Report n. 179

Volatility Estimation via Fourier Analysis

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Pisa, Maggio 2000

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Abstract

In this paper we apply a new method, based on Fourier analysis, to compute cross volatilities on historical data. The main feature of such a method is to be based on an integration procedure instead of a differentiation one. On equally spaced data (daily, weekly) it provides the same result than the classical method. However it is best suited for high frequency data, since it does not need to aggregate or interpolate data. The method has been tested on Monte Carlo simulations, and some results on real data are shown.

1 Introduction

In this paper we implement the methodology proposed in [9] to estimate multivariate volatility for financial time series when the data are observations of a vector of continuous time semi-martingales. The methodology is based on Fourier Analysis, it allows for time varying, eventually stochastic coefficients. Volatility for diffusion processes is a concept well defined in theory but very difficult to be estimated empirically for financial time series. In fact, financial data are not observed in continuous time. Many estimators of the volatility with constant coefficients have been constructed in the econometric-statistic of processes literature. The methods are based on *differentiation* of the time series, i.e., the expectation of the quadratic movements of the financial data is computed; instead, our approach is based on integration. This feature will allow us to avoid some difficulties encountered with classical methods, expecially with high frequency data, see [3].

Our method is almost model free, it makes very weak assumption on the market model. The method is semi-parametric, no assumption on the functional form of the volatility is done. These features render our method well suited to detect the dynamics of volatility.

To test the methodology, we provide both a Monte Carlo analysis reconstructing the volatiltiy of a given process and we apply it to real data. In both cases we compare the results to those obtained with the classical methods. We perform both univariate and multivariate tests.

The Monte Carlo analysis establishes that the method performs well in computing the volatility of a theoretical process and in reconstructing the cross volatilities of two processes.

In Section 2 we present the methodology developed in [9]. In Section 4 we present some technical points associated with the implementation of the methodology. In Section 3 we relate our approach to related literature. In Section 5 we perform a Monte Carlo analysis of the method testing the capability of the method to reconstruct the volatility of a theoretical process.

2 The Methodology

The methodology developed in [9] makes a simple Assumption about the market model. The time series of interest (prices, returns, volumes, etc.) are discrete time observations of a set of semimartingales:

$$du_i = \sum_j \sigma_{ij}(t)dW_j(t) + \mu_i(t)dt \quad i = 1, \ldots, N. \tag{1}$$

where σ and μ are random, time dependent functions.

The volatility matrix is defined as:

$$\Sigma_{ij}(t) := \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \boldsymbol{E}^{N_t}\Big[(u_i(t+\epsilon)-u_i(t))\cdot(u_j(t+\epsilon)-u_j(t))\Big] \quad i,j=1,\ldots,N. \quad (2)$$

where $\boldsymbol{E}^{N_t}[\cdot]$ denotes the expectation operator and N_t is the $\sigma-$field generated by the full observation of the economic data until time t. The relation between $\Sigma(t)$ and the model (1) is given by:

$$\Sigma_{ij}(t) = \sum_{k=1}^{N} \sigma_{ik}(t)\sigma_{jk}(t).$$

We normalize the time window for the computation of the volatility to $[0, 2\pi]$. In [9, Theorem 1.2] it is shown that the Fourier coefficients of Σ can be computed using the Fourier coefficients of du, then classical results of Fourier theory allows us to reconstruct $\Sigma(t)$ $\forall t \in [0, 2\pi]$.

First of all, we consider a univariate setting. The Fourier coefficients of du are

$$\begin{aligned} a_0(du) &= \tfrac{1}{2\pi}\textstyle\int_0^{2\pi} du(t) \\ a_k(du) &= \tfrac{1}{\pi}\textstyle\int_0^{2\pi} \cos(kt)du(t) \\ b_k(du) &= \tfrac{1}{\pi}\textstyle\int_0^{2\pi} \sin(kt)du(t) \end{aligned} \quad (3)$$

Then we obtain the Fourier coefficients of Σ through the formulas:

$$a_0(\Sigma) = \lim_{n\to\infty} \frac{\pi}{n+1-n_0} \sum_{s=n_0}^{n} \frac{1}{2}(a_s^2(du) + b_s^2(du)) \quad (4)$$

$$a_k(\Sigma) = \lim_{n\to\infty} \frac{2\pi}{n+1-n_0} \sum_{s=n_0}^{n} a_s(du)a_{s+k}(du) \quad (5)$$

$$b_k(\Sigma) = \lim_{n\to\infty} \frac{2\pi}{n+1-n_0} \sum_{s=n_0}^{n} b_s(du)b_{s+k}(du). \quad (6)$$

By the classical Fourier-Féjer inversion formula, we can reconstruct $\Sigma(t)$:

$$\Sigma(t) = \lim_{n\to\infty} \sum_{k=0}^{n} (1 - \frac{k}{n}) \cdot (a_k(\Sigma)\cos(kt) + b_k(\Sigma)\sin(kt)) \quad (7)$$

The generalization to multivariate volatility is straightforward. The Fourier coefficients of Σ_{ij} $(i, j = 1, \ldots, N)$ are

$$a_k(\Sigma_{ij}) = \lim_{N\to\infty} \frac{\pi}{N+1-n_0} \sum_{s=n_0}^{N} \frac{1}{2}(a_s(du_i)a_{s+k}(du_j) + a_s(du_j)a_{s+k}(du_i)). \quad (8)$$

Similar formulas hold for (4) and (6).

Note that this method allows us to reconstruct the volatility inside the interval $[0, 2\pi]$.

We observe that the Fourier coefficients for cross-volatilities are computed through the coefficients computed for the single time series.

3 Related Literature

Estimation of volatility for continuous time processes is a difficult task.

The classical way to estimate the volatility is to adapt the formula (2) to the data observed with the given frequency without considering the limit. The quadratic variation estimated empirically in this way provides an unbiased estimator in case of constant volatility. Many estimators have been proposed in this setting.

The literature on the estimation of volatility has grown up recently for three main reasons: a) tick by tick time series are now available, b) risk management techniques, such as VAR, based on the estimation of the volatility are now heavily used in many financial institutions, c) observation that the volatility for financial time series is non constant and in many cases nonstationary.

The reference framework foresees a parametrization of the volatility through constant coefficients and then an estimator is employed. In this setting unbiased estimators of the volatility have been proposed by simply considering the quadratic variation of the financial data and using closing data for securities prices, more refined estimators have been proposed by considering high&low prices, on this point see for example [2]. The three reasons pointed out above motivated recent developments in the literature.

First of all, financial time series analysis has shown undoubtly that volatility is not constant. This conclusion is reached by considering time series with different frequencies, see [11] for monthly volatility. In particular it is observed a clustering of volatility wich also appears to be highly persistent. To model these phenomena GARCH and ARCH models have been proposed.

Tick by tick observations of a time series pose problems for the computation volatility and in particular of cross volatilities, see [6] for a survey on this type of time series. Two time series reporting every price of the transactions for two assets are characterized by inequally and irregularly spaced data, this fact leads to problems in computing cross volatilities. To avoid them, interpolation or imputation methods are employed. In the first case a time horizon is fixed, the time axis is split according to that horizon and inside each interval the last observation is considered. This procedure gives us homogeneous and equally spaced time series but it entails two main drawbacks: in some intervals of time no observation is available, some observations are thrown away. Interpolation requires to aggregate data observaions, centering them at some fixed points.

Recently new methods have been developed which are not based on interpolation and imputation, see [7, 4]. They avoid manipulation of the data by assuming that the underlying true returns are serially uncorrelated (in the first case) and that the process generating the transaction times and the prices are independent (in the first and in the second case). Our method does not make these assumptions.

The literature on tick by tick observations provided us with some well established regularities. In [5, 8] it is observed that cross volatilities tend to vanish as the interval of time employed for the computation of the volatility goes to zero.

Much work in the recent literature on the estimation of the volatility of diffusion processes is due to the fact that high frequency data are now at the disposal of researchers. Model-free estimates of volatilities have been proposed recently. Among them we recall [1].

4 Implementation

Applying the method to real data, we have at our disposal a time series $(t_i, S(t_i))$, $i = 1, \dots K$ of N observations at time t_i. In what follows we will consider $u(t) = \log S(t)$.

We will compress the data in the interval $[0, 2\pi]$ and compute the integrals (3) through *integration by parts*:

$$a_k(du) = \frac{1}{\pi}\int_0^{2\pi} \cos(kt)du(t) = \frac{u(2\pi) - u(0)}{\pi} - \frac{k}{\pi}\int_0^{2\pi} \sin(kt)u(t)dt \quad (9)$$

