t)\right]W_{01}(a',b,t)\,da'+\\
&+\int_{\mathbb{R}_+}\left[\mu^f(c,t)+v_{10}(a,c,t)\right]W_{10}(a,c,t)\,dc+\\
&+\int_{\mathbb{R}_+^2}v_{21}(a,a',b,t)\,W_{21}(a,a',b,t)\,dada'+\\
&+\int_{\mathbb{R}_+}\left[\sigma_2(a,a',t)+\mu^f(a',t)\right]W_2(a,a',t)\,da'+\\
&+\int_{\mathbb{R}_+^2}\sigma_{21}^m(a,a',b,t)\,W_{21}(a,a',b,t)\,dbda'+\\
&+\int_{\mathbb{R}_+^2}\sigma_{20}^m(a,a',c,t)\,W_{20}(a,a',c,t)\,dcda'+\\
&-\mu^m(A,t)\,M(A,t)-\sum_{j\in J_m}R_j(A,t)\,.
\end{aligned}
$$

If we suppose that $\beta,\sigma$ and $v$ depend only on $t$ we obtain the ODE problem

$$
\begin{aligned}
\dot{M}^\star(t) &= 2\left[\mu^m(t)+v_{11}(t)\right]W_{11}^\star(t)+\left[\mu^f(t)+v_{01}(t)\right]W_{01}^\star(t)+\left[\mu^f(t)+v_{10}(t)\right]W_{10}^\star(t)+\\
&\quad+\left[v_{21}(t)+\sigma_{21}^m(t)\right]W_{21}^\star(t)+\left[\sigma_2(t)+\mu^f(t)\right]W_2^\star(t)+\sigma_{20}^m(t)\,W_{20}^\star(t)+\\
&\quad-\mu^m(t)\,M^\star(t)-\sum_{j\in J_m}R_j^\star(t)\,.
\end{aligned}
$$

## 4.7 Female single

The dynamics is:

$$
\partial_{A',x,t}f(A',x,t) = -\mu^f(A',t)\,f(A',x,t)-\sum_{j\in J_f}R_j(A',x,t)
$$

where

$$
J_f=\{1,3,5,19,22,23\}
$$

with the boundary conditions:

$f(A',x,t)=0$ if $A'\leq x$;

$$
\begin{aligned}
f(\tau,0,t) = & \int_{\mathbb{R}_+^2} \left[\mu^m(b,t)+\nu_{01}(b,t)\right] w_{01}(\tau,b,x,t)\, dxdb+ \\
& + \int_{\mathbb{R}_+^2} \left[\mu^f(a',t)+v_{00}(a',\tau,x,t)\right] w_{00}(a',\tau,x,t)\, dxda'+ \\
& + \int_{\mathbb{R}_+^2} \left[\mu^f(c,t)+v_{00}(\tau,c,x,t)\right] w_{00}(\tau,c,x,t)\, dxdc+ \\
& + \int_{\mathbb{R}_+^2} \left[\nu_{10}(a,\tau,x,t)+\mu^m(a,t)\right] w_{10}(a,\tau,x,t)\, dxda+ \\
& + \int_{\mathbb{R}_+^3} \nu_{20}(a,a',\tau,x,t)\, w_{20}(a,a',\tau,x,t)\, dada'dx+ \\
& + \int_{\mathbb{R}_+^2} \left[\sigma_2(a,\tau,x,t)+\mu^m(a,t)\right] w_2(a,\tau,x,t)\, dxda+ \\
& + \int_{\mathbb{R}_+^3} \sigma_{21}^f(a,\tau,b,x,t)\, w_{21}(a,\tau,b,x,t)\, dbdadx+ \\
& + \int_{\mathbb{R}_+^3} \sigma_{20}^f(a,\tau,c,x,t)\, w_{20}(a,\tau,c,x,t)\, dcdadx.
\end{aligned}
$$

If we suppose that $\beta, \sigma$ and $v$ do not depend on $x$, integrating with respect to $x$, we obtain

$$
\begin{aligned}
\partial_{A',t} F(A',t) = & \int_{\mathbb{R}_+} \left[\mu^m(b,t)+v_{01}(a',b,t)\right] W_{01}(a',b,t)\, db + \\
& + \int_{\mathbb{R}_+} \left[\mu^f(a',t)+v_{00}(a',c,t)\right] W_{00}(a',c,t)\, da' + \\
& + \int_{\mathbb{R}_+} \left[\mu^f(c,t)+v_{00}(a',c,t)\right] W_{00}(a',c,t)\, dc + \\
& + \int_{\mathbb{R}_+} \left[\mu^m(a,t)+v_{10}(a,c,t)\right] W_{10}(a,c,t)\, da + \\
& + \int_{\mathbb{R}_+^2} v_{20}(a,a',c,t)\, W_{20}(a,a',c,t)\, dada' + \\
& + \int_{\mathbb{R}_+} \left[\sigma_2(a,a',t)+\mu^m(a,t)\right] W_2(a,a',t)\, da + \\
& + \int_{\mathbb{R}_+^2} \sigma_{21}^m(a,a',b,t)\, W_{21}(a,a',b,t)\, dbda + \\
& + \int_{\mathbb{R}_+^2} \sigma_{20}^f(a,a',c,t)\, W_{20}(a,a',c,t)\, dcda + \\
& - \mu^f(A',t)\, F(A',t) - \sum_{j\in J_f} R_j(A',t).
\end{aligned}
$$

If we suppose that $\beta, \sigma$ and $v$ depend only on $t$ we obtain the ODE problem

$$\begin{aligned}\dot{F}^{\star}(t) &= \left[\mu^{m}(t)+v_{01}(t)\right]W_{01}^{\star}(t)+2\left[\mu^{f}(t)+v_{00}(t)\right]W_{00}^{\star}(t)+\left[\mu^{m}(t)+v_{10}(t)\right]W_{10}^{\star}(t)+\\&+\left[v_{20}(t)+\sigma_{20}^{f}(t)\right]W_{20}^{\star}(t)+\left[\sigma_{2}(t)+\mu^{m}(t)\right]W_{2}^{\star}(t)+\sigma_{21}^{m}(t)W_{21}^{\star}(t)+\\&-\mu^{f}(t)F^{\star}(t)-\sum_{j\in J_f}R_{j}^{\star}(t).\end{aligned}$$

# 5 Conclusions

In this paper a general dynamical multistate model of family dynamics for nuclear "one-child" family has been developed. The model is to be considered an extension of the traditional pair formation models of traditional mathematical demography, in that, as in pair formation models, the unique nonlinearity lies in the process of pair formation. In this paper we have only considered backbone and have disregarded the problems connected with the mixing restraints (inherited from the two-sex problem) and the family restraints (typical of every household problem). The complexity of the family problem even in the simlest case of "one-child" families suggests that our model should be considered as a starting point from which to formulate simplified sub-problems (for instance by disregarding age,or one sex, or by simplifying the demographic patterns permitted) which could potentially be capable of shed some light on the dynamics of family formation and dissolution). This aspect will be the object of our forthcoming work.

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