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ABSTRACT

Population heterogeneities are a well acknowledged determinant of observed population facts. Here the dynamical role played by fertility heterogeneity between employed and unemployed individuals in an age structured demo-economic model à la Goodwin is considered. The employed individuals are assumed to obey a classical malthusian fertility schedule, whereas the fertility of the unemployed is assumed to be fully exogenous. Several noteworthy dynamical feature arise. First, even in the simplest case, i.e. the symmetric case in which the fertility of unemployed is identical to that of the employed at zero wage, the dynamical interaction of heterogeneous fertility with age structure may lead to sustained oscillations. Moreover, when the symmetry is broken down, and the fertility of the unemployed is allowed to exceed that of the employed at least at low levels of the wage, robust chaotic phenomena may appear.

1 Introduction

In recent times there has been an increasing interest for the mechanisms governing demo-economic interactions. Several contributions (Lee (1974), Samuelson (1976), Frauenthal and

Swick (1983). Feichtinger and Sorger (1989, 1990). Feichtinger and Dockner (1990). Chu Lu (1995)) have been devoted to provide sound mathematical representation of the popular "Easterlin effect". Other areas attracting the interest of modelers are the problem of the escape from the malthusian trap (Komlos and Artzrouni 1990. Steinmann et al. 1998), and the interaction between population. the economy and the environment (see Milik and Prskawetz (1996). Prskawetz et al. (1999), and references therein). Despite their different specific issues, all these contributions share some common features. one of the main of which is the aim to discover sound demo-economic forces capable to generate sustained oscillations and complex dynamics.

Also this paper is aimed to the investigation of demo-economic mechanisms leading to persistent oscillations and chaos. but from a different perspective, starting from the classical Goodwin's (1967) growth cycle model. The Goodwin's model represents a rich framework embedding the overall macroeconomic structure. Moreover, as recently pointed out (Chiarella and Flaschel 1999), the Goodwin's model has some general features which make it a constitutive building block for highly realistic macro-economic models. In our previous work (Manfredi and Fanti 1999a,b) we have started a systematic investigation of the effects of a fully endogenous supply of labour within Goodwin's model. To this end we developed a general demographic framework considering a simplified "mechanistic" representation of age structure. which appears a suitable and parsimonious tool for embedding population dynamics within macro-economic systems (a similar framework is used in Feichtinger and Sorger 1989). We have been able to show (Manfredi and Fanti 1999b) that in presence of "minimal age structure ingredients" the existence of malthusian fertility has purely destabilising effects on Goodwin's model. and moreover that persistent oscillations may appear when "reasonable" participation behaviour are assumed.

A possible weakness in the previous results lies in the fact that Goodwin's model allows. as a fundamental ingredient. the existence of structural unemployment. Hence. the fact to postulate. as done in our previous work. a unique overall fertility schedule for both employed and unemployed may appear unrealistic. In this paper we continue our investigation by postulating heterogeneity in the fertility of employeds and unemployeds. The role of heterogeneities is well ackowledged in demography (for instance Keyfitz 1985). Recently, studies on the transition to adulthood in some western countries (Italy for instance. see Billari 1998). where most of births take place within the marriage. and where to find a (first) regular job is often a necessary condition to get marry. reveal the delicate impact on fertility played by long delays spent in seeking the first job. Obviously Becker's work (Becker 1960) gives support to the possibility that the individuals who are employed may have a lower fertility compared to unemployed. In a Goodwin-type context the possible effects of fertility heterogeneity are intringuing: as shown in section two of this paper. even in the simplest case. without age structure. heterogeneity may have both stabilising and destabilising effects on the Goodwin's equilibrium. In particular. stabilising effects arise when the fertility of those who are employed exceeds that of the unemployed. and vice-versa.

In the age structured case things are more complicated. Here. as a starting point we assume that malthusian behaviour (fertility is an increasing function of the standard of living) is only relevant for those who actually receive a wage (the employed): in other terms we assume that whereas the fertility of the employed is wage dependent. that of the unemployed is fully exogenous. To be honest we consider in depth only the simplest case. namely the case in which the fertility of the employed at very low wage levels is identical to that of the unemployed. This assumption appears to be a very basic step toward a "realistic" formulation of fertility in presence of structural unemployment. In absence (as far as we know) of stylised empirical results on the fertility patterns of employed and unemployed, simple assumptions as the present one may be useful in order to get some insight on their possible dynamical effects.

Several interesting dynamical results appear. First of all, the main result obtained in our previous work, the destabilising role played by age structure in presence of malthusian fertility, becomes more complex when a realistic fertility schedule is considered. In particular, when the correct heterogeneous model has been specified, the dynamical interaction between heterogeneous fertility and age structure appears to be "essentially" stabilising. We say "essentially" because the model shows that stabilisation fails when the recruitment into the labour force is strongly delayed (as it often occurs in developed economies). This fact shows that, in the final analysis, age structure, and in particular the process of recruitment into the labour force (here given by process of transition to adulthood) is a main source of instability in the demo-economic environment. In turn, the rate of transition to adulthhod appears to be the main critical parameter. Once the model has been destabilised, stable sustained oscillations appear via Hopf bifurcation of the positive equilibrium. This latter result is of interest because it has been obtained via a "minimal" set of dynamical ingredients: the Lotka-Volterra-Goodwin model plus a simple demographic equation. In particular oscillations (and instability) appear when the (average) age of transition to adulthhood exceeds a critical threshold. This result therefore gives clear prescriptions for what concerns demo-economic stabilization policies.

Finally, once one allows the fertility of the unemployed to exceed that of the employed at very low wages (for instance due to the existence of high unemployment benefits) then the model can display chaotic potential as well. The appearance of chaos seems to be a consequence of the instability window which appears as soon as we allow an "uncontrolled" fertility of the unemployed. In this paper we report some preliminary simulation results on the chaotic behaviour of our model which will be investigated more deeply in subsequent work. The richness of results displayed by the model, despite the minimal set of dynamical ingredients, shows that it can become a useful prescriptive tool, once adequately "operationalised".

The present paper is organised as follows. Section two is pedagogical: it first recalls the main features of the classical Goodwin's model, and subsequently discusses the effects of differential fertility between employed and unemployed within the basic Goodwin's framework. In section three we introduce a general age-structured Goodwin-type model with fertility heterogeneity and study its main static and dynamical properties with special reference to the generation of persistent oscllations. Section four deals with chaotic behaviour. Conclusive remarks and directions for future work follow in section five.

2 The Goodwin's growth-cycle model and heterogeneity fertility

2.1 The classical Goodwin's growth cycle

The Goodwin's model (Goodwin 1967), sometimes known as the Lotka-Volterra-Goodwin (LVG) model, is described by the system :

$$\begin{aligned} \frac{\dot{V}}{V} &= -(\alpha+\gamma)+\rho U \\ \frac{\dot{U}}{U} &= m(1-V)-(\alpha+n_s) \end{aligned} \qquad (1)$$

where $U = U(t)$ is the employment rate at time t, defined as the ratio between the labour force actually employed $L(t)$ and the supply of labour $N_s(t)$, while $V = V(t)$ is the distributive share of labour, given by the ratio $w(t)L(t)/Q(t)$, where w is the real wage and Q the total product. V can be expressed as: $V = w/A$ where A is the average productivity of labour. Moreover $\gamma > 0, \rho > 0$ are characteristic parameters of the labour market, $m > 0$ is the output-capital ratio, and $\alpha > 0, n_s > 0$ respectively denote the rate of change of the productivity of labour and of the labour supply. The assumptions underlying (1) are: i)the labour market is driven by the Phillips relation: $\dot{w} = w(-\gamma + \rho U)$ $(0 < \gamma < \rho)$; ii) the accumulation rules are such that: a)the wage earners do not save, b)profits are entirely reinvested, c)the technology is Leontief-type, d)the capital output ratio $K/Q = m^{-1}$ is constant. Thanks to assumptions a)....d) the rate of growth of the output $g = \dot{Q}/Q$ obeys: $g = m(1-V) > 0$; iii) the supply of labour and the productivity of labour grow exogenously at the constant rates $n_s > 0$ and $\alpha > 0$.

Model (1) describes a typical Lotka-Volterra system in which the labour share acts as the predator of the employment (the prey). In particular, when $m < \alpha + n_s$ the system has as a unique equilibrium the zero equilibrium $E_0 = (0,0)$ which is globally asymptotically stable (GAS)[1]. Viceversa, provided $m > \alpha + n_s$, system (1) exhibits the traditional Lotka-Volterra conservative oscillations around the positive equilibrium E_1 of coordinates: $U_1 = (\alpha+\gamma)/\rho$; $V_1 = (m-\alpha-n_s)/m$ (notice that E_1 is economically meaningful provided $\rho > \alpha+\gamma$). The equilibrium values U_1, V_1 are the average values of (U, V) during the fluctuation period.

The inequality $m > \alpha+n_s$ provides a threshold (Manfredi and Fanti 1999b) which at the same time governs whether: i)the E_0 equilibrium may be unstable, therefore providing the conditions for the economic "take off" which is a necessary "precondition" in a process of economic growth; ii)the positive equilibrium E_1 exists and is locally stable. In brief: the inequality $m > \alpha + n_s$ governs the stability switches between E_0 and E_1, therefore permitting the establishment of the conditions for a "structured" economic activity. This suggests, borrowing from the demographic dictionary, the following definition:

DEFINITION 1. *Let* $m, \alpha, n_s > 0$. *The ratio:*

[1] The stability of the "zero" equilibrium corresponds to a situation in which accumulation is too weak t[illegible]mit the birth of a "structured" economic activity, as stated by a labour market plus a production structure.

$$R_E = \frac{m}{\alpha + n_s} \tag{2}$$

is the reproduction ratio of the economy.

The previous definition. similar to the classical demographic notion of net reproduction ratio (NRR) R_0, is suggested from the fact that, provided $\alpha + n_s > 0$, the growth of the economy is possible IFF $R_E > 1$. This definition is natural in Goodwin-type economies in which all the profit is reinvested in new labour. Notice that (still take $\alpha = 0$) the definition of R_E is trivial when $n_s < 0$ (in this case the economy is always "productive").

2.2 Differential fertility of employed and unemployed: basic facts

In model (1) the supply of labour N_s is fully exogenous. As a first step toward more realism N_s may be defined as the product between the total number of individuals in the working age span $N(t)$, and the participation rate $s(t)$: $N_s(t) = s(t)N(t)$. In this paper we disregard participation effects (considered in Manfredi and Fanti 1999b), by putting $s = 1$ (everybody in the work age span offers his/her labour force). Let us write $\dot{N}(t)/N(t) = n(t)$ for the rate of change of the population in working age span. Obviously $n(t) = b(t) - \mu(t)$ where $b(t), \mu(t)$ respectively denote the enrollment and the exit rate in the working age-span. As a first coarse approximation, one may assume that $b(t)$ is the birth rate (this amounts to postulate that the enrollment in the labour force occurs without delay, i.e. at birth, and it is to be intended just as a preliminary step) and μ the mortality rate of the population. By explicitly assuming that employed and unemployed do not have identical demographic behaviours we may write, using standard demographic arguments:

$$n(t) = (b_e(t) - d_e(t))\,U(t) + (b_u(t) - d_u(t))\,(1 - U(t)) = n_e(t)U(t) + n_u(t)\,(1 - U(t)) \tag{3}$$

where b_e, b_u, μ_e, μ_u respectively denote the fertility and mortality rates observed in the employed and unemployed population (so that n_eU and $n_u(1 - U)$ denote the contributions to the overall rate of change of the population due, respectively, to employed and unemployed individuals).

2.3 The case of wage-independent vital rates

As a first step we assume that the rates b_e, b_u, μ_e, μ_u are constant. By using (3) model (1) becomes:

$$\begin{aligned} \frac{\dot{V}(t)}{V(t)} &= -(\alpha + \gamma) + \rho U \\ \frac{\dot{U}(t)}{U(t)} &= (m - \alpha) - (n_e(t)U(t) + n_u(t)\,(1 - U(t))) - mV \end{aligned} \tag{4}$$

The employment equation may also be written as

$$\dot{U}(t) = U\,((m - \alpha - n_u) - (n_e - n_u)U(t) - mV) = \tag{5}$$

The model (4) is a quadratic Lotka-Volterra-Goodwin model. Here we only consider the case $m - \alpha - n_u > 0$ (the case $m - \alpha - n_u < 0$ seems less interesting as it needs an implausibly large value of n_u). The analysis of equilibria shows that (4) always has the zero equilibrium $E_0 = (0, 0)$. We then have two distinct situations depending on whether the quadratic term in the prey equation has a negative sign or not, i.e. depending on whether $n_e > n_u$ or $n_e < n_u$:

i) $n_e > n_u$. From the V equation we find $U = U_1 = (\alpha + \gamma) / \rho$. Let $n(U_1) = \bar{n}_1 = n_e U_1 + n_u (1 - U_1)$ denote the overall rate of growth of the supply of labour for $U = U_1$. If $m - \alpha - \bar{n}_1 > 0$, a strictly positive and economically meaningful equilibrium $E_1 = (U_1, V_1)$ exists with: $V_1 = (m - \alpha - \bar{n}_1) / m$. Moreover, for $V = 0$ we get: $U_2 = (m - \alpha - n_u) / (n_e - n_u)$. Hence, an axis equilibrium $E_2 = (U_2, 0)$ always exists, though it is economically meaningful ($U_2 < 1$) only for: $m - \alpha - n_u < n_e - n_u$, i.e. for $m - \alpha - n_e < 0$.

ii) $n_e < n_u$. In this case no axis equilibrium is possible. An economically meaningful positive equilibrium E_1 always exists as $V_1 = (m - \alpha - \bar{n}) / m$. is always positive.

A local stability analysis at E_1 gives

$$J_{(V,U)}(E_1) = \begin{pmatrix} 0 & \rho V \\ -mU & -(n_e - n_u)U \end{pmatrix} \tag{6}$$

Hence: $Tr(J(E_1)) = -(n_1 - n_2)U$; $Det(J(E_1)) = \rho V m U > 0$. As $Det(J) > 0$, when E_1 exists it is LAS as long as $n_e > n_u$ and vice-versa when $n_e > n_u$. This leads to the following

PROPOSITION 2. *in presence of fertility heterogeneity Goodwin's conservative oscillations disappear. In particular the positive equilibrium E_1 is stabilised for $n_1 > n_2$ and destabilised in the opposite case.*

In other words: in presence of demographic heterogeneity of employed and unemployed Goodwin's conservative oscillations are replaced by regimes of stability or instability[2], depending on whether the fertility rate of the employed individuals is greater or smaller of the fertility rate of the unemployed. This agrees with the traditional "policy" consideration that to avoid instability we should keep under control the demographic behaviour of the "poors" (here identified with the unemployed).

2.4 The effects of wage-related fertility

Here we enlarge (4) by assuming that the fertility of the employed individuals is wage-dependent: $b_e = b_e(w)$, and follows a "classical" (Day et al. 1989) malthusian relation: $b_e(0) \geq 0, b'(w) > 0$[3]. Our work assumption here is that the fertility of employed at zero wage is identical to the fertility of unemployed: $b_e(0) = b_u$.[4] By assuming that the mortality rate is identical for both employed and unemployed individual, the overall rate of change of the labour supply is:

[2] These regimes are essentially of oscillatory type.

[3] We could easily assume that the mortality rate as well is related to the standard of living, but this is a unnecessary complication.

[4] The "zero" wage could be interpreted here as the subsistence wage.

$$n(t) = b_e(w)U + b_u(1-U) - \mu = n(w,U) \tag{7}$$

The introduction of (7) in (1) would lead to an increase of the dimension of the system. To avoid this complication we assume that the productivity is constant over time ($\alpha = 0$) at a prescribed value A_0. Still for simplicity we rescale A_0 to one, so that w may be also interpreted as the wage share (so that in normal situations it should hold: $0 \le w \le 1$). In this manner we obtain the following system in the variables (w, U)

$$\begin{aligned} \dot{w} &= w(-\gamma + \rho U) = wf(U) \\ \dot{U} &= U[m(1-w) - n(w,U)] = Ug(w,U) \end{aligned} \tag{8}$$

It is convenient to write:

$$\dot{U} = U(m - n_u - mw - (b_e(w) - b_u)U) \tag{9}$$

where $n_u = b_u - \mu$. As before $m - n_u$ is not necessarily positive (it certainly does when $b_u - \mu < 0$) but negativity requires implausibly high levels of the fertility of the unemployed.

It is sometimes useful to speak in terms of reproduction ratios. Let us define (Manfredi and Fanti 1999b): $R_0^u = b_u/\mu$ =the Net Reproduction Rate (NRR) of those who are in the state of unemployed[5]; $R_0^e(w) = b_e(w)/\mu$=the NRR of the employed when the actual wage is w:$R_0(w,U) = [b_e(w)U + b_u(1-U)]/\mu = R_0^e(w)U + R_0^u(1-U)$ =the overall NRR when the actual wage and employment are respectively w, U: $R_0^E(w,U) = m/n(w,U)$ =the Net Reproduction Rate of the Economy when the wage and employment are respectively set to their current levels. Notice that the assumption $b_e(0) = b_u$ implies $b(0,U) = b_e(0)U + b_u(1-U) = b_u$, and so: $R_0(0,U) = R_0(0,0)$. In other words: when $b_e(0) = b_u$ the overall NRR at zero wage is always equal to $R_0(0,0)$.

2.4.1 The analysis of equilibria and their local stability

As usual $E_0 = (0,0)$ is an equilibrium. The jacobian evaluated at E_0 is

$$J(E_0) = \begin{pmatrix} -\gamma & 0 \\ 0 & m - n_u \end{pmatrix} \tag{10}$$

leading to a saddle point or a stable node depending on whether $m \gtrless n_u$. Hence, unless the fertility of the unemployed is implausibly high, the zero equilibrium is unstable (a saddle point). This fact reveals that the case $m \gtrless n_u \Leftrightarrow R_0^E(0,U) = R_0^E(0,0) > 1$ (whatever be U) is the most interesting one. Moreover in the case $b_e(0) = b_u$ no axis equilibria with zero wage are possible (In fact: $g(0,U) = m - (b_e(0)U + b_u - b_uU - \mu) = m - n_u > 0$). Finally, the system may have up to one strictly positive equilibrium $E_1 = (w_1, U_1)$, where $U_1 = \gamma/\rho$, while economically meaningful equilibrium value w_1 of the wage are solutions of

$$m(1-w) - (b_e(w)U_1 + b_u(1-U_1) - \mu) = m(1-w) - n(w, U_1) = 0$$

[5] To avoind misunderstanding: for the Net Reproduction Rate of the unemployed we simply mean the average number of offspring that a woman would have during her fertile life span when the whole fertile age span is spent in the state of unemployed (hence with a characteristic fertility rate b_u).

satisfying: $0 \leq w_1 \leq 1$. As: $m(1-w_1) > 0$, at equilibrium we need: $n(w_1, U_1) > 0$. Hence, if $n(w, U_1) < 0$ for all w, no positive equilibrium is possible. For $n(w, U_1) > 0$ (remember that $n(w, U_1)$ is an increasing function of the wage), then, if $n(0, U_1) > m$ (i.e. if $R_0^E(0, U_1) = R_0^E(0,0) < 1$) no positive equilibrium is possible. Vice-versa, if $n(0, U_1) < m$ (i.e.: $R_0^E(0, U_1) = R_0^E(0,0) > 1$), then a unique economically meaningful positive equilibrium exists provided that $n(1, U_1) > 0$.

The stability analysis of the positive equilibrium gives the jacobian

$$J(E_1) = \begin{pmatrix} 0 & \rho w_1 \\ -\left(m + U_1 b_e'(w_1)\right) U_1 & -U\left(b_e(w_1) - b_u\right) \end{pmatrix}$$

We have:

$$Tr(J(E_1)) = -U\left(b_e(w_1) - b_u\right) < 0\ ;\ Det(J(E_1)) = \rho\left(m + U_1 b_e'(w_1)\right) w_1 U_1 > 0$$

showing that the positive equilibrium, provided it exists, is always LAS. Let us summarise our findings by the following:

PROPOSITION 3. *When $b_e(0) = b_u$ the system (7) always admits the zero equilibrium $E_0 = (0,0)$, which is a saddle point or a stable node depending on whether: $m \gtrless (b_u - \mu) = n_u \Leftrightarrow R_0^E(0, U) = R_0^E(0,0) \gtrless 1$. Hence, unless the fertility of the unemployed is implausibly high, the zero equilibrium is unstable and the condition for the economic take-off is met. Moreover, as long as $R_0^E(0,0) < 1$ (i.e. as long as E_0 is stable) the system can not have positive equilibria. Vice-versa, when $R_0^E(0,0) > 1$, E_0 becomes unstable. At the same time a positive equilibrium E_1 appears, which is economically meaningful provided that $n(1, U_1) > 0$ (i.e. provided that influence of wage on fertility is capable to generate population growth at least when the wage is set up to its maximal value.). When it exists, the positive equilibrium E_1 is always LAS (it inherits the stability of E_0).*

REMARK 1. *It is useful to ground the latter result with the one in the previous sub-section in which fertility was wage-independent. In presence of malthusian fertility the assumption $b_e(0) = b_u$ implies that the fertility of the employed is always, necessarily, higher of that of unemployed individuals, thus preventing the instability factor caused by the higher fertility of the unemployed.*

3 Heterogeneous fertility and age structure

As a further step toward greater realism, in this section age structure is explicitly introduced in the basic framework of the previous section. Following Manfredi and Fanti (1999a,b) the population process is modelled by means of a simplified three-stage structure aimed to represent the most relevant stages of the individual life-cycle: pre-work ages ("young"), working age span ("adult"), retirement (let us call them stage 1, stage 2 and stage 3 for short). We assume that only adult individuals contribute to natality. Table 1 reports the description for the involved demographic parameters.

Parameter Name	Description
μ_1, μ_2, μ_3	mortality rates in stages 1,2,3
v_1	rate of transition from stage 1 to stage 2
	(rate of transition to adulthood)
v_2	rate of transition from stage 2 to stage 3
	(rate of retirement)
b	rate of fertility of adults

Table 1 Demographic parameters employed in the model

As the result of these assumptions, the rate of change $n(t)$ of the population in the working age span (details in appendix 1) follows the dynamical equation:

$$\dot{n} = (-1)\left[n^2 + Pn - B(w, U)\right] \tag{11}$$

where:

$$P = (\mu_1 + v_1) + (\mu_2 + v_2) > 0 ; \quad B(w, U) = v_1 b(w, U) - Q \ : \ Q = (\mu_1 - v_1)(\mu_2 + v_2) > 0 \tag{12}$$

and $b(w, U)$ denotes the fertility rate along the form already used in the previous section:

$$b = b(w, U) = b_e(w)U + b_u(1 - U) \tag{13}$$

where the function $b_e(w)$ is "malthusian". By embedding the formulation (11)-(12)-(13) within the model (1) we arrive to our final model, which is defined by the three state equations:

$$\begin{aligned} \dot{w} &= w(-\gamma + \rho U) \\ \dot{U} &= U(m(1 - w) - n) \\ \dot{n} &= (-1)\left[n^2 + Pn - B(w, U)\right] \end{aligned} \tag{14}$$

plus (13) and (12).

The system (14) defines a flexible demo-economic framework, in which the economic subsystem is influenced by the demographic one via the labour supply term $n = \dot{P}_2/P_2$, whereas several inputs are possible from the economic system to the demographic subsystem: virtually all the demographic parameters may be influenced by the living conditions and other economic factors. The potential applications of model (14) go beyond Goodwin-type economies: for instance it is directly applicable to the Solow's neoclassical growth model. In "ecological" terms (14) adds to the basic LVG predator-prey system a third "actor", the rate of change of the labour supply, which acts as a predator of employment. Hence (14) is a "two predators- one prey" system in which the two predators mutually experience a one-way symbiotic relation.

3.1 Analysis of the age structured model

Here we investigate the main properties of the model (14) - (13) - (8). Again, though we will not systematically use the language of reproduction ratios in our discussion, in view of a possible "operationalisation" of the model, it is useful to point out that most of the results of the model depend on a set of thresholds expressible as reproduction ratios. Let $R_0(w,U) = \frac{v_1 b(w,U)}{Q} = \frac{v_1 b_e(w)}{Q}U + \frac{v_1 b_u}{Q}(1-U)$ the overall net reproduction rate (NRR) prevailing in the population when the actual wage and employment are respectively (w,U), where $R_0^1(w) = \frac{v_1 b_e(w)}{Q} = \frac{v_1 b_e(w)}{(\mu_2+v_2)(\mu_1+v_1)}$ is the NRR of the employed population at wage w, and $R_0^2 = \frac{v_1 b_u}{Q} = \frac{v_1 b_u}{(\mu_2+v_2)(\mu_1+v_1)}$ the NRR of the unemployed population.

3.1.1 The equilibria

The analysis of equilibria shows that the system (14) always has an equilibrium $E_0 = (0,0,n_0)$. For $U = w = 0$ we get:

$$n^2 + Pn - B(0,0) = 0 \tag{15}$$

where: $B(0,0) = v_1 b_u - Q$. Therefore we have (notice that only the positive solution of the quadratic is acceptable):

$$n_0 = \frac{1}{2}\left(-P + \sqrt{P^2 + 4B(0,0)}\right)$$

It holds $n_0 \gtrless 0$ depending on whether $B(0,0) \gtrless 0$, which in turn means: $v_1 b_u - Q > 0 \rightarrow R_0^u \gtrless 1$, where R_0^u is the NRR of the unemployed part of the population. The Jacobian of (14) evaluated at E_0 has the eigenvalues:

$$\lambda_{1,2,3} = (-\gamma, m - n_0, -(2n + P))$$

Hence as in the basic case without age structure, stability and instability of E_0 essentially depend on the mutual magnitude of the accumulation rate m and the rate of growth of the population n_0 at zero wage and employment. When n_0 is negative (i.e. when: $R_0^u < 1$) then E_0 is always (trivially) unstable as a consequence of population decay (as in Goodwin's model). When n_0 is positive (i.e. when: $R_0^u > 1$) nontrivial instability appears when $m - n_0 > 0$.

For what concerns other equilibria, it is possible to show that as long as $b_e(0) = b_u$ the system (14) has no axis equilibria (a feature often present in LVG models) and admits at most one strictly nonzero equilibrium $E_1 = (w_1, U_1, n_1)$. To see this latter point, from the w equation we get: $U = U_1 = \gamma/\rho$. Hence we have the system:

$$m(1-w) - n = 0 \quad ; \quad n^2 + Pn - B(w,U_1) = 0$$

the nonzero solutions ' which are the intersections between the curves:

$$n_A(w) = m(1-w) \quad ; \quad n_B(w) = \frac{1}{2}\left(-P + \sqrt{P^2 + 4B(w,U_1)}\right)$$

where: $B(w, U_1)$ is an increasing function of the wage. The two intercepts at zero wage are:

$$n_A(0) = m \quad : \quad n_B(0) = \frac{1}{2}\left(-P + \sqrt{P^2 + 4B(0, U_1)}\right) = \frac{1}{2}\left(-P + \sqrt{P^2 + 4B(0, 0)}\right)$$

But: $B(0, U_1) = B(0, 0)$ so that: $n_B(0) > 0 \leftrightarrow R_0^u > 1$. Hence: A) when $n_B(0) > 0 \leftrightarrow R_0^2 > 1$ then there is one (and only one) economically meaningful positive equilibrium provided that:

$$n_B(0) < n_A(0) \quad \Leftrightarrow \quad n_B(0) < m$$

Hence E_1 certainly exists in the parameter region in which E_0 is unstable. Moreover: B) when $n_B(0) < 0 \leftrightarrow R_0^2 < 1$ then, as $n_A(w)$ is a decreasing function of w while n_B is an increasing function of w, there is certainly a unique nonzero equilibrium E_1. The equilibrium value of the wage is economically meaningful ($0 \le w_1 \le 1$) provided that $n_B(1) > 0$, i.e.

$$B(1, U_1) = v_1\left(b_1(1)U_1 + b_2(1 - U_1)\right) - Q > 0$$

or, finally: $R_0(1, U_1) > 1$. Vice-versa, if $n_B(1) < 0$, the two nullclines intersect for a wage greater than one, and this is not economically meaningful (it is the analog of the case $n < 0$ in the basic Goodwin's model which was not interesting as it described an unstable, i.e. purely explosive, economy).

Summarizing we have the following proposition on the existence of equilibria which extends, *mutatis mutandis* (i.e. taking into account the effects of age structure) the corresponding proposition used in section 3:

PROPOSITION 1. *When the fertility of employed individuals at very low wages is identical to the fertility of unemployed, the system (14) always admits the zero equilibrium $E_0 = (0, 0)$, which is a saddle point or a stable node depending on whether: $m \gtrless n_0$. Hence, unless the fertility of the unemployed is implausibly high, the zero equilibrium is unstable and the condition for the economic take-off is met. Moreover as long as E_0 is stable the system can not have positive equilibria. Vice-versa, when $m > n_0$, E_0 becomes unstable. At the same time a positive equilibrium E_1 appears, which is economically meaningful provided that $n(1, U_1) > 0$, i.e. provided that influence of wage on fertility is capable to generate population growth at least when the wage is set up to its maximal value.).*

3.1.2 Stability analysis of the positive equilibrium: the case of linear fertility

For simplicity, the fertility schedule of the employed is now taken as linear: $b_e(w) = c_0 + c_1 w$ $c_0 \ge 0$, $c_1 > 0$. We point out that our conclusions are unaffected if we choose a general monotonically increasing (possibly saturating) function of the wage. As in this paper we assume $c_0 = b_u$, we have

$$b_e(w) = b_u + c_1 w \quad b_u \ge 0, \; c_1 > 0 \tag{16}$$

The positive equilibrium E_1, when it exists, has the explicit coordinates:

$$U_1 = \frac{\gamma}{\rho} \; : \; w_1 = \frac{m - n_1}{m} \; : \; n_1 = \frac{1}{2}\left(-\left(P + \frac{v_1 c_1}{m}U\right) + \sqrt{\left(P + \frac{v_1 c_1}{m}U\right)^2 + 4\left(v_1 b(1, U_1) - Q\right)}\right) \tag{17}$$

The following facts hold (see appendix 2):

$$\frac{\partial n_1}{\partial b_2} > 0 \quad \frac{\partial n_1}{\partial c_1} > 0 \quad \frac{\partial n_1}{\partial U} > 0 \quad \frac{\partial n_1}{\partial v_1} > 0 \tag{18}$$

The stability analysis of E_1 leads to the jacobian:

$$J_{w,U,n}(E_1) = \begin{pmatrix} 0 & \rho w & 0 \\ -mU & 0 & -U \\ v_1 U c_1 & v_1 c_1 w & -(2n+P) \end{pmatrix}$$

where all the quantities (w, U, n) are intended to be evaluated at E_1. From the Routh-Hurwicz theorem E_1 is LAS (see the appendix 3) provided that the following condition holds

$$2n_1 + P - \gamma > 0 \tag{19}$$

The latter inequality shows that, as opposite to section 2.3 (where under the present assumptions the system was always LAS in absence of age structure), stability does not always hold. This motivates the investigation of the stability condition by using v_1 (the reciprocal of the mean age at entrance into the labour force) as the "pivot" parameter. In fact v_1 is the parameter that tunes the effect of the recruitment into the labour force, which is the true source of instability. Define

$$H(v_1) = 2n(v_1) + P(v_1) \tag{20}$$

As shown in the appendix the function $H(v_1;..)$ is a strictly increasing function of v_1 with a strictly positive intercept $H(0) = |(\mu_1 + \mu_2) - v_2|$. The following proposition therefore holds (the proof is postponed in appendix 3):

PROPOSITION 2. *If $H(0) > \gamma$ then the E_1 equilibrium is always LAS. Vice-versa, when $H(0) < \gamma$ the E_1 equilibrium is LAS only for $v_1 > v_1^*$ where v_1^* is the unique solution of the equation $H(v_1) - \gamma = 0$. The region $0 < v_1 < v_1^*$ is an instability region. The point $v_1 = v_1^*$ is a Hopf bifurcation point for E_1.*

The previous proposition shows several interesting points. First, when γ is, in relative terms, small, stability always prevails. But $1/\gamma$ is a pure labour market parameter measuring the capability of the working class to defend the wage in situation of low employment. This fact therefore provides a counter-intuitive result on the interaction existing between the rules prevailing on the labour market and the demographic patterns of the population. Vice-versa, when γ is large enough instability may occur only when ν_1 is below a prescribed threshold v_1^*, i.e. when the average age of transition to adulthood is above $1/v_1^*$. This sharply confirms the role of age structure, through the delay of transition to the age of recruitment into the labour force, as a critical destabilising factor of the economy. As expected, there is a whole window of values of ν_1: $(v_1^* - \delta, v_1^*)$ in which the aforementioned Hopf bifurcation is supercritical[6] and generates therefore (stable) sustained demoeconomic oscillations. The involved limit cycles are fully viable (i.e. bounded in the admissible region of wage and employment), as showed in fig. 1. All these facts are fully confirmed by numerical simulation.

[6] There is also a subcriticity window, in a right neighbourhood of the bifurcation value.

Fig. 1. A stable limit cycle in the 2-dim plane (U, n) emerging after destabilisation of the positive equilibrium in model (14). The parameter constellation is $\gamma = 0.17$, $\rho = 0.6$, $m = 0.2$, $\mu_1 = \mu_2 = 0.013$, $\nu_2 = 0.02$, $b_2 = 0.05$, $b_1 = .1$, $\nu_1 = 0.075$.

4 The emergence of chaos

The purpose of this section is that of illustrating very briefly, via numerical simulation, the possibility of chaos in the age structured model of the previous section; more deep consideration are the object of a forthoming paper. The results of the previous section are based on a strong "symmetry", i.e. the equality between the fertility of employed and unemployed when the wage is very "low". More complex facts appear when this symmetry is broken down, in particular when a further instability window is made possible by allowing $b_e(0) < b_u$. This latter situation (fertility rates of the unemployed which are higher than those of the employed) can well be justified in developed countries according to a microfounded "view" à la Becker in which the opportunity-cost of the wage and preference for child quality play a role unfavourable for the fertility both of the wage earners and of the middle-class (that is the employed individuals). This view is obviously opposite to the malthusian one adopted in section four.

When one allows $b_e(0) < b_u$ then the system (14) looses its nature of a "two predators-one prey" system and takes a more complicated nature, in which, depending on the state of the system, the interaction between the variables U and n may even become of a competitive nature. It then becomes natural to consider what happens in the case of pure competition between U and n, which occurs when the fertility of the employed is a constant b_e satisfying $b_e < b_u$ (this case is a prototype of a "non-malthusian" situation). This is useful in order to characterise the "minimal" factors which are responsible of complex behaviour. In particular in this case the critical parameter appears to be b_u (or the difference $b_u - b_e$) which is a measure of the instability "stress" caused by the uncontrolled fertility of unemployed.

When $b_u < b_e$ the main equilbrium E_1 of the system is (oscillatory) stable, a fact already clear since the second section. At $b_u = b_e$ stability is broken down, and unstable oscillations appear. Surprinsingly, further increase of b_u do not simply accelerate instability: the increasing oscillations diverging from E_1 are at some stage "stopped" by some global bifurcation and wander erratically in the transients before converging to a stable limit cycle well disjoint from the equilibrium (Fig. 2).

Fig. 2. The emergence of chaos: a phase plane view in the space (n, w)

This occurs in an extremely wide window of values b_2 (from 0.08 to 0.70 for the parameter configuration: $\gamma = 1.9, \rho = 2, \mu_1 = 0.005, \mu_2 = 0.008, \nu_1 = 0.135, \nu_2 = 0.01, b_e = 0.01$), hence even for reasonable values from the demographic standpoint. Moreover both the shape and the location of the attractor change only slightly as the bifurcation parameter is stressed. Hence the result is a "robust" one. The symptoms of transitional chaos are evident from the time path of the population growth in fig. 3.[7]

Fig.3. The emergence of chaos. A plot of a time path of the rate of change of the supply of labour

The previous simulative analysis opens the following question: the transient chaos - which is the major symptom of impending chaos and of the existence of a chaotic set (not necessarily attracting) - can denote that a chaotic behaviour can be a robust feature of the overall class of models we are considering? In order to answer such question we consider a slight generalisation of the previous model (which leaves unaltered the main qualitative features of the system). We assume that the rate of transition to adulthood is a linear function of the state of employment, i.e. that the timing of entry of the young in the labour market at every time depends on the current state of the labour market itself: $v_1 = vU$.

The global properties of this system (ascertained via simulations) show that: 1) as regards the positive equilibrium, the Hopf bifurcation arising close to the bifurcation value of the control parameter (the fertility of the unemployed) can be supercritical (a stable limit cycle can emerge). 2) when the bifurcation parameter is further increased this limit cycle vanishes and the orbits "lands" on a fully chaotic attracting set. We conjecture that the transition to chaos depends on a blue sky catastrophe. Figures 4-5 report a pictorial example of the above facts, by using the v coefficient tuning the speed of transition to adulthood as a control parameter: in fig. 4 when $b_u = 0.10$ and $v = 0.06$ the trajectories approach a completely "viable" limit cycle; in fig. 5 when $b_u = 0.10$ and $v = 1.1$ the trajectories wander erratically in an attracting set. The "robustness" of the cyclical and chaotic feature of this class of model is confirmed by the fact that for all the reasonable (demographically speaking) values of v belonging to the parametric region of instability of E_1 the system always shows a fluctuating behaviour in a bounded region of phase space.

Fig. 4. The stable limit cycle emerging after destabilisation in the model with state dependent recruitment into the labour force

Fig. 5. The stable attractor emerging after destruction of the limit cycle in the model with state dependent recruitment

5 Conclusive remarks

This paper shows some unexpected consequences of the dynamical interaction of age structure (in the simplest form, i.e. as a delayer of the process of recruitment into the adult state) and heterogenous fertility in a demo-economic extension of the classical Lotka-Volterra-Goodwin model.

[7] The mechanism through which complex behaviour appears seems to be a saddle connection, i.e. a "blue sky" catastrophe, an entirely global bifurcation that it leaves unhaltered the properties of the existing equilibria. These aspects are anyway the object of a forthcoming paper.

First. even "minimally realistic" formulations of fertility are capable, in presence of heterogeneity, of leading to sustained oscillations. Second, the fertility of unemployed has. when uncontrolled. a destabilising action. which is capable to trigger chaotic phenomena. The characterisation of this "demographic" chaos. the exploration of some unexpected relations between the rules prevailing on the labour market and the demographic patterns. and an "operationalisation" of the model discussed here, represent possible directions for future work.

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7 Appendix 1. Modelling the dynamics of the population in the working age span

We summarise here the derivation of our general Goodwin-type demo-economic framework (see also Manfredi and Fanti 1999b). The demographic part of our framework is based on an ordinary differential equation representation of the full age structure problem which is often used in mathematical biology (Li and Hallam 1988). Let $N(a,t)$ be the age-time density of a given population, defining the density of individuals aged a at time t. The dynamics of a (one-sex) age-structured population is governed by its Von Foerster partial differential equation (see Keyfitz 1985):

$$\left[\frac{\partial}{\partial a} + \frac{\partial}{\partial t}\right] N(a,t) = -\mu(a,t)N(a,t) \tag{21}$$

plus the usual boundary and initial conditions:

$$N(0,t) = B(t) \ ; \ N(a,0) = \varphi(a) \tag{22}$$

where $\mu(a,t)$ is the death rate at age a, $B(t)$ the birth function at time t, and $\varphi(a)$ is a prescribed function of age assigning the age density at time zero. To derive from (21)-(22) an ODE formulation let us now subdivide our population into the following three broad age classes: $(0, A_1)$=pre-work age (for instance $A_1 = 15$ years), (A_1, A_2)=working age span (for instance: $(15, 65)$), (A_2, ∞)=retirement ages, and let the functions $P_1(t), P_2(t), P_3(t)$ respectively denote the number of individuals (young, adult and retired) in the three groups at time t[8]. By performing separate integrations of the basic PDE over the three age groups, and by remembering that, by definition: $P(0,t) = B(t)$, $P(\infty, t) = 0$, we get the following relations:

$$\begin{aligned} \dot{P}_1(t) &= B(t) - D_1(t) - P(A_1,t) \\ \dot{P}_2(t) &= [P(A_1,t) - P(A_2,t)] - D_2(t) \\ \dot{P}_3(t) &= P(A_2,t) - D_3(t) \end{aligned} \tag{23}$$

where the $D_i(t)$ terms denote the number of deaths within each class at time t, while $P(0,t)$, $P(A_1,t)$, $P(A_2,t)$ are the flows from one class to the next one due to the aging process. The interpretation of (23) is straightforward. For instance the dynamics of the number $P_2(t)$ of individuals in the working age span is the outcome of the balance between the entries from the pre-work class $(P(A_1,t))$ and the exits due to aging $(P(A_2,t))$ and mortality $(D_2(t))$. Notice that: $D_i(t) = \bar{\mu}_i(t) P_i(t)$ where $\bar{\mu}_i(t)$ is the average death rate in class i (which is usually time-varying unless the population is in a stable state). By assuming that i) the death rate is constant in each class $(\bar{\mu}_i(t) = \mu_i)$ we get $D_i(t) = \mu_i P_i(t)$. Let us now consider the quantities $P(A_1,t), P(A_2,t)$ describing transitions from one age-class to the next one. Clearly: $P(A_1,t) = B(t - A_1)\, p(A_1)$, $P(A_2,t) = B(t - A_2)\, p(A_2)$ where $p(a)$, is the probability of surviving up to age a. Let us now assume that:

$$B(t) = b(t)\, P_2(t) \tag{24}$$

i.e. that all births take place in the adult population, where b is the fertility rate of the adults, assumed age-independent. In this case: $P(A_1,t) = b(t - A_1)\, P_2(t - A_1) p(A_1)$ and therefore:

$$\dot{P}_2(t) = b(t - A_1)\,[P_2(t - A_1) p(A_1) - P_2(t - A_2) p(A_2))] - \mu_2 P_2(t) \tag{25}$$

where: $p(A_1) = e^{-\mu_1 A_1} = k_1$ and $p(A_2) = p(A_1)\, e^{-\mu_2 (A_2 - A_1)} = \varepsilon k_1$. The relation (25) provides the exact mathematical representation of the dynamics of the population in the working age span. Hence, even if we assume that both fertility and mortality are age-independent (mortality is state-dependent), such "exact" description needs differential-difference equations, the tractability of which, once embedded within a macro-economic model, may be limited.[9] A strong simplification comes about if we assume that: $P(A,t) = v_1 P_1(t)$, $P(B,t) = v_2 P_2(t)$. In this case (23) collapses into the ODE system:

---

[8] We could of course introduce an arbitrary number of stages (and this could be of interest from other points of view) but this is the simplest reasonable choice.

[9] The analysis of population frameworks as (25) within macro-economic models is a target of our future work.

$$\begin{aligned}\dot{P}_1(t) &= bP(t) - (\mu_1 + v_1)\,P_1(t)\\ \dot{P}_2(t) &= v_1 P_1 - (\mu_2 + v_2)\,P_2(t)\\ \dot{P}_3(t) &= v_2 P_2(t) - \mu_3 P_3(t)\end{aligned} \tag{26}$$

where $\mu_i (i = 1, 2, 3)$ are the death rates and $v_i (i = 1, 2)$ are the transition rates between the three classes. Both types of rates will be assumed as constant since now on.

Some remarks are useful. The system (26) provides, being derived under several assumptions, a necessarily simplified view of the demographic reproduction process. Fertility is age-independent. Moreover, by working on broad age stages our description can not give information on facts concerning smaller time scales, i.e. it disregards age structure inside $(0, A)$ and (A, B). On the other side, it has the advantage of capturing in a sufficiently realistic way the main effects of age-structure whilst mantaining at the same time, being based on ODE's, simplicity of treatment. Feichtinger and Sorger (1990) used the same ODE system but without explicit reference to the underlying PDE for age structure, and in any case they did not derive any equation resembling our equation (30). As stated by Feichtinger and Sorger (1990, pp. 279): ".. *the approach may not be suited for quantitative calculations but it has avdantages if one wants to get insights into the qualitative population dynamics.*"

Let us now go one step further and introduce explicitly (26) within the basic Goodwin's formulation. This leads to the following 4-dimensional system in the (w, U) variables:

$$\begin{aligned}\dot{w}(t) &= w(t)\,(-\gamma + \rho U)\\ \dot{U}(t) &= U(t)\,(m(1 - w) - n)\\ \dot{P}_1(t) &= B(t) - (\mu_1 + v_1)\,P_1(t)\\ \dot{P}_2(t) &= v_1 P_1(t) - (\mu_2 + v_2)\,P_2(t)\end{aligned} \tag{27}$$

(the equation of the retired population does not provide any input to the system and can be disregarded). System (27) can be simplified. The rate of change of the adult population $n = \dot{P}_2/P_2$ satisfies:

$$\frac{\dot{P}_2}{P_2} = v_1 \frac{P_1}{P_2} - (\mu_2 + v_2) \tag{28}$$

It is thus possible to reduce the system one further dimension by introducing the new variable $Z = P_1/P_2$. As: $\dot{Z} = Z\left(\dot{P}_1/P_1 - \dot{P}_2/P_2\right)$ we obtain the dynamical equation for Z:

$$\dot{Z} = b - HZ - v_1 Z^2 \tag{29}$$

where: $H = (\mu_1 + v_1) - (\mu_2 + v_2) \gtreqless 0$. Hence (27) has been reduced to a 3-dimensional system in which the equations for P_1 and P_2 are replaced by the equation (29). The ensuing system is the "minimal" modelling structure for the investigation of demoeconomic interactions within Goodwin's model. The demographic subsystem appears through the ratio $Z = Z(t)$ between the number of young and adult individuals in the population. An alternative formulation is obtained by directly deriving a dynamical equation for $n(t) = \dot{P}_2/P_2$ (instead of Z). By still using (28) we get the dynamical equation:

$$\dot{n}(t) = (-1)\left[n^2 + Pn - B\right] \tag{30}$$

where:

$$P = (\mu_1 + v_1) + (\mu_2 + v_2) > 0 \; : \; B = v_1 b - Q \; ; \; Q = (\mu_2 + v_2)(\mu_1 + v_1) \tag{31}$$

The formulation (30)-(31) is the one presented in section 4 of the main text. In particular the quantity:

$$R_0 = \frac{v_1 b}{Q} = \frac{v_1 b}{(\mu_1 + v_1)(\mu_2 + v_2)} \tag{32}$$

is the net reproduction ratio of the population in our three stages demographic system.

8 Appendix 2. The positive equilibrium in the age structured model: the case of linear fertility

8.1 Computation of the positive equilibrium

All the considerations on equilibria valid for the general model (14) hold in the case of linear fertility. In this case the discussion of the existence of positive equilibria leads to the system:

$$m(1-w) - n = 0 \quad n^2 + Pn - (v_1 (b_2 + c_1 w U_1) - Q) = 0 \tag{33}$$

where U_1 is the positive equilibrium value of employment. From the first equation $w = (m-n)/m$; a substitution into the second one gives the quadratic in n:

$$mn^2 + (mP + v_1 c_1 U_1) n - (v_1 (b_2 + c_1 U_1) - Q) m = 0 \tag{34}$$

As $b_2 + c_1 U_1$ is the fertility function $b(w, U)$ evaluated when the wage takes its maximal level ($w = 1$) and the employment takes its equilibrium level: $b_2 + c_1 U_1 = b(1, U_1)$ we obtain the equation:

$$mn^2 + (mP + v_1 c_1 U) n - (v_1 b(1, U_1) - Q) m = 0 \tag{35}$$

This quadratic has a positive coefficient of the first order term. The intercept must be negative in order to have a positive value of the rate of change of the supply of labour compatible with a wage equilibrium w_1 such that $0 \leq w_1 \leq 1$. This quadratic has a unique economically meaningful solution whic h is given in the main text

$$n_1 = \frac{1}{2}\left(-\left(P + \frac{v_1 c_1}{m} U\right) + \sqrt{\left(P + \frac{v_1 c_1}{m} U\right)^2 + 4(v_1 b(1, U_1) - Q)} \right) \tag{36}$$

8.2 Some static results

Here we prove the inequalities (18) of the main text. Let us start from the equation (35). By using $n_1 < m$ we quickly get

$$\frac{\partial n}{\partial b_2} = (-1) \frac{-v_1 m}{2mn + (mP + v_1 c_1 U)} > 0$$

$$
\begin{aligned}
\frac{\partial n}{\partial c_1} &= (-1)\,\frac{v_1 U\,(n-m)}{2mn+(mP+v_1c_1U)} > 0 \\
\frac{\partial n}{\partial U} &= (-1)\,\frac{v_1 c_1\,(n-m)}{2mn+(mP+v_1c_1U)} > 0
\end{aligned}
$$

Let us now consider more in detail $\partial n_1/\partial v_1$. We have

$$
\frac{\partial n_1}{\partial v_1}=\frac{1}{2}\left(-\left(P'(v_1)+\frac{c_1}{m}U\right)+\frac{\left(2\left(P(v_1)+\frac{v_1c_1}{m}U\right)\left(P'(v_1)+\frac{c_1}{m}U\right)+4\left(b(1,U)-Q'(v_1)\right)\right)}{2\sqrt{\Delta}}\right)
$$

where

$$
\Delta=\left(P(v_1)+\frac{v_1c_1}{m}U\right)^2+4\left(v_1b(1,U)-Q(v_1)\right)
$$

As $P'(v_1)=1$; $Q'(v_1)=\mu_2+v_2$, we get

$$
\frac{\partial n_1}{\partial v_1}=\frac{1}{2}\left(-\left(1+\frac{c_1}{m}U\right)+\frac{1}{\sqrt{\Delta}}\left(\left(P(v_1)+\frac{v_1c_1}{m}U\right)\left(1+\frac{c_1}{m}U\right)+2\left(b(1,U)-(\mu_2+v_2)\right)\right)\right)
$$

Hence $\partial n_1/\partial v_1>0$ when

$$
\left(P(v_1)+\frac{v_1c_1}{m}U\right)\left(1+\frac{c_1}{m}U\right)-2\left(b(1,U)-(\mu_2+v_2)\right)>\left(1+\frac{c_1}{m}U\right)\sqrt{\Delta}
$$

As $b(1,U)-(\mu_2+v_2)>0$ we may square: by simplifying and writing for simplicity $b(1,U)=b$, $P(v_1)=P$, $Q(v_1)=Q$, we get the condition

$$
F=F_1+F_2-F_3>0 \tag{37}
$$

where:

$$
\begin{aligned}
F_1 &= (b-(\mu_2+v_2))^2 \quad ; \quad F_2=\left(P+\frac{v_1c_1}{m}U\right)\left(1+\frac{c_1}{m}U\right)(b-(\mu_2+v_2)) \\
F_3 &= \left(1+\frac{c_1}{m}U\right)^2(v_1b-Q)
\end{aligned}
$$

But

$$
P+\frac{v_1c_1}{m}U=v_1+\frac{v_1c_1}{m}U+(\mu_1+\mu_2+v_2)=v_1+\frac{v_1c_1}{m}U+q_1
$$

where $q_1=\mu_1+\mu_2+v_2>0$. Threfore, after some algebra F_2 may be written in the form

$$
F_2=F_3+\mu_1(\mu_2+v_2)\left(1+\frac{c_1}{m}U\right)^2+q_2
$$

This finally shows that:

$$
F=F_1+F_2-F_3=(b-(\mu_2+v_2))^2+\mu_1(\mu_2+v_2)\left(1+\frac{c_1}{m}U\right)^2+q_2>0
$$

i.e. that $\partial n_1/\partial v_1>0$ as stated in the (18) of the main text.

9 Appendix 3. Stability and bifurcation analysis of the positive equilibrium

The jacobian of system (14) evaluated at E_1 is

$$J_{w,U,n}(E_1) = \begin{pmatrix} 0 & \rho w & 0 \\ -mU & 0 & -U \\ v_1 U c_1 & v_1 c_1 w & -(2n+P) \end{pmatrix}$$

The characteristic polynomial $P(X) = X^3 + a_1^2 + a_2 X + a_3$ is

$$X^3 + (2n+P) X^2 + Uw (v_1 c_1 + \rho m) X + ((2n+P) m + v_1 U c_1) \rho U w = 0 \quad (38)$$

Thanks to the Routh-Hurwicz theorem, as all the coefficients are strictly positive, E_1 is stable provided that: $\Delta_2 = a_1 a_2 - a_3 > 0$. It holds

$$\begin{aligned} \Delta_2 &= (2n+P) Uw (v_1 c_1 + \rho m) - ((2n+P) m + v_1 U c_1) U \rho w = \\ &= Uw ((2n+P) v_1 c_1 - v_1 c_1 \rho U) = v_1 c_1 U w ((2n+P) - \rho U) \end{aligned} \quad (39)$$

The condition for the LAS of E_1 has the form:

$$H(v_1) - \gamma > 0 \quad (40)$$

where $H(v_1) = 2n_1(v_1) + P(v_1)$. Let us discuss the previous condition by using v_1 as the pivot parameter. A result proved in appendix 2 shows that $H(v_1)$, being the sum of two strictly increasing functions of v_1, is a strictly increasing function of v_1. In particular $H(0) > 0$. In fact:

$$\begin{aligned} H(0) &= 2n_1(0) + P(0) = -P(0) + \sqrt{(P(0))^2 - 4Q(0)} + P(0) = \sqrt{(P(0))^2 - 4Q(0)} = \\ &= \sqrt{((\mu_1 + \mu_2) + v_2)^2 - 4v_2(\mu_1 + \mu_2)} = \sqrt{((\mu_1 + \mu_2) - v_2)^2} > 0 \end{aligned}$$

Hence, if $H(0) > \gamma$ then the E_1 equilibrium is always LAS. As $H(0) = |(\mu_1 + \mu_2) - v_2|$, the condition $H(0) > \gamma$ may be written

$$\gamma < |(\mu_1 + \mu_2) - v_2|$$

Vice-versa, when $H(0) < \gamma$ the E_1 equilibrium is LAS only for $v_1 > v_1^*$ where v_1^* is the unique solution of the equation $H(v_1) - \gamma = 0$. The region $0 < v_1 < v_1^*$ is an instability region. The point $v_1 = v_1^*$ is a Hopf bifurcation point for E_1.

To prove that at the point $v_1 = v_1^*$ where stability is lost, a Hopf bifurcation occurs, we need to show that: i) purely imaginay eigenvalues exist for the linearised system at $a = a_1, a = a_2$ due to a "continuous" movement of a pair of complex eigenvalues: ii)the crossing of the imaginary axis by the involved complex pair occurs with nonzero speed. The proof of i) is evident, see for instance Liu (1994) or Fanti and Manfredi (1998). To show that the crossing of the imaginary axis by the bifurcating eigenvalues occurs with nonzero speed, we have to consider (Liu, 1994) the sign of the derivative of the higher order determinant of the Routh-Hurwicz theorem with respect to the chosen bifurcation parameter (v_1), evaluated at the bifurcation point. From (39):

$$\Delta_2 = c_1 U v_1 w(v_1)(H(v_1) - \gamma)$$

we get

$$\frac{d\Delta_2}{dv_1} = c_1 U_1 \left(w(v_1)(H(v_1) - \gamma) + v_1 \frac{dw(v_1)}{dv_1}(H(v_1) - \gamma) + v_1 w(v_1) \frac{dH(v_1)}{dv_1} \right)$$

Remembering that

$$w_1 = (m - n_1)/m \quad ; \quad \frac{dw(v_1)}{dv_1} = -\frac{1}{m}\frac{dn_1}{dv_1} \quad ; \quad \frac{dH_1}{dv_1} = 2\frac{dn_1}{dv_1} + 1$$

we have:

$$\frac{d\Delta_2}{dv_1} = c_1 U_1 \left(\frac{m - n_1}{m}(H(v_1) - \gamma) - \frac{v_1}{m}\frac{dn_1}{dv_1}(H(v_1) - \gamma) + v_1 w(v_1) \frac{dH(v_1)}{dv_1} \right) = c_1 U_1 L(v_1)$$

Let us now consider the last expression at the bifurcation point $v_1 = v_1^*$. We get

$$\begin{aligned} L(v_1^*) &= \frac{m - n_1(v_1^*)}{m}(H(v_1^*) - \gamma) - \frac{v_1}{m}\left(\frac{dn_1}{dv_1}\right)_{v_1^*}(H(v_1^*) - \gamma) + v_1^* \frac{m - n_1(v_1^*)}{m}\left(\frac{dH(v_1)}{dv_1}\right)_{v_1^*} \\ &= v_1^* \frac{m - n_1(v_1^*)}{m} H'(v_1^*) > 0 \end{aligned}$$

as $H'(v_1) > 0$ and $m - n_1(v_1) > 0$. The last result shows that the nonzero speed condition is satisfied and therefore that v_1^* is a Hopf bifurcation point.

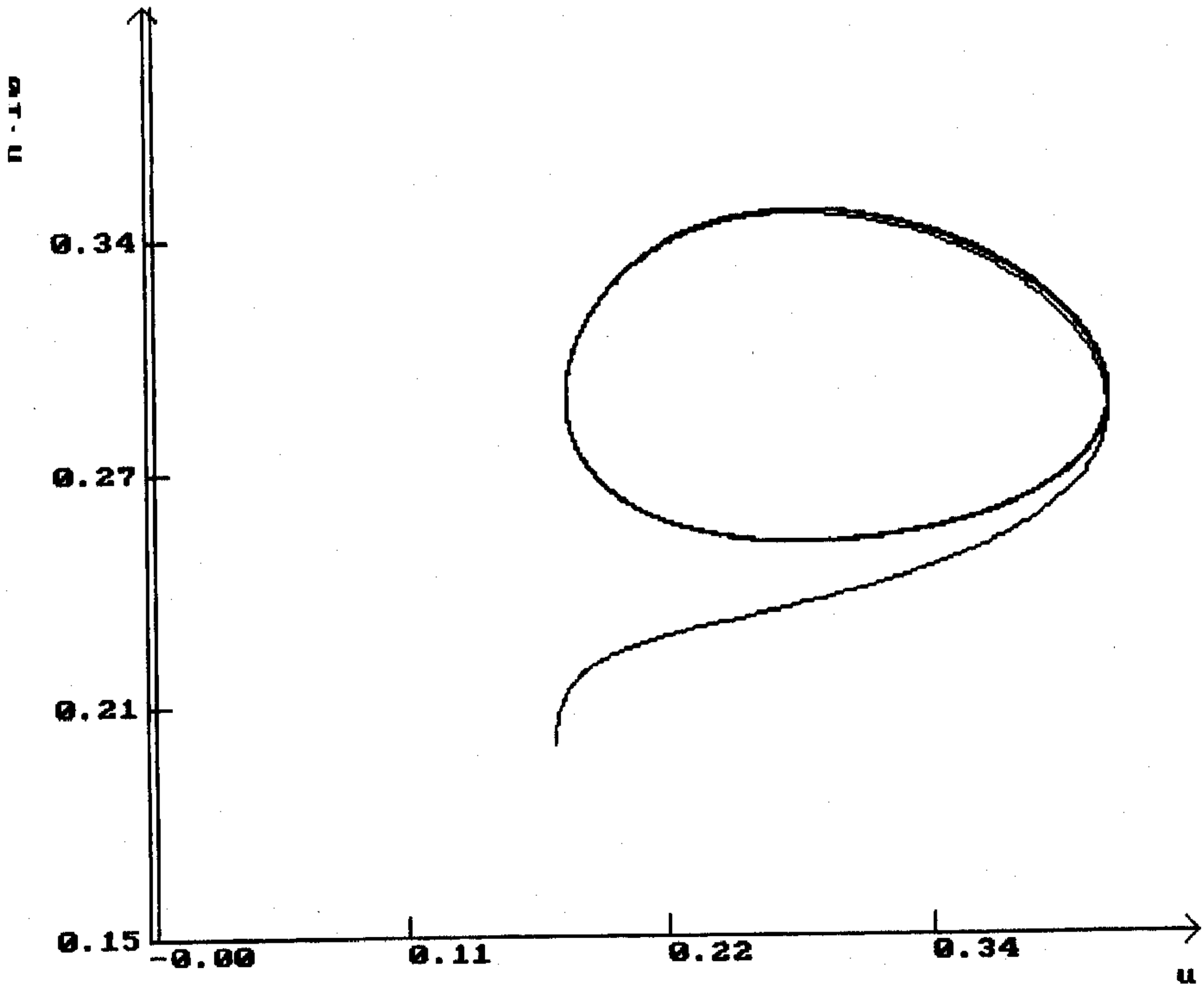

Fig. 1

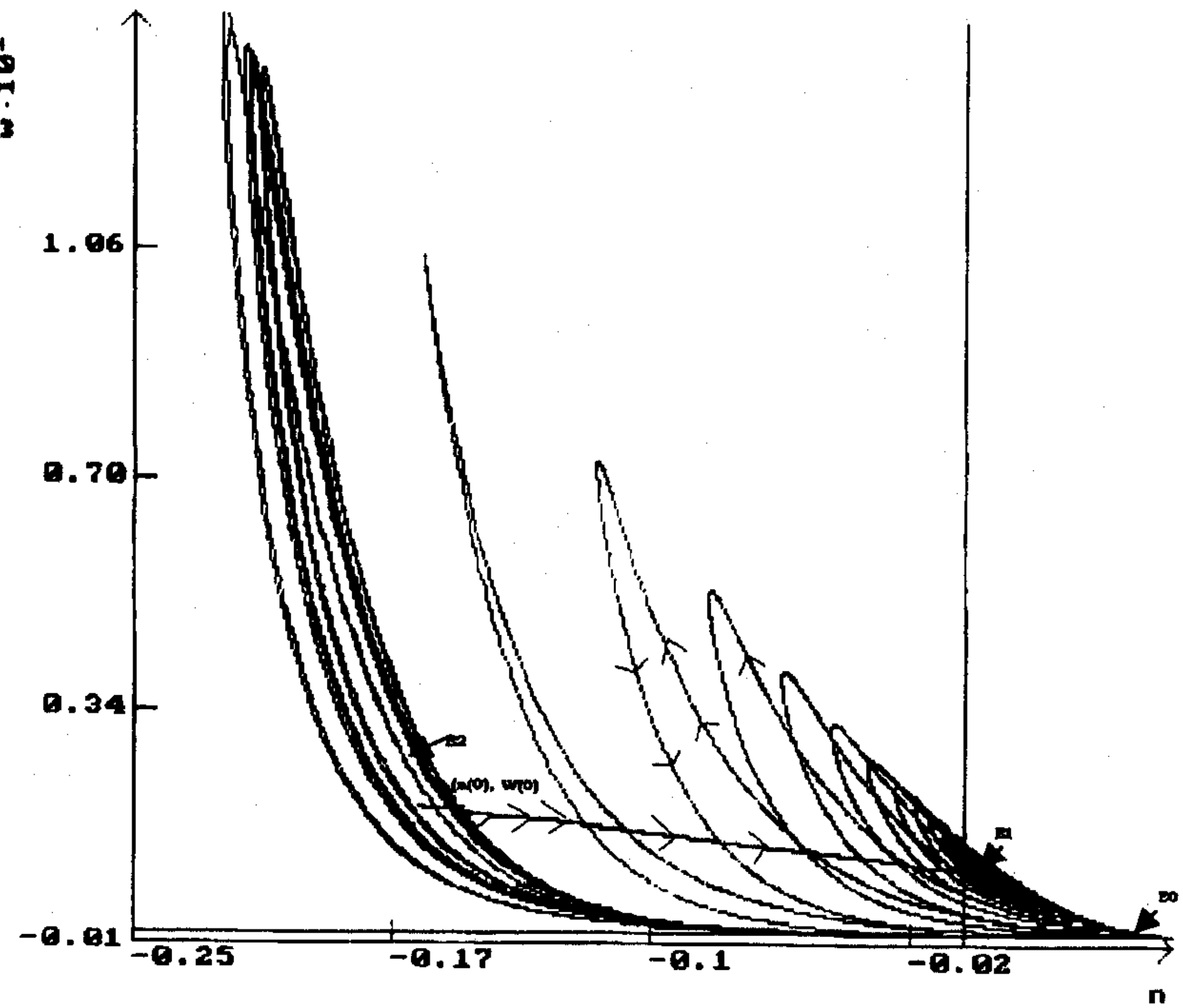

FIG. 2 Global picture of the trajectories in the phase space (n,W).

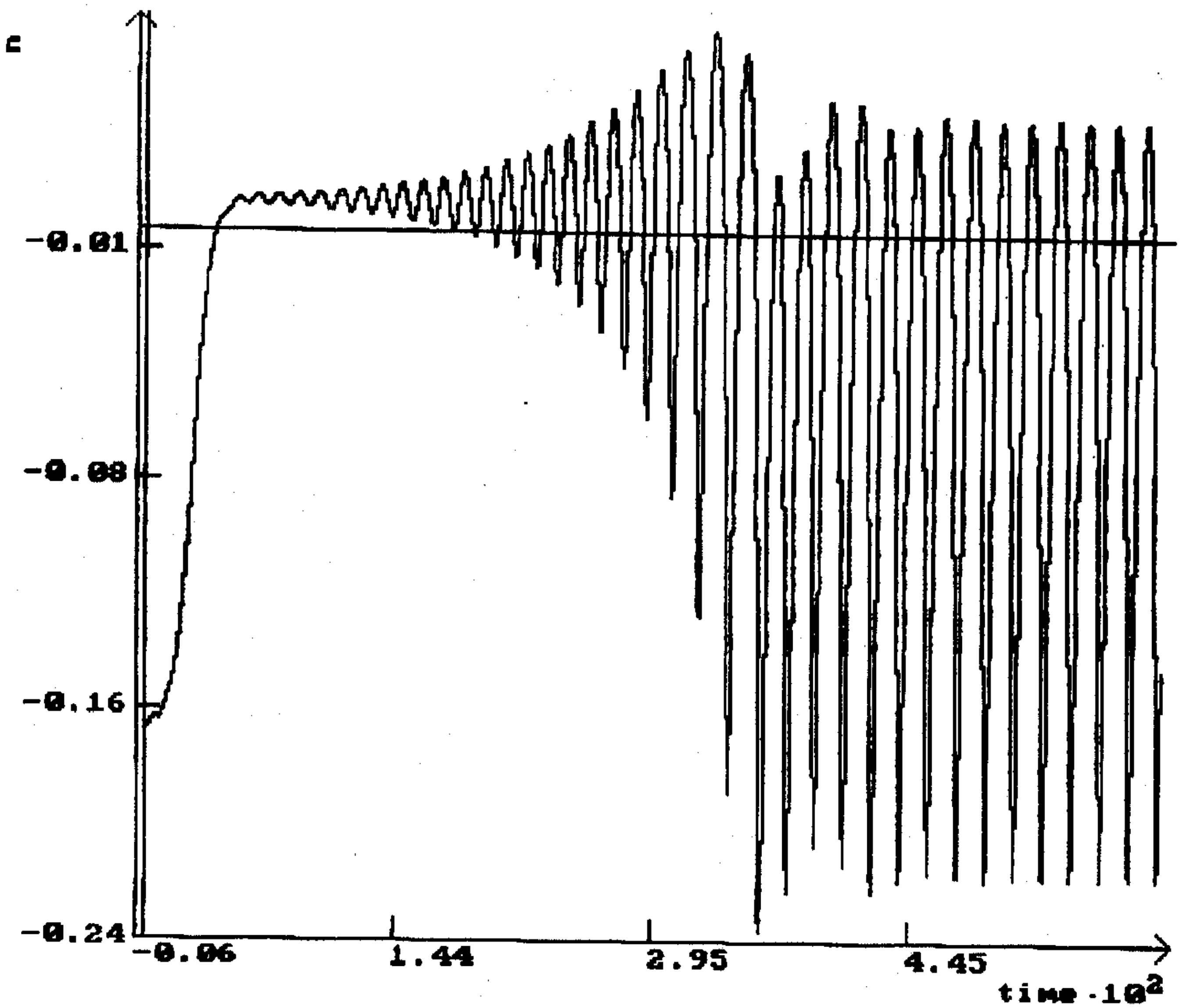

FIG. 3. Plot of the time path of the population growth rate.

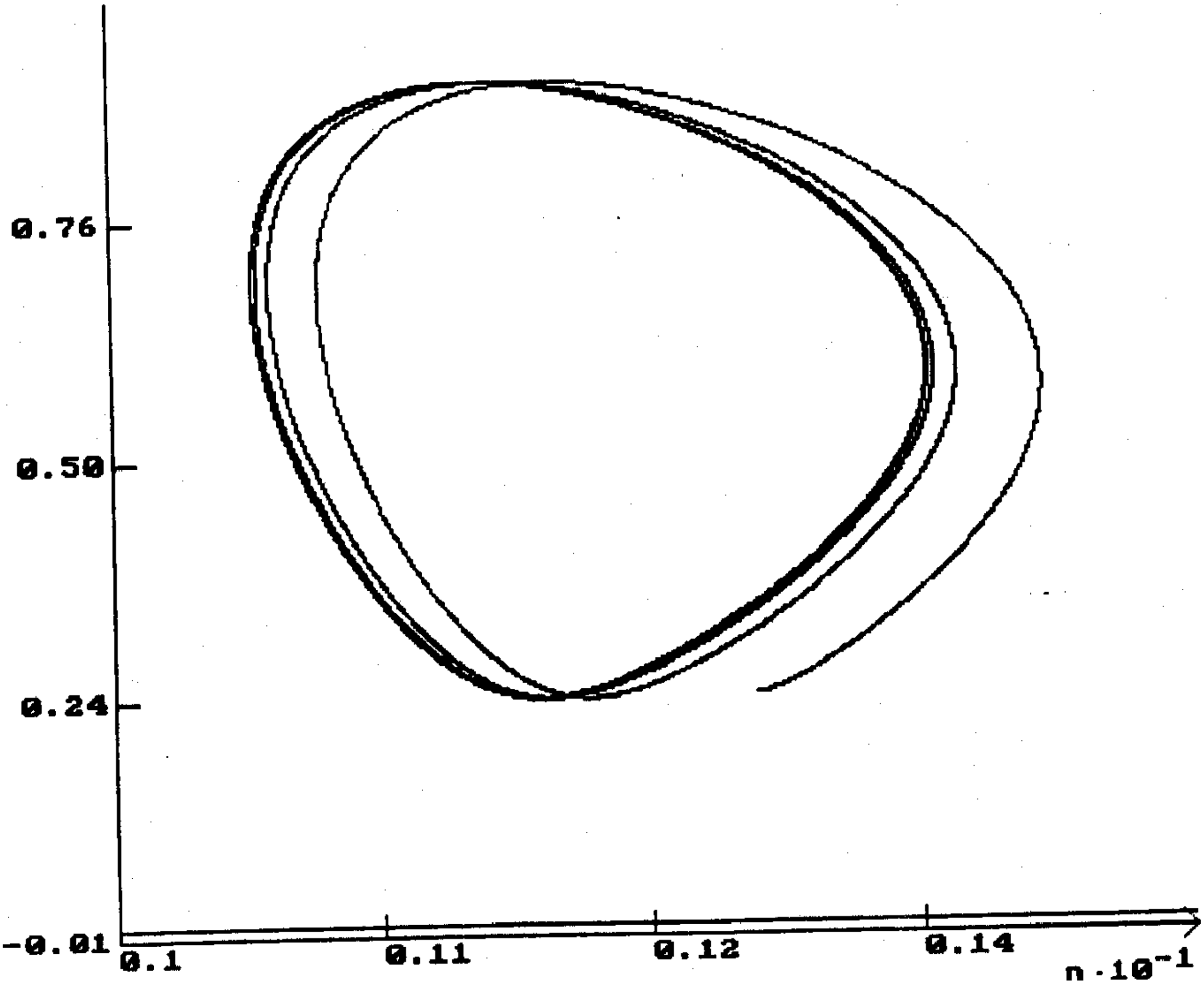

Fig. 4

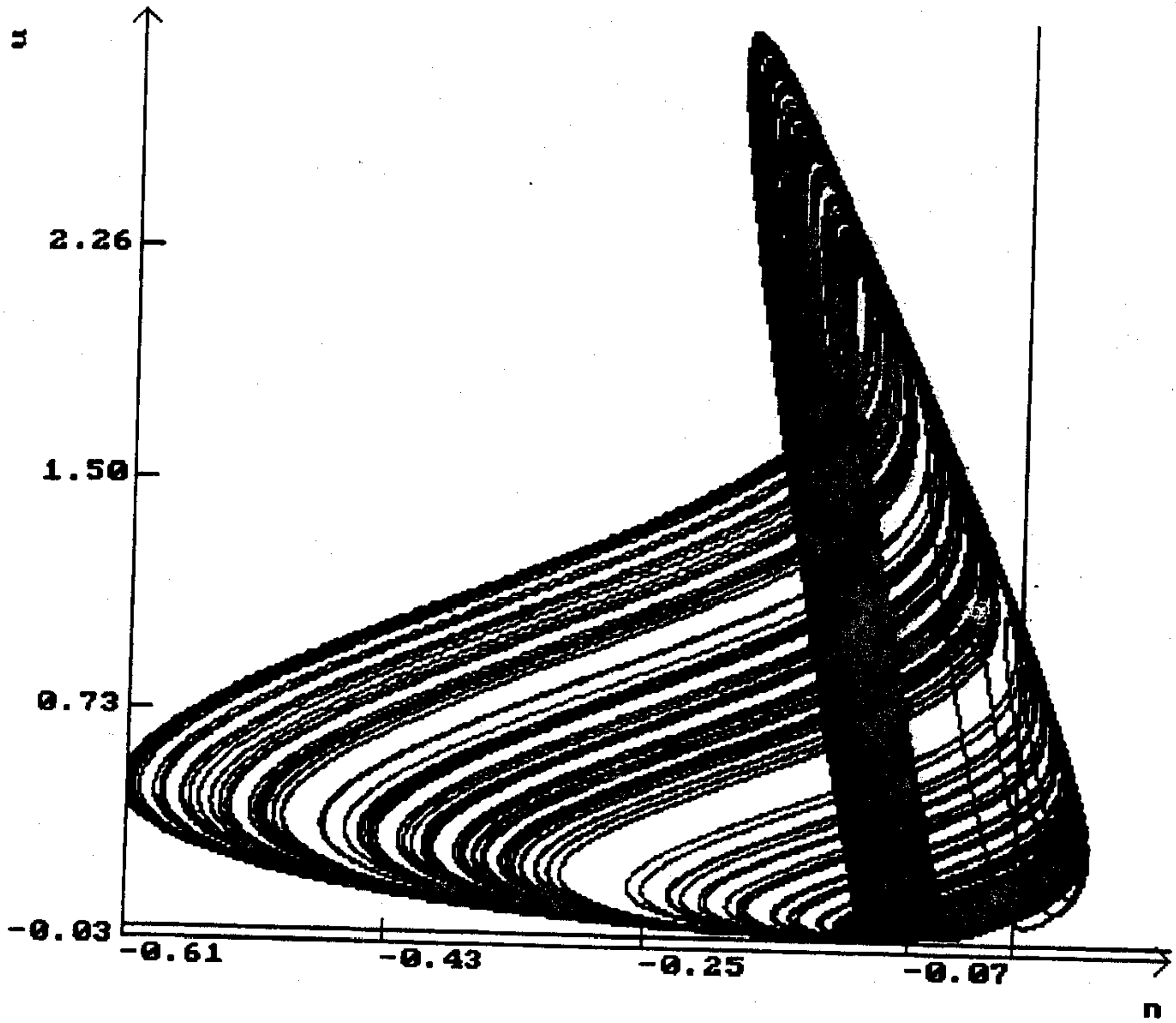

Fig. 5