The smallest wavelength that can be evaluated is twice the smallest distance between two consecutive prices; in the case of equally spaced data, it will correspond to $k = N/2$ (Nyquist frequency), see [10]. We will always choose $N/2$ as the largest frequency. Then the Fourier coefficients (4),(5),(6) will be evaluated for $0 \le k \le J$ and with $n = M$ such that:

$$M + J = \frac{N}{2} \quad (10)$$

As the observations are finite, to implement the method and in particular the integration we need an assumption on how data are connected. Our choice is the function $u(t)$ be equal to $u(t_i)$ in the interval $[t_i, t_{i+1}]$, piecewise constant. With this choice, the integral in equation (9) in the interval $[t_i, t_{i+1}]$ becomes:

$$\frac{k}{\pi}\int_{t_i}^{t_{i+1}} \sin(kt)u(t)dt = u(t_i)\frac{k}{\pi}\int_{t_i}^{t_{i+1}} \sin(kt)dt = u(t_i)\frac{1}{\pi}(cos(kt_i) - cos(kt_{i+1})) \quad (11)$$

thus avoiding the multiplication by k which amplifies cancellation errors when k becomes large.

In the integration by parts formula (9) the constant term $(u(2\pi) - u(0))/2\pi$ appears in every a_k; this could make the formulas (9) too strongly dependent from such a random term so we add to du the drift term $-\frac{u(2\pi)-u(0)}{2\pi}dt$, so that the transformed u will have $u(2\pi) = u(0)$. In fact, adding drift term to du will not change the volatility.

5 Monte Carlo Analysis

We start with a single diffusion process, which in the following will represent the price of an asset. To simplify the analysis, we assume no drift and therefore we simulate the price model:

$$dS(t) = \sigma(t)dW(t) \qquad t \in [0, 2\pi]$$

in the following way:

$$\begin{cases} S_1 = 100. \\ S_{i+1} = S_i \cdot e^{r_i} \\ r_i \sim \mathcal{N}(0, \sigma_i) \end{cases} \tag{12}$$

where as $\mathcal{N}(\mu, \sigma)$ we mean the normal distribution of mean μ and standard deviation σ.

We take $u(t) = \log S(t)$, therefore $u(t)$ will follow the stochastic path:

$$du(t) = -\frac{\sigma^2(t)}{2}dt + \sigma(t)dW(t)$$

so that it will have the same volatility as dS. We can now make a test with different choices of $\sigma(t)$.

5.1 $\sigma(t)$ = constant

The first natural choice is to set $\sigma(t)$ to a constant. If σ is constant, then, as can be readily obtained:

$$a_k, b_k \sim \mathcal{N}(0, \frac{\sigma}{\sqrt{\pi}}), \qquad k \geq 1$$

This distribution is the theoretical one, but since we take $u(t)$ on a discrete lattice and make assumptions on how $u(t)$ is connected between two adjoining points, then such distributions could have in general a broader variance.
To test this point, we perform the Fourier analysis on 500 Monte Carlo series of N data, results are shown in figure 1 for $N = 100$. The result is quite good; the variance is the expected one, and the mean is consistent with zero.
We remark that taking N data with a volatility σ means to take $\sigma_i = \sigma\sqrt{\frac{2\pi}{N}}$ when drawing random numbers in (12).
We can now calculate the moments of the Σ-coefficients.
We know that:

$$\boldsymbol{E}\left[a_k^2(du), b_k^2(du)\right] = \sigma_k^2$$

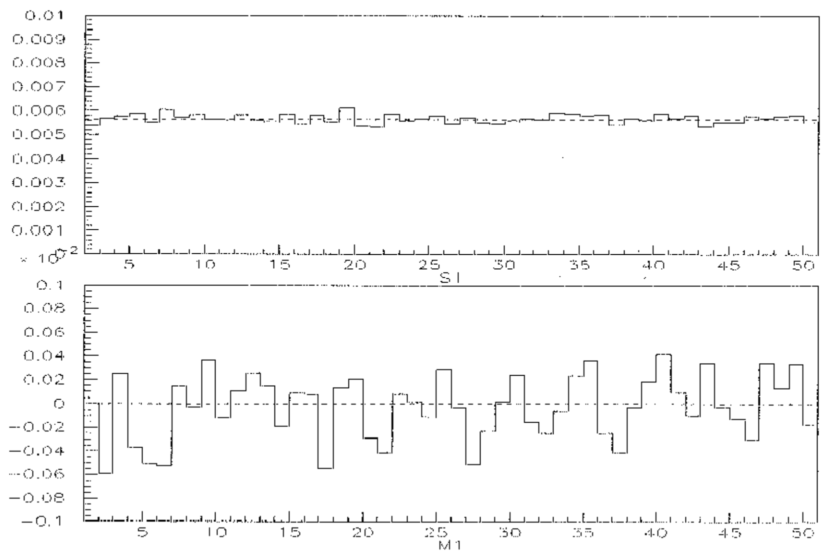

FIGURE 1: Moments of a_k for $N = 100$, $\sigma = 0.01$: for $1 \leq k \leq 50$ (a) Mean μ_k, it is consistent with zero; (b) Variance σ_k; it is consistent with $\sigma/\sqrt{\pi}$

$$Var[a_k^2(du), b_k^2(du)] = 2\sigma_k^4$$

Starting from these equation, we get:

$$\boldsymbol{E}\left[a_0(\Sigma)\right] = \sigma^2 \tag{13}$$

$$Var[a_0(\Sigma)] = 2\frac{\sigma^4}{N} \tag{14}$$

To estimate the moments of $a_k(\Sigma)$,$b_k(\Sigma)$ we will make the assumption that the coefficients $a_k(du)$,$b_k(du)$ are indipendent, which is true if σ is constant but should be checked in any other case. Making this assumption we readily obtain:

$$\boldsymbol{E}\left[a_k(\Sigma), b_k(\Sigma)\right] = 0 \tag{15}$$

$$Var[a_k(\Sigma), b_k(\Sigma)] = 8\frac{\sigma^4}{N} \tag{16}$$

To check the validity of such formulas one can use again Monte Carlo data. So we can see that the coefficients of σ are again gaussian variables, with the first two moments following formulas (13),(14),(15) and (16).
Eventually, using the above results we can give an estimate of the precision of the algorithm. We remark that if you are interested only in the mean

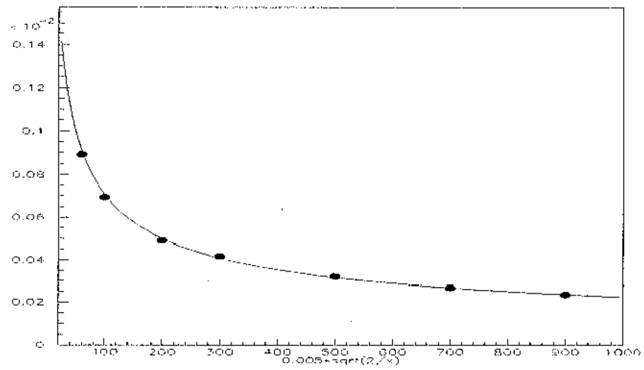

FIGURE 2: Dots: variance if the distribution obtained on the MC sample for different choices of N. Line: expected variance as a function of N: $\Delta\Sigma/\Sigma = \sqrt{2/N}$; it gives the correct explanation of the observed function

volatility of the interval chosen, then we have $< \Sigma >_t \; = a_0$, which will give a precision of $\simeq \sqrt{2}/\sqrt{N}$, equal to the precision of the classical case (figure 2).

5.2 $\sigma(t)$ = piecewise constant

In the following we will choose as $\sigma(t)$:

$$\sigma(t) = \begin{cases} \sigma_1 & 0 \le t \le \pi \\ \sigma_2 & \pi < t \le 2\pi \end{cases} \tag{17}$$

With this choice we want to check if the algorithm can reveal volatility variations *inside* the time window.
This time, in the perfect world scenario we should have:

$$a_k(du), b_k(du) \sim \mathcal{N}\Big(0, \sqrt{\frac{\sigma_1^2 + \sigma_2^2}{2\pi}}\Big) \tag{18}$$

As before, we find the following equation for σ_k:

$$\sigma_k = \sqrt{\frac{\sigma_1^2 + \sigma_2^2}{2\pi}} \tag{19}$$

It means that formulas (13),(14),(16) remain the same with $\sigma = \sqrt{(\sigma_1^2 + \sigma_2^2)/2}$. It is enough to draw a conclusion. Indeed, in order to reproduce $\Sigma(t)$ as in (17) we should have:

$$\begin{aligned} a_k(\Sigma) &= 0 \\ b_k(\Sigma) &= \frac{1}{k\pi}(\sigma_2^2 - \sigma_1^2)((-1)^k - 1) \end{aligned} \tag{20}$$

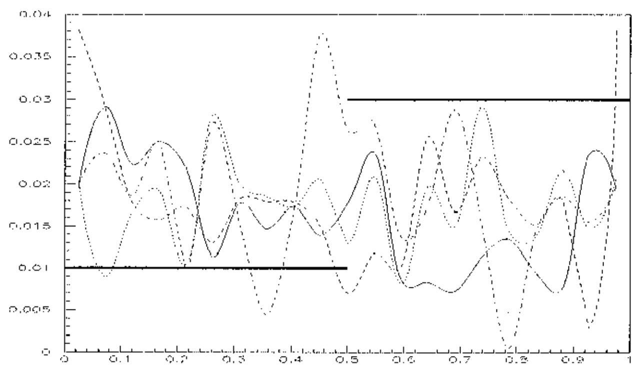

FIGURE 3: Four different volatility reconstructions; the generated one is the bold line. The algorithm is unable to reconstruct the volatility inside the interval

so that the largest coefficient is $b_1 = \frac{-2}{\pi}(\sigma_2^2 - \sigma_1^2)$. But the variance of b_1 (16) is larger than its expected mean. For example, if we have $\sigma_1 = 0.01$, $\sigma_2 = 0.03$, which is an unusual case because actually the difference in a time window will be even smaller, we have from (16) a standard deviation of $6 \cdot 10^{-4}$, and from (20) an expected value of $6.34 \cdot 10^{-4}$. For larger k, the estimate of b_k will be even poorer. This makes us understand that it is nearly impossible to reconstruct $\Sigma(t)$ for any t, since from a unique realization of the market we cannot hope to obtain the Fourier coefficients with the needed precision. This fact is illustrated in figure 3, where five different reconstructions of the same generated Monte Carlo sequence are shown.

5.3 Multivariate analysis

To test if the algorithm works well with multi-variate analysis we generate two Monte Carlo process with a given degree of correlation ρ. This is easily done in the following way: we choose two random numbers, ϵ_1 and ϵ_2 with distribution $\mathcal{N}(0, \sigma)$ and then transform them as:

$$\begin{aligned} r_1 &= \epsilon_1 \\ r_2 &= \rho\epsilon_2 + \sqrt{1-\rho^2}\epsilon_1 \end{aligned} \tag{21}$$

As can be easily checked, r_1 and r_2 are still normally distributed with correlation ρ.

Then we test the algorithm on thr Monte Carlo series, and the result is shown in figure 4 for $\rho = 0.5$. The distribution is the same as the one expected from the classical way of estimating the correlation, as in the univariate case. The

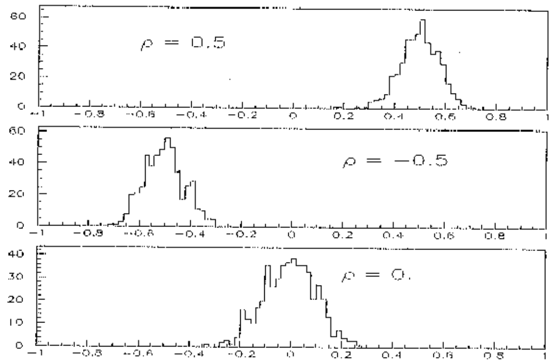

FIGURE 4: Distibution of measured correlation with the Fourier algorithm on three Monte Carlo simulations with generated correlation $0.5, -0.5, 0$.

approximate formula for the error is:

$$\Delta\rho = (1-\rho^2)\sqrt{\frac{1}{N}} \tag{22}$$

where N is the number of data used.

5.4 Conclusions

From the above analysis, we have that given a time window, we can compute the mean of Σ on such an interval with the same precision as the variance of returns, and this is done through:

$$\sqrt{<\Sigma>} = a_0 \tag{23}$$

with:

$$\frac{\Delta\Sigma_{ii}}{\Sigma_{ii}} = \sqrt{\frac{2}{N}} \tag{24}$$

where N is the number of data used, and the error on correlations is given by formula (22).

6 Results on real data

We applied the method to a time series of daily data of Dow Jones Industrial and Dow Jones Transportation from 1896 ro 1998 (28000 data). We divided

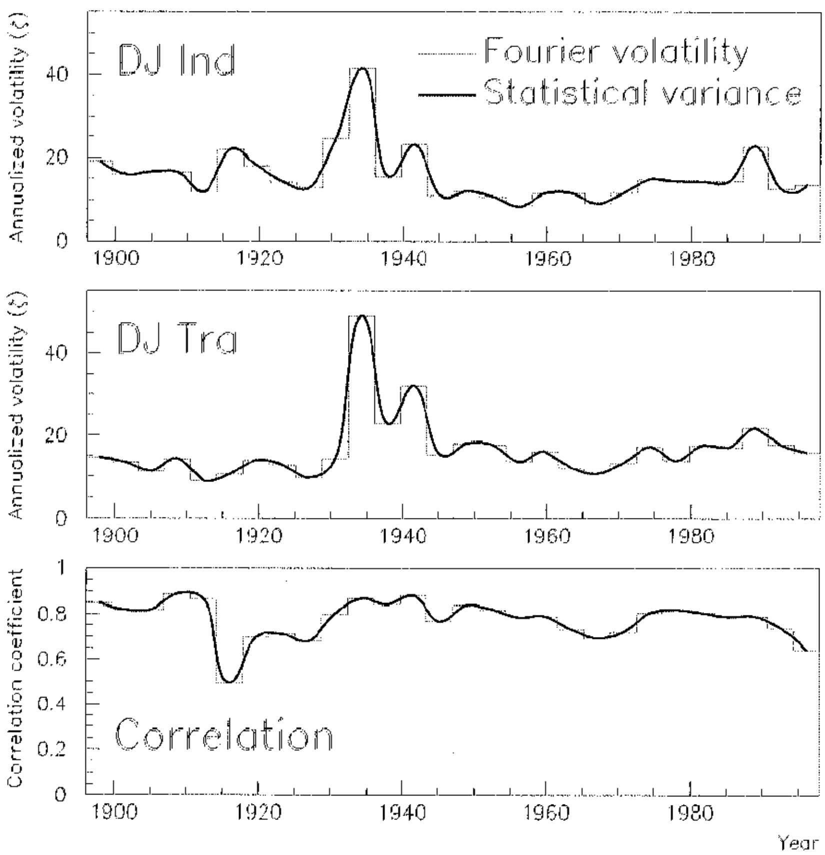

FIGURE 5: Volatility Matrix for two indexes: Dow Jones Industrial and Dow Jones Transportation. The agreement is quite good.

this sample into 28 periods of 1000 data. The result is shown in figure 5; the comparison is very good.

7 High Frequency Calibration

When we go in the high frequency regime, we encounter the difficulty that time intervals are not equally spaced. However, the Fourier algorithm, being based on an integration procedure instead of a differentiation one, should provide the necessary robustness to address this point. We will show this is

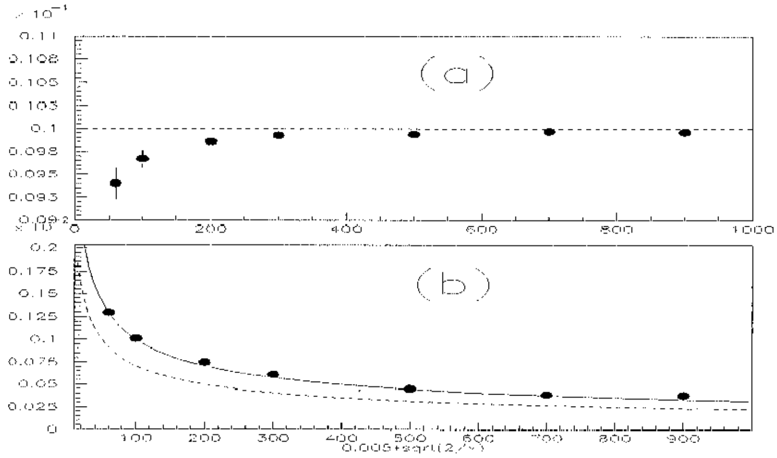

FIGURE 6: (a) Dots: Mean of the distribution obtained for different N; the dashed line is the generated σ. (b) Dots: Variance of the distribution obtained for different N. Dashed line: expected variance, $\Delta\sigma/\sigma = \sqrt{2/N}$. Solid line: expected variance times $\sqrt{2}$; it approximately explains the observed variance.

true.

An other problem that one has to face when trying to perform a multivariate analysis on high frequency data, is that they are not synchronous, e.g. the price of two stocks changes at different times.
Again, we will resort to Monte Carlo analysis to provide more insight on the Fourier method. In this case one has to be careful, because choosing the Monte Carlo model means having in mind the market model, and results might be influenced by such a choice.
We will choose the following approach to mimic HF data: at first we will choose a time scale corresponding to the smallest time interval between trades (one second), then reproduce the Monte Carlo model with the method depicted in equation (21) using as the fixed time interval such a time scale; then we will choose the trade times randomly from an exponential distribution with a given decay time τ (say 5 seconds).
Figure 6 shows the results obtained on the Monte Carlo simulation of high frequency data; for different choices of N, mean and variance of the obtained distributions are shown. The mean is somewhat underestimated of a few percents, expecially when N is small. The variance is larger of a factor $\sqrt{2}$ than the previous case. No dipendence on τ has been observed, as expected.
The syncronization effect can be shown in figure 7. We generate two high frequency series with correlation coefficient 0.5 and $\tau = 5$. If we apply the Fourier algorithm to all data, we get the solid curve: it exhibits positive

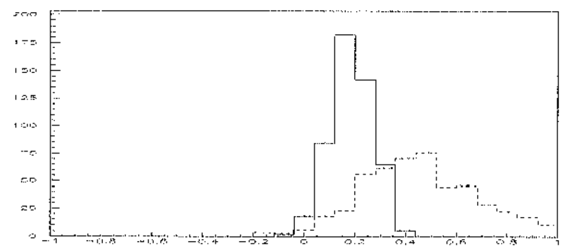

FIGURE 7: Syncronization effect: the solid curve is the measured correlation distribution when taking all data; the dashed curve when taking only syncronized data. The widening of such a distribution is due to the loss in statistics.

correlation but it is less than the generated one. If we apply Fourier analysis only to data points which occurr at the same time, we get the dashed curve, which has the right correlation (but with a larger variance due to loss in statistics).

This means that when reckoning correlations, only that data which come in the same time are meaningful, not syncronous data points cannot fully reveal correlations. This is, however, only a limitation in statistics: the fact that high frequency data are not equally spaced does not affect the power of the algorithm.

We have to remark that also for correlations an extra $\sqrt{2}$ term appears in the variance (22).

7.1 Conclusions

Also for high frequency data, given a time window, we can compute the mean of Σ on such an interval computing a_0, with the precision

$$\frac{\Delta\Sigma_{ii}}{\Sigma_{ii}} = \sqrt{\frac{2}{N}}\sqrt{2} \tag{25}$$

where N is the number of data used, and the error on correlations is given by formula (22) multiplied by $\sqrt{2}$.

8 Results on real data

We applied the method to high frequency data registered at the New York Stock Exchange for ten stocks in the month of January, 1995. The stocks are

General Electrics	AA
J.P. Morgan	Coca-Cola
Merril Lynch	Mobil
Exxon	Mercks
Pepsi	AT & T

TABLE 1: The ten stocks traded at NYSE used in this analysis in the month of January 1995

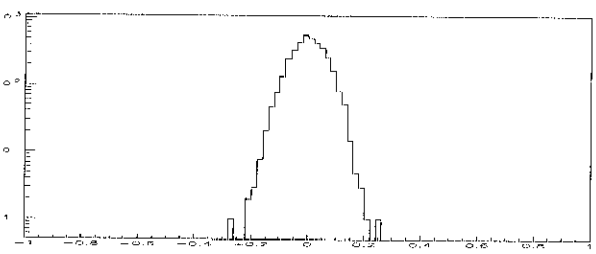

FIGURE 8: Cumulative distributions of the correlations between the 10 stocks in 84 time windows. The correlations are not economically significant.

listed in table 1.

We divided any trading day into 4 periods, anyone 5800 seconds long. Within each of such periods, for all the 21 trading days in the month, we computed with the Fourier Analysis the cross correlations between the 10 stocks. The distribution of such correlations is shown in figure 8. Correlations are very close to zero when the frequency is very high. It is a fact well known in literature: when time goes to zero correlations go to zero too (Epps effect) [5]. We confirm this result.

We also employed the Fourier method to compute the autocorrelation coefficient of every stocks. Figure 9 shows the result for J.P. Morgan on January 3^{th}, fitted with an exponential of decay time $\tau = 5$ minutes. Table 8 shows the autocorrelation decay time obtained for the other stocks in the same day. They agree with the ones already observed in financial literature, confirming a positive autocorrelation function in the first $5 - 10$ minutes. Moreover we can observe that the autocorrelation function remains positive for a longer time when the stocks are less liquid.

Also lagged correlations between stocks can be measured with this times. Figure 10 shows such a lagged correlation between Exxon and Mobil on January 3^{th} 1995. The bump in the upper figure between 50 seconds shows

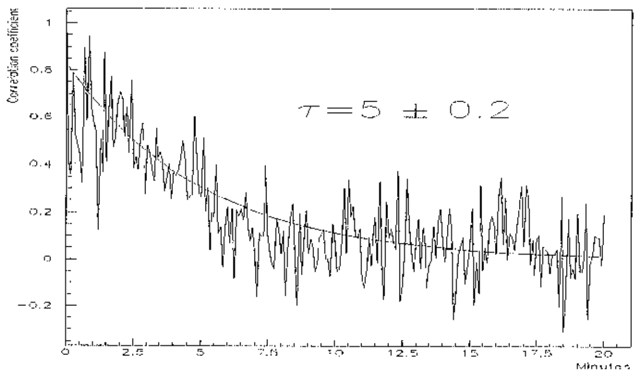

FIGURE 9: Autocorrelation function of J.P. Morgan on january 3^{th}. It is well fitted with an exponential with decay time $\simeq$ 5 minutes.

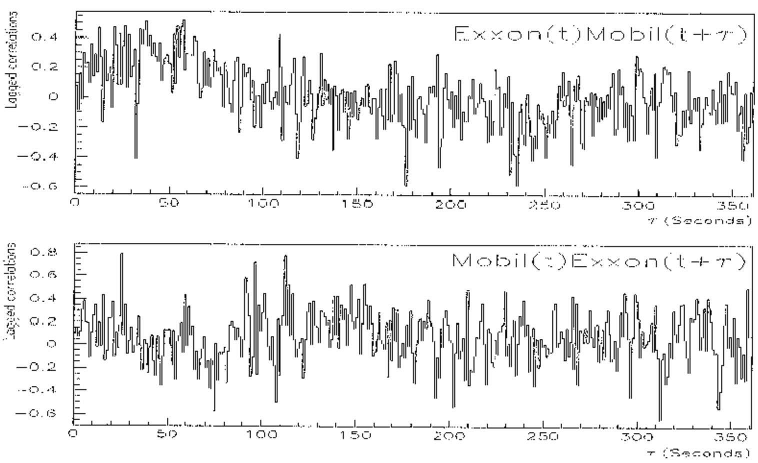

FIGURE 10: Upper figure: correlation between Exxon and Mobil (lagged) on January 3^{th} 1995. It is significantly positive in the first minute. Lower figure: correlation between Mobil and Exxon (lagged). No significant correlation appears.

that that day Exxon price changes influenced Mobil price changes with a delay of about 1 minute. The opposit is not true, as seen in the lower figure.

Stocks	Decay time (minutes)	Number of trades
AA	14.4 ± 0.2	1130
GE	1.5 ± 0.2	2467
JPM	5.0 ± 0.2	1369
KO	7.8 ± 1.0	1407
MER	13 ± 0.7	732
MOB	7.2 ± 0.2	1255
MRK	1.2 ± 0.2	1922
PEP	5.8 ± 0.2	1326
T	6 ± 0.2	1717
XON	0.85 ± 0.05	1416

TABLE 2: Decay time of the autocorrelation coefficients of the ten stocks considered

9 Factor Analysis

10 Conclusions

References

[1] Andersen, T. Bollerslev, T., Diebold, F. and Labys, P. (1999) The Distribution of Exchange Rate Volatility. NBER Working Paper, 6961.

[2] Ball, Torous, W. (1984) The Maximum Likelihood Estimation of security Price Volatility: Theory, Evidence, and Application to Option Pricing, *Journal of Business*, 57: 97-112.

[3] Campbell J., Lo A. and MacKinlay (1997), *The econometrics of financial markets.* Princeton University Press, Princeton University Press.

[4] de Jong, F. and Nijman, T. (1997) *High frequency analysis of lead-lag relationships between financial markets.* Journal of Empirical Finance, 4: 259-277.

[5] Epps, T. (1979) Comovements in Stock Prices in the Very Short Run. *Journal of the American Statistical Association*, 74: 291-298.

[6] Guillaume, D., Dacorogna, M., Davé, R., Muller, U., Olsen,R. and Pictet, O. (1997) From the bird's eye to the microscope: A survey of new stylized facts of the intra-daily foreign exchange markets. *Finance and Stochastics.*

[7] Lo, A. and MacKinlay, C. (1990) An Econometric Analysis of Nonsynchronous Trading, *Journal of Econometrics*, 45: 181-211.

[8] Lundin, M., Dacorogna, M. and Muller, U. (1999) Correlation of high-frequency financial time series. *Financial Markets Tick by Tick*, Lequeux ed..

[9] P. Malliavin, M. Mancino, *Fourier series method in Statistical Determination of Multivariate Volatilities, Infinitesimal Principal Component Analysis and a geometric vision of Market evolution*, Dimadefas, Firenze, December 1999, 99-12.

[10] M.Priestley, *Spectral Analysis and Time Series*, Academic Press, 1989

[11] W.Schwert (1989) Why Does Stock Market Volatility Change over Time? *Journal of Finance*, XLIV: 1115-1153.