Report n. 180

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Pisa, June 2000

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June 2000

Abstract The aim of this paper is to deep on the study of first order necessary optimality conditions of the minimum principle type by means of the image space approach; several results known in literature will be generalized to the vector case and extended for nondifferentiable functions; the given approach allow us to state, even in the scalar case and under differentiability hypothesis, some new results.

Keywords Vector Optimization, Minimum Principle Optimality Conditions, Image Space.
AMS - 2000 Math. Subj. Class. 90C29, 90C46, 90C30
JEL - 1999 Class. Syst. C61, C62

1 Introduction

The first order necessary optimality conditions of the minimum principle type are usually related in finite dimensional spaces to scalar optimization problems of the following kind ([1]):

$$\begin{cases} \max\ f(x) \\ g(x) \in \Re^m_+, \\ h(x) = 0 \\ x \in X \subseteq A \subseteq \Re^n \end{cases}$$

where $f(x)$ is a scalar differentiable function and $g(x)$ and $h(x)$ are vector differentiable functions. Under some particular hypothesis it has been

*This paper has been partially supported by M.U.R.S.T.; some very preliminary results of this research appeared in [12, 13].

[1] Minimum principle optimality conditions are used also in infinite dimensional spaces for instance regarding to optimal control theory [14, 18, 23, 25].

proved that if a feasible point x_0 is an optimal solution of the problem then:

$$[\alpha_f \nabla f(x_0) + \alpha_g^T J_g(x_0) + \alpha_h^T J_h(x_0)](x - x_0) \leq 0 \quad \forall x \in X$$

where $\alpha_f \geq 0$, $\alpha_g \in \Re_+^m$, $\alpha_h \in \Re^p$, $(\alpha_f, \alpha_g, \alpha_h) \neq 0$, are multipliers. The main feature of these conditions is that they are related to problems where the set X is not necessarily open and to a point x_0 which is not necessarily an interior point of X; for this reason they have been also called "generalized Lagrange multiplier rule", since they extend the Fritz John criterion:

$$[\alpha_f \nabla f(x_0) + \alpha_g^T J_g(x_0) + \alpha_h^T J_h(x_0)] = 0$$

Obviously, the latter condition follows from the former one when X is open, while when X is not open the Fritz John condition does not hold in general. In this paper we are going to deep on the study of the first order necessary optimality conditions of the minimum principle type stated in [3, 22, 20] by means of the image space approach [5, 6, 7, 8, 9, 10, 11]; this approach will let us to generalize the results to vector optimization problems and to extend the conditions for nondifferentiable functions. In particular, in Section 3 the efficiency of a feasible point will be characterized in the image space with no use of assumptions regarding the functions of the problem; in Section 4 some necessary optimality conditions in the image space will be stated under subdifferentiability hypothesis; in Section 5 several necessary optimality conditions of the minimum principle type will be studied under subdifferentiability hypothesis analyzing in the decision space the conditions stated in the image space; finally in Section 6 the necessary conditions will be specified under differentiability assumptions, obtaining results more general than the ones known in the literature.

2 Definitions and preliminary results

The image space approach, originally suggested by Hestenes [19], will allow us to extend and generalize to the vector case the known results concerning the optimality conditions of the "minimum principle" type for scalar optimization problems.

From now on, we will consider the following vector optimization problem:

$$P : \begin{cases} C - \max\ f(x) \\ g(x) \in V, \\ h(x) = 0 \\ x \in X \subseteq A \subseteq \Re^n \end{cases} \equiv \begin{cases} C - \max\ f(x) \\ g(x) \in V, \\ x \in (X \cap S) \subseteq A \subseteq \Re^n \\ S = \{x \in A : h(x) = 0\} \end{cases} \tag{2.1}$$

where $f : A \longrightarrow \Re^s$, $g : A \longrightarrow \Re^m$ and $h : A \longrightarrow \Re^p$ are vector valued functions, with A open set, and $C \subset \Re^s$ and $V \subset \Re^m$ are closed convex cones with nonempty interior. Note that the set X is not necessarily open and that not necessarily $\text{Int}(X) \neq \emptyset$. From now on, we will study optimality conditions for a feasible point $x_0 \in X$ which will be assumed, without loss of generality, to be binding all the constraints, that is to say that $g(x_0) = 0$. The feasible point $x_0 \in X$ is said to be a *local efficient point* if there exists a suitable neighbourhood I_{x_0} of x_0 such that:

$$\nexists y \in I_{x_0} \cap X \text{ such that } f(y) \in f(x_0) + C^0, \;\; g(y) \in V, \;\; h(y) = 0 \qquad (2.2)$$

where $C^0 = C \setminus \{0\}$. For a sake of simplicity we will sometimes use also the following function:

$$F : A \to \Re^{s+m+p} \text{ such that } F(x) = (f(x), g(x), h(x))$$

which allow us to say that $x_0 \in X$ is a local efficient point if and only if there exists a suitable neighbourhood I_{x_0} of x_0 such that:

$$\nexists y \in I_{x_0} \cap X \text{ such that } F(y) \in F(x_0) + (C^0 \times V \times 0) \qquad (2.3)$$

The study of optimality conditions in the image space will be carried on by means of the *Bouligand Tangent cone to X at x_0*, denoted with $T(X, x_0)$ and defined as follows:

$$\begin{aligned} T(X, x_0) \;\; = \;\; & \{x \in \Re^n : \exists \{x_k\} \subset X, x_k \to x_0, \exists \{\lambda_k\} \subset \Re^{++}, \lambda_k \to +\infty, \\ & x = \lim_{k \to +\infty} \lambda_k (x_k - x_0)\}. \end{aligned}$$

In the following we will denote with $U(X, x_0)$ any subcone of $T(X, x_0)$. Particular subcones of $T(X, x_0)$ are the so called *cone of feasible directions to X at x_0* ([2]), denoted with $F(X, x_0)$, and the *cone of interior directions to X at x_0*, denoted with $I(X, x_0)$ (see [3, 16, 17]).

The optimality conditions in the decision space will be stated by means of separating theorems and the use of multipliers, hence the concept of

[2]Let $X \subseteq \Re^n$ be a nonempty set and let $x_0 \in \text{Cl}(X)$. The *cone of feasible directions to X at x_0* $F(X, x_0)$ and the *cone of interior directions to X at x_0* $I(X, x_0)$ are defined as follows:

$$\begin{aligned} F(X, x_0) \;\; &= \;\; \{x \in \Re^n : \exists \delta > 0 \text{ such that } x_0 + \lambda x \in X \quad \forall \lambda \in (0, \delta]\}; \\ I(X, x_0) \;\; &= \;\; \{x \in \Re^n : \exists \epsilon > 0, \exists \delta > 0 \text{ such that } \lambda \in (0, \delta), \|y - x\| < \epsilon \text{ imply} \\ & \qquad x_0 + \lambda y \in X\}. \end{aligned}$$

Remind that $I(X, x_0) \subseteq \text{Int}(F(X, x_0)) \subseteq F(X, x_0) \subseteq \text{Cl}(F(X, x_0)) \subseteq T(X, x_0)$.

positive polar of a cone K, denoted with K^+, will be used. With this aim, a key tool for the proofs of the next sections is the following.

Theorem 2.1 *Let C_i, $i = 1, \ldots, n$, be cones and let $K_{C_i} \subseteq C_i$, $i = 1, \ldots, n$, be any subcones of C_i, $i = 1, \ldots, n$, respectively. Then:*

$$(K_{C_1} \times \ldots \times K_{C_n})^+ = (K_{C_1}^+ \times \ldots \times K_{C_n}^+). \tag{2.4}$$

Proof It is sufficient to prove this property for $n = 2$. We firstly prove that $(K_{C_1}^+ \times K_{C_2}^+) \subseteq (K_{C_1} \times K_{C_2})^+$; assuming $(\alpha_1, \alpha_2) \in (K_{C_1}^+ \times K_{C_2}^+)$ it yields that $\alpha_1^T c + \alpha_2^T v \geq 0$ $\forall c \in K_{C_1}$ and $\forall v \in K_{C_2}$ so that $(\alpha_1, \alpha_2) \in (K_{C_1} \times K_{C_2})^+$. Let us prove now that $(K_{C_1} \times K_{C_2})^+ \subseteq (K_{C_1}^+ \times K_{C_2}^+)$ and with this aim assume $(\alpha_1, \alpha_2) \in (K_{C_1} \times K_{C_2})^+$. Suppose by contradiction that $\alpha_1 \notin K_{C_1}^+$ $[\alpha_2 \notin K_{C_2}^+]$ so that $\exists \bar{c} \in K_{C_1}$ $[\exists \bar{v} \in K_{C_2}]$ such that $\alpha_1^T \bar{c} < 0$ $[\alpha_2^T \bar{v} < 0]$; since K_{C_1} $[K_{C_2}]$ is a cone then $\lambda \bar{c} \in K_{C_1}$ $[\lambda \bar{v} \in K_{C_2}]$ $\forall \lambda > 0$ so that, given $v \in K_{C_2}$ $[c \in K_{C_1}]$, for $\lambda > 0$ great enough we have $\alpha_1^T(\lambda \bar{c}) + \alpha_2^T v < 0$ $[\alpha_1^T c + \alpha_2^T(\lambda \bar{v}) < 0]$ and this contradicts that $(\alpha_1, \alpha_2) \in (K_{C_1} \times K_{C_2})^+$. □

Finally, it's worth reminding some known minimum principle optimality conditions which are going to be generalized in the next sections.

Theorem 2.2 **[20, 22]** *Consider problem P with a scalar objective function $f(x)$, suppose $f(x)$, $g(x)$ and $h(x)$ to be differentiable at the feasible point $x_0 \in X$ and suppose the Jacobian matrix $J_h(x_0)$ to be continuous at x_0. Suppose finally that:*

$$X \text{ is convex, with } \mathrm{Int}(X) \neq \emptyset$$

If $x_0 \in X$ is a local maximizer then $\exists \alpha_f \geq 0$, $\exists \alpha_g \in V^+$, $\exists \alpha_h \in \Re^p$, $(\alpha_f, \alpha_g, \alpha_h) \neq 0$, such that:

$$[\alpha_f \nabla f(x_0) + \alpha_g^T J_g(x_0) + \alpha_h^T J_h(x_0)](x - x_0) \leq 0 \quad \forall x \in X$$

or equivalently:

$$[\alpha_f \nabla f(x_0) + \alpha_g^T J_g(x_0) + \alpha_h^T J_h(x_0)]v \leq 0 \quad \forall v \in F(X, x_0)$$

Theorem 2.3 **[3]** *Consider problem P with a scalar objective function $f(x)$, suppose $f(x)$, $g(x)$ and $h(x)$ to be differentiable at the feasible point $x_0 \in X$ and suppose the Jacobian matrix $J_h(x_0)$ to be continuous at x_0. Suppose finally that:*

$I(X, x_0)$ is a convex cone

If $x_0 \in X$ is a local maximizer then $\exists \alpha_f \geq 0$, $\exists \alpha_g \in V^+$, $\exists \alpha_h \in \Re^p$, $(\alpha_f, \alpha_g, \alpha_h) \neq 0$, such that:

$$[\alpha_f \nabla f(x_0) + \alpha_g^T J_g(x_0) + \alpha_h^T J_h(x_0)]v \leq 0 \quad \forall v \in I(X, x_0)$$

Note that both the previous results are based on a sort of convexity hypothesis regarding to problem P, since the convexity of the set X or of the cone $I(X, x_0)$ is required.

3 Approach in the Image Space: the nonsmooth case

In this section the efficiency of x_0 will be characterized without the use of any subdifferentiability hypothesis. Following an approach similar to the one used in [5, 6, 7, 8, 9, 10, 11], we introduce the following subset of the Bouligand tangent cone at $F(x_0)$ in the image space:

$$\begin{aligned} T_1 &= \{t \in \Re^{s+m+p} : \exists \{x_k\} \subset X, x_k \to x_0, h(x_k) = 0, \exists \{\lambda_k\} \subset \Re^{++}, \\ &\quad \lambda_k \to +\infty, t = \lim_{k \to +\infty} \lambda_k (F(x_k) - F(x_0))\}. \end{aligned} \tag{3.1}$$

We will see that, by means of the cone T_1, it is possible to state the following optimality conditions in the image space which extend the ones stated in [6, 7, 8, 10] with respect to problems P having an open set X or having x_0 belonging to the interior of X.

Theorem 3.1 *Consider problem P. If $x_0 \in X$ is a local efficient point then:*

$$T_1 \cap (\mathrm{Int}(C) \times \mathrm{Int}(V) \times 0) = \emptyset \tag{3.2}$$

Proof We will prove the result by contradiction. Suppose that $\exists t^* \in T_1 \cap (\mathrm{Int}(C) \times \mathrm{Int}(V) \times 0)$; then $\exists \{x_k\} \subset X$, $x_k \to x_0$, $h(x_k) = 0$, $\exists \{\lambda_k\} \subset \Re^{++}$, $\lambda_k \to +\infty$, such that $t^* = \lim_{k \to +\infty} \lambda_k (F(x_k) - F(x_0))$.
Being $t^* \in (\mathrm{Int}(C) \times \mathrm{Int}(V) \times 0)$ and being $h(x_k) = 0$ $\forall k$ then for a known limit theorem:

$$\exists \bar{k} > 0 \text{ such that } \lambda_k (F(x_k) - F(x_0)) \in (\mathrm{Int}(C) \times \mathrm{Int}(V) \times 0) \quad \forall k > \bar{k}$$

so that, being $\lambda_k > 0$, $F(x_k) \in F(x_0) + (\mathrm{Int}(C) \times \mathrm{Int}(V) \times 0)$ $\forall k > \bar{k}$ and this contradicts the local efficiency of x_0. □

In the next theorem we will show that it is possible to fully characterize in the image space the optimality of x_0.

Theorem 3.2 *Consider problem P. The point $x_0 \in X$ is a local efficient point if and only if the following condition holds:*

> $\forall t \in T_1 \cap (C \times V \times 0)$, $t \neq 0$, *and* $\forall \{x_k\} \subset X$, $x_k \to x_0$, $h(x_k) = 0$, *such that* $\exists \{\lambda_k\} \subset \Re^{++}$, $\lambda_k \to +\infty$, *with* $t = \lim_{k \to +\infty} \lambda_k (F(x_k) - F(x_0))$, *there exists an integer* $\bar{k} > 0$ *such that:*
>
> $$F(x_k) \notin F(x_0) + (C^0 \times V \times 0) \ \forall k > \bar{k}.$$

Proof $\Rightarrow$) If x_0 is a local efficient point then, for (2.3), $\forall \{x_k\} \subset X$, $x_k \to x_0$, $h(x_k) = 0$, there exists an integer $\bar{k} > 0$ such that $F(x_k) \notin F(x_0) + (C^0 \times V \times 0) \ \forall k > \bar{k}$, and this is true also for particular sequences such that $t = \lim_{k \to +\infty} \lambda_k (F(x_k) - F(x_0))$ with $t \in T_1 \cap (C \times V \times 0)$.
$\Leftarrow$) We will prove the result by contradiction. Suppose that $x_0 \in X$ is not a local efficient point, then by means of (2.3) $\exists \{x_k\} \subset X$, $x_k \to x_0$, such that $F(x_k) \in F(x_0) + (C^0 \times V \times 0) \ \forall k$, so that in particular $h(x_k) = 0 \ \forall k$. Let us consider now the sequence $\{d_k\} \subset \Re^{s+m+p}$ with $d_k = \frac{F(x_k) - F(x_0)}{\|F(x_k) - F(x_0)\|}$; since the unit ball is a compact set, we can suppose (substituting $\{d_k\}$ with a suitable subsequence, if necessary) that $\lim_{k \to +\infty} d_k = t^* \neq 0$, $t^* \in T_1$. On the other hand, $d_k = \frac{F(x_k) - F(x_0)}{\|F(x_k) - F(x_0)\|} \in (C^0 \times V \times 0)$ so that its limit $t^* \in (C \times V \times 0)$. It then results that $t^* \in T_1 \cap (C \times V \times 0)$, $t \neq 0$, and this contradicts the hypothesis since $t^* = \lim_{k \to +\infty} \frac{F(x_k) - F(x_0)}{\|F(x_k) - F(x_0)\|}$ and $F(x_k) \in F(x_0) + (C^0 \times V \times 0) \ \forall k$. □

Directly from Theorem 3.2 we can state the following sufficient optimality condition.

Corollary 3.1 *Consider problem P. If the following condition holds then $x_0 \in X$ is a local efficient point:*

$$T_1 \cap (C \times V \times 0) = \{0\} \tag{3.3}$$

4 Approach in the Image Space: the subdifferentiable case

The previous optimality conditions, based on the cone T_1, are not easy to be applied being T_1 not trivial to be determined. Some more useful optimality conditions in the image space, involving both the inequality and the equality constraints, can be stated under the following subdifferentiability assumptions, which will be assumed from now on:

Subdifferentiability Hypothesis

Functions f, g and h are Hadamard directionally differentiable at $x_0 \in X$ (3).

See [15] for the definition and a complete study of Hadamard directionally differentiable functions (see also [1, 2, 24, 27]). The necessary optimality conditions in the image space will be stated by means of the following cones:

$$\begin{aligned}
Ker_{\partial h} &= \{0\} \cup \{d \in \Re^n : d = \mu v, \frac{\partial h}{\partial v}(x_0) = 0, \mu \geq 0, \|v\| = 1, v \in \Re^n\} \\
Ker^C_{\partial h} &= \Re^n \setminus Ker_{\partial h} \\
K_L &= \{t \in \Re^{m+s+p} : t = \mu(\frac{\partial f}{\partial v}(x_0), \frac{\partial g}{\partial v}(x_0), \frac{\partial h}{\partial v}(x_0)), \mu \geq 0, \|v\| = 1, \\
&\quad v \in (T(X \cap S, x_0) \cup Ker^C_{\partial h})\} \\
K_U &= \{t \in \Re^{m+s+p} : t = \mu(\frac{\partial f}{\partial v}(x_0), \frac{\partial g}{\partial v}(x_0), \frac{\partial h}{\partial v}(x_0)), \mu \geq 0, \|v\| = 1, \\
&\quad v \in U(X, S, x_0) \subseteq (T(X \cap S, x_0) \cup Ker^C_{\partial h})\} \subseteq K_L
\end{aligned}$$

where $U(X, S, x_0)$ is any subcone of $T(X \cap S, x_0) \cup Ker^C_{\partial h}$ (4).

Remark 4.1 Note that in general it is:

$$T(X \cap S, x_0) \subseteq T(S, x_0) \subseteq Ker_{\partial h}$$

[3]Let $f : A \to \Re$, with $A \subseteq \Re^n$ open set. The limit:

$$\lim_{\lambda \to 0^+, h \to v} \frac{f(x_0 + \lambda h) - f(x_0)}{\lambda}$$

is called the *Hadamard directional derivative of* $f(x)$ *at* $x_0 \in A$ *in the direction* v; if this derivative exists and is finite for all v then $f(x)$ is *Hadamard directionally differentiable at* $x_0 \in A$. In order to verify the Hadamard directional derivability, remind that a function $f(x)$ is Hadamard directionally differentiable at x_0 (see [15]) if and only if its derivative $\frac{\partial f}{\partial v}(x_0) \stackrel{\text{def}}{=} \lim_{\lambda \to 0^+} \frac{f(x_0+\lambda v) - f(x_0)}{\lambda}$ is continuous as a function of direction and the function itself is *Dini uniformly directionally differentiable at* x_0 (hence directionally differentiable at x_0), that is to say that:

$$\lim_{\|v\| \to 0} \left| f(x_0 + v) - f(x_0) - \frac{\partial f}{\partial v}(x_0) \right| = 0$$

Remind also that if a function $f(x)$ is Hadamard directionally differentiable at x_0 then it is also continuous at x_0. A vector valued function $F : A \to \Re^m$ is Hadamard directionally differentiable at x_0 if all its components verify this property.

[4]Remind that in the literature [5, 6, 7, 8, 9, 10, 11] it has been defined with K_L the cone $K_L = \{t \in \Re^{s+m} : t = [J_f(x_0), J_g(x_0)]v, v \in \Re^n\}$, which is nothing but the image of $[J_f(x_0), J_g(x_0)]$.

In order to verify this property, firstly note that $T(X \cap S, x_0) \subseteq T(S, x_0)$ being $X \cap S \subseteq S$. Since $t = 0 \in T(S, x_0) \cap Ker_{\partial h}$ let us consider just $t \in T(S, x_0)$, $t \neq 0$; then $\exists \{x_k\} \subset S$, $x_k \to x_0$, $\exists \{\lambda_k\} \subset \Re^{++}$, $\lambda_k \to +\infty$, such that $t = \lim_{k \to +\infty} \lambda_k(x_k - x_0)$; we can also suppose (eventually substituting $\{x_k\}$ with a proper subsequence) that $v = \lim_{k \to +\infty} \frac{x_k - x_0}{\|x_k - x_0\|}$. Since $\{x_k\} \subset S$ it yields $h(x_0) = h(x_k) = 0$ $\forall k > 0$ so that, by means of the Hadamard directional differentiability of $h(x)$, we have:

$$0 = \lim_{k \to +\infty} \frac{h(x_k) - h(x_0)}{\|x_k - x_0\|} = \lim_{\gamma_k \to 0^+, d_k \to v} \frac{h(x_0 + \gamma_k d_k) - h(x_0)}{\gamma_k} = \frac{\partial h}{\partial v}(x_0)$$

where $\gamma_k = \|x_k - x_0\|$ and $d_k = \frac{x_k - x_0}{\|x_k - x_0\|}$, so that $v \in Ker_{\partial h}$.
Let us now prove that $t \in Ker_{\partial h}$ too. By means of the definition it results:

$$t = \lim_{k \to +\infty} \lambda_k(x_k - x_0) = \lim_{k \to +\infty} \lambda_k \|x_k - x_0\| \lim_{k \to +\infty} \frac{x_k - x_0}{\|x_k - x_0\|} = \mu v$$

where $\mu = \lim_{k \to +\infty} \lambda_k \|x_k - x_0\| \geq 0$ and $\|v\| = 1$. Being $Ker_{\partial h}$ a cone and being $v \in Ker_{\partial h}$ we then have that $t \in Ker_{\partial h}$. □

By means of the previously defined cones we are now able to state the following necessary optimality condition in the image space.

Theorem 4.1 *Consider Problem P; if the feasible point $x_0 \in X$ is a local efficient point then:*

$$K_L \cap (\text{Int}(C) \times \text{Int}(V) \times 0) = \emptyset \tag{4.1}$$

and for any cone $U(X, S, x_0) \subseteq (T(X \cap S, x_0) \cup Ker^C_{\partial h})$ it is:

$$K_U \cap (\text{Int}(C) \times \text{Int}(V) \times 0) = \emptyset \tag{4.2}$$

Proof We will prove condition (4.1) by contradiction. Suppose that there exists $t = (t_f, t_g, t_h) \in K_L \cap (\text{Int}(C) \times \text{Int}(V) \times 0)$, so that $\exists \mu > 0$, $\exists v \in (T(X \cap S, x_0) \cup Ker^C_{\partial h})$, $\|v\| = 1$, such that $t = \mu(\frac{\partial f}{\partial v}(x_0), \frac{\partial g}{\partial v}(x_0), \frac{\partial h}{\partial v}(x_0))$ with $(\frac{\partial f}{\partial v}(x_0), \frac{\partial g}{\partial v}(x_0), \frac{\partial h}{\partial v}(x_0)) \in (\text{Int}(C) \times \text{Int}(V) \times 0)$. Being $\frac{\partial h}{\partial v}(x_0) = 0$ then $v \in Ker_{\partial h}$ which implies that $v \notin Ker^C_{\partial h}$ and $v \in T(X \cap S, x_0)$. By means of the definition of $T(X \cap S, x_0)$ we then have that $\exists \{x_k\} \subset (X \cap S)$, $x_k \to x_0$, $\exists \{\lambda_k\} \subset \Re^{++}$, $\lambda_k \to +\infty$, such that $v = \lim_{k \to +\infty} v_k$ where $v_k = \lambda_k(x_k - x_0)$. Being functions f and g Hadamard directionally differentiable it results:

$$\lim_{k \to +\infty} \frac{f(x_k) - f(x_0)}{\frac{1}{\lambda_k}} = \lim_{k \to +\infty} \frac{f(x_0 + \frac{1}{\lambda_k} v_k) - f(x_0)}{\frac{1}{\lambda_k}} = \frac{\partial f}{\partial v}(x_0) \in \text{Int}(C)$$

and, in the same way:

$$\lim_{k \to +\infty} \frac{g(x_k) - g(x_0)}{\frac{1}{\lambda_k}} = \frac{\partial g}{\partial v}(x_0) \in \mathrm{Int}(V)$$

By means of a well known limit theorem it then exists $\bar{k} > 0$ such that $f(x_k) - f(x_0) \in \mathrm{Int}(C)$ and $g(x_k) - g(x_0) \in \mathrm{Int}(V)$ for any $k > \bar{k}$; this means that the sequence $\{x_k\} \subset (X \cap S)$, $x_k \to x_0$, is feasible for $k > \bar{k}$ and that x_0 is not a local efficient point, which is a contradiction.
Condition (4.2) follows directly from condition (4.1) being $U(X, S, x_0) \subsetneq (T(X \cap S, x_0) \cup Ker^C_{\partial h})$. □

Remark 4.2 For the sake of completeness, note that the previous results can be obtained as a corollary of Theorem 3.1.
Denoting with $B = \{t = (t_f, t_g, t_h) \in \Re^{s+m+p} : t_h \neq 0\}$ we have, directly from Theorem 3.1, that the efficiency of x_0 implies that:

$$(T_1 \cup B) \cap (\mathrm{Int}(C) \times \mathrm{Int}(V) \times 0) = \emptyset.$$

We now just have to verify that $K_L \subseteq (T_1 \cup B)$. Let $t = \mu \frac{\partial F}{\partial v}(x_0) \in K_L$, $v \in (T(X \cap S, x_0) \cup Ker^C_{\partial h})$, $\|v\| = 1$, $\mu \geq 0$; if $\mu = 0$ then $t = \mu \frac{\partial F}{\partial v}(x_0) = 0 \in T_1$ while if $\mu \neq 0$ and $v \in Ker^C_{\partial h}$ then $\frac{\partial h}{\partial v}(x_0) \neq 0$ and $t \in B$. Suppose now $\mu \neq 0$ and $v \in T(X \cap S, x_0)$, then $\exists \{x_k\} \subset X$, $x_k \to x_0$, $h(x_k) = 0$, such that $v = \lim_{k \to +\infty} \frac{x_k - x_0}{\|x_k - x_0\|}$; let also $\lambda_k = \|x_k - x_0\|^{-1}$. By means of the Hadamard directional differentiability of $F(x)$ at x_0 we have:

$$\frac{\partial F}{\partial v}(x_0) = \lim_{k \to +\infty} \frac{F(x_k) - F(x_0)}{\|x_k - x_0\|} = \lim_{k \to +\infty} \lambda_k (F(x_k) - F(x_0)) \in T_1;$$

being T_1 a cone it then follows that $t = \mu \frac{\partial F}{\partial v}(x_0) \in T_1$ too. □

Remark 4.3 Note that the necessary optimality conditions (4.1) and (4.2), stated in the image space, hold without any convexity hypothesis (on $U(X, S, x_0)$, $T(X \cap S, x_0)$, K_U or K_L) as it is shown in the following Example 4.1; this property points out that these necessary optimality conditions are more general than the ones stated in [3, 20, 22].

Example 4.1 Let us consider the following problem:

$$P : \{\max f(x_1, x_2) = x_1, g(x_1, x_2) = x_2 \geq 0, x \in X\}$$

where $X = X_1 \cup X_2 \cup X_3$ with:

$$\begin{aligned} X_1 &= \{(x_1, x_2) \in \Re^2 : x_1 + x_2 \geq 0, 2x_1 + x_2 \leq 0\}, \\ X_2 &= \{(x_1, x_2) \in \Re^2 : x_1 \leq 0, x_2 \leq 0\}, \\ X_3 &= \{(x_1, x_2) \in \Re^2 : x_1 + x_2 \geq 0, x_1 + 2x_2 \leq 0\} \end{aligned}$$

and $x_0 = (0,0)$; since there are no equality constraints involved in the problem, we can consider $S = \Re^2$. Note that $(\text{Int}(C) \times \text{Int}(V)) = \Re^2_{++}$ and $X = T(X \cap S, x_0) = K_L$ since $[J_f(x_0), J_g(x_0)]$ is equal to the identity matrix. The point x_0 is the global efficient point of the problem and the necessary optimality condition (4.1) is verified being $X \cap \Re^2_{++} = \emptyset$; on the other hand, none of the convexity hypothesis required in Theorems 2.2 and 2.3 hold, being X, $I(X, x_0)$, $T(X \cap S, x_0)$ and K_L not convex.

5 Optimality conditions in the Decision Space: the subdifferentiable case

In this section some necessary optimality conditions of the "minimum principle" type, involving both equality and inequality constraints, will be obtained specifying the results stated in the previous section.
Firstly note that the problem of Example 4.1 does not verify any of the thesis of the Theorems 2.2 and 2.3; this emphasize that in order to achieve a necessary optimality condition in the decision space, that is an optimality condition involving the directional derivatives and some multipliers, some additional hypothesis are required, so that the previously stated optimality conditions in the image space results to be more general than the forthcoming ones.
In our study we will use the following set $Im_{\partial h}$ which is nothing but the image of the directional derivative $\frac{\partial h}{\partial v}(x_0)$ with respect to the directions:

$$Im_{\partial h} = \{t \in \Re^p : t = \mu \frac{\partial h}{\partial v}(x_0), v \in \Re^n, \|v\| = 1, \mu \geq 0\}$$

The following theorem points out that, in order to state optimality conditions in the decision space, an additional hypothesis implicitly based on the separation between $(\text{Int}(C) \times \text{Int}(V) \times 0)$ and the convex hull of K_U or K_L, denoted with $\text{Co}(K_U)$ and $\text{Co}(K_L)$ respectively, is necessary.

Theorem 5.1 *Consider Problem P and let $U(X, S, x_0) \subseteq \Re^n$ be a cone. If the following condition holds:*

$$\text{Co}(K_U) \cap (\text{Int}(C) \times \text{Int}(V) \times 0) = \emptyset$$

then $\exists \alpha_f \in C^+$, $\exists \alpha_g \in V^+$, $\exists \alpha_h \in \Re^p$, $(\alpha_f, \alpha_g, \alpha_h) \neq 0$, *such that:*

$$\alpha_f^T \frac{\partial f}{\partial v}(x_0) + \alpha_g^T \frac{\partial g}{\partial v}(x_0) + \alpha_h^T \frac{\partial h}{\partial v}(x_0) \leq 0 \ \forall v \in U(X, S, x_0), v \neq 0.$$

Proof If $\text{Co}(Im_{\partial h}) \neq \Re^p$ there exists a support hyperplane for the convex cone $\text{Co}(Im_{\partial h})$, so that $\exists \alpha_h \in \Re^p$, $\alpha_h \neq 0$, such that $\alpha_h^T t \leq 0 \ \forall t \in \text{Co}(Im_{\partial h})$; this implies that $\alpha_h^T \frac{\partial h}{\partial v}(x_0) \leq 0 \ \forall v \in \Re^n$, $v \neq 0$. Assuming $\alpha_f = 0$ and $\alpha_g = 0$ we then have that:

$$\alpha_f^T \frac{\partial f}{\partial v}(x_0) + \alpha_g^T \frac{\partial g}{\partial v}(x_0) + \alpha_h^T \frac{\partial h}{\partial v}(x_0) \leq 0 \quad \forall v \in \Re^n, v \neq 0$$

and the thesis is proved. Suppose now $\text{Co}(Im_{\partial h}) = \Re^p$ and $\text{Co}(K_U) \cap (\text{Int}(C) \times \text{Int}(V) \times 0) = \emptyset$; by means of a well known separation theorem between convex sets, $\exists (\alpha_f, \alpha_g, \alpha_h) \in (\text{Int}(C) \times \text{Int}(V) \times 0)^+$, $(\alpha_f, \alpha_g, \alpha_h) \neq 0$, such that $(\alpha_f, \alpha_g, \alpha_h)^T t \leq 0 \ \forall t \in \text{Co}(K_U) \supseteq K_U$. By means of Theorem 2.1 it results $(\text{Int}(C) \times \text{Int}(V) \times 0)^+ = \text{Int}(C)^+ \times \text{Int}(V)^+ \times \Re^p$ and the thesis follows being C and V convex cones ([5]). □

Remark 5.1 Note that the proof of Theorem 5.1 points out that the case $\text{Co}(Im_{\partial h}) \neq \Re^p$ is a trivial one, since a minimum principle like condition holds with no additional hypothesis, such as convexity ones, optimality assumptions on x_0, regularity conditions for the problem.

The previous Theorem 5.1 allow us to introduce the following concept of regularity condition, which will be defined with respect to any of the subcones of $(T(X \cap S, x_0) \cup Ker^C_{\partial h})$.

Definition 5.1 Consider Problem P and a cone:

$$U(X, S, x_0) \subseteq (T(X \cap S, x_0) \cup Ker^C_{\partial h}).$$

A *first order U-regularity condition* is any condition verifying the following logical implication:

$$K_L \cap (\text{Int}(C) \times \text{Int}(V) \times 0) = \emptyset \implies \text{Co}(K_U) \cap (\text{Int}(C) \times \text{Int}(V) \times 0) = \emptyset \quad (5.1)$$

By means of the concept of U-regularity condition and the previously stated results, we are now able to prove the following necessary optimality condition in the decision space.

[5]Let C be a cone; it is known (see [26]) that $C^+ = \text{Cl}(C)^+$ so that $\text{Int}(C)^+ = \text{Cl}(\text{Int}(C))^+$ too. If C is a convex cone we also have (see [4]) that $\text{Cl}(\text{Int}(C)) = \text{Cl}(C)$ so that $\text{Int}(C)^+ = C^+$.

Theorem 5.2 *Consider Problem P; If the feasible point $x_0 \in X$ is a local efficient point and a first order U-regularity condition holds, with:*

$$U(X, S, x_0) \subseteq (T(X \cap S, x_0) \cup Ker^C_{\partial h}),$$

then $\exists \alpha_f \in C^+$, $\exists \alpha_g \in V^+$, $\exists \alpha_h \in \Re^p$, $(\alpha_f, \alpha_g, \alpha_h) \neq 0$, *such that:*

$$\alpha_f^T \frac{\partial f}{\partial v}(x_0) + \alpha_g^T \frac{\partial g}{\partial v}(x_0) + \alpha_h^T \frac{\partial h}{\partial v}(x_0) \leq 0 \quad \forall v \in \text{Cl}(U(X, S, x_0)), v \neq 0 \quad (5.2)$$

Proof The thesis follows from Theorem 4.1, Condition (5.1) and Theorem 5.1 since, being f, g and h Hadamard directionally differentiable at x_0, the directional derivatives $\frac{\partial f}{\partial v}(x_0)$, $\frac{\partial g}{\partial v}(x_0)$ and $\frac{\partial h}{\partial v}(x_0)$ are continuous as functions of direction. □

Example 4.1 points out that the optimality condition in the decision space expressed in Theorem 5.2 is less general (being based on a U-regularity condition) than the one stated in the image space, since assuming $U(X, S, x_0) = T(X \cap S, x_0)$ we have, even if $x_0 \in X$ is a local efficient point, that $\text{Co}(K_U) = \Re^2$ so that the U-regularity condition does not hold.
It is easy to prove the following first order U-regularity conditions.

Theorem 5.3 *Consider problem P and a cone:*

$$U(X, S, x_0) \subseteq (T(X \cap S, x_0) \cup Ker^C_{\partial h}).$$

The following conditions are first order U-regularity conditions:
i) $\text{Co}(K_U) \cap (\text{Int}(C) \times \text{Int}(V) \times 0) = \emptyset$,
ii) $\text{Co}(K_U) \subseteq K_L$,
iii) K_U *is a convex cone.*

The result stated in Theorem 5.2, based on the U-regularity property with respect to a cone $U(X, S, x_0) \subseteq (T(X \cap S, x_0) \cup Ker^C_{\partial h})$, can be deeped on studying subcones of $T(X \cap S, x_0)$ ([6]).

For a sake of simplicity, from now on we will use the following notations:

$$I_X = I(X, x_0), \quad T_X = T(X, x_0), \quad F_X = F(X, x_0)$$
$$I_S = I(S, x_0), \quad T_S = T(S, x_0), \quad F_S = F(S, x_0).$$

[6]It is known (see [3]) that:
$F(X, x_0) \cap F(S, x_0) = F(X \cap S, x_0) \subseteq T(X \cap S, x_0) \subseteq T(X, x_0) \cap T(S, x_0)$.

Lemma 5.1 *Let us consider Problem P; it results:*

$$\mathrm{Cl}(I_X \cap T_S) \cup \mathrm{Cl}(T_X \cap I_S) \cup \mathrm{Cl}(F_X \cap F_S) \subseteq T(X \cap S, x_0). \qquad (5.3)$$

If $T(S, x_0) = Ker_{\partial h}$ *then:*

$$I_X \cup \mathrm{Cl}(T_X \cap I_S) \cup \mathrm{Cl}(F_X \cap F_S) \cup Ker^C_{\partial h} \subseteq (T(X \cap S, x_0) \cup Ker^C_{\partial h}). \quad (5.4)$$

Proof We firstly prove that $I(X, x_0) \cap T(S, x_0) \subseteq T(X \cap S, x_0)$. If $I_X \cap T_S = \emptyset$ the result is trivial, otherwise let $t \in I(X, x_0) \cap T(S, x_0)$, $t \neq 0$ (note that if $t = 0$ then $t \in T(X \cap S, x_0)$ trivially), so that $\exists \{x_k\} \subset X$, $x_k \to x_0$, $\exists \{\lambda_k\} \subset \Re^{++}$, $\lambda_k \to +\infty$, such that $t = \lim_{k \to +\infty} \lambda_k (x_k - x_0)$. Since $t \in I(X, x_0)$ then $\exists \bar{k} > 0$, $\exists \delta > 0$ such that $\mu \in (0, \delta)$, $k > \bar{k}$ imply $x_0 + \mu(\lambda_k(x_k - x_0)) \in X$. Being $x_k = x_0 + \frac{1}{\lambda_k}(\lambda_k(x_k - x_0))$ and $\lambda_k \to +\infty$, then $\exists \tilde{k} > \bar{k}$ such that $\forall k > \tilde{k}$ it results $\frac{1}{\lambda_k} < \delta$ and $x_k = x_0 + \frac{1}{\lambda_k}(\lambda_k(x_k - x_0)) \in X$. This means that $\forall k > \tilde{k} > \bar{k} > 0$ we have $x_k \in X \cap S$ so that $t \in T(X \cap S, x_0)$ and hence $I(X, x_0) \cap T(S, x_0) \subseteq T(X \cap S, x_0)$. Being $T(X \cap S, x_0)$ a closed cone we finally have $\mathrm{Cl}(I_X \cap T_S) \subseteq T(X \cap S, x_0)$. In the same way we can also prove that $\mathrm{Cl}(T_X \cap I_S) \subseteq T(X \cap S, x_0)$. Since $F(X, x_0) \cap F(S, x_0) = F(X \cap S, x_0) \subseteq T(X \cap S, x_0)$ (see [3]) it results $\mathrm{Cl}(F(X, x_0) \cap F(S, x_0)) \subseteq T(X \cap S, x_0)$ being $T(X \cap S, x_0)$ a closed cone. Condition (5.3) is then proved; in order to prove condition (5.4) we just have to verify that $I(X, x_0) \subseteq (T(X \cap S, x_0) \cup Ker^C_{\partial h})$. Let $t \in I(X, x_0)$; if $t \in Ker^C_{\partial h}$ the inclusion is trivial, if $t \notin Ker^C_{\partial h}$ then $t \in Ker_{\partial h} = T(S, x_0)$ so that $t \in I(X, x_0) \cap T(S, x_0) \subseteq T(X \cap S, x_0)$. □

Directly from Theorem 5.2 and Lemma 5.1 we state the following corollary.

Corollary 5.1 *Consider Problem P; if the feasible point $x_0 \in X$ is a local efficient point and a first order U-regularity condition holds with:*

$$U(X, S, x_0) \subseteq \mathrm{Cl}(I_X \cap T_S) \cup \mathrm{Cl}(T_X \cap I_S) \cup \mathrm{Cl}(F_X \cap F_S) \cup Ker^C_{\partial h},$$

or with $T(S, x_0) = Ker_{\partial h}$ *and:*

$$U(X, S, x_0) \subseteq I_X \cup \mathrm{Cl}(T_X \cap I_S) \cup \mathrm{Cl}(F_X \cap F_S) \cup Ker^C_{\partial h},$$

then $\exists \alpha_f \in C^+$, $\exists \alpha_g \in V^+$, $\exists \alpha_h \in \Re^p$, $(\alpha_f, \alpha_g, \alpha_h) \neq 0$, *such that:*

$$\alpha_f^T \frac{\partial f}{\partial v}(x_0) + \alpha_g^T \frac{\partial g}{\partial v}(x_0) + \alpha_h^T \frac{\partial h}{\partial v}(x_0) \leq 0 \quad \forall v \in \mathrm{Cl}(U(X, S, x_0)), v \neq 0$$

Remind that in Lemma 5.1 the hypothesis $T(S, x_0) = Ker_{\partial h}$ is not trivial since in general it is just $T(S, x_0) \subseteq Ker_{\partial h}$, as it has been proved in Remark 4.1 and it is pointed out in the next Example 5.1. The next Example 5.1 points out also that under subdifferentiability hypothesis it is possible to have $I(S, x_0) \neq \emptyset$ even when $\text{Co}(Im_{\partial h}) = \Re^p$.

Example 5.1 Let us consider the point $x_0 = (0,0)$ and the following function $h : \Re^2 \to \Re$:

$$h(x) = \begin{cases} 0 & \text{if } x_1 \geq 0, x_2 \leq 0 \\ \min(x_1, x_2) & \text{if } x_1 \geq 0, x_2 > 0 \\ x_1 x_2 & \text{if } x_1 < 0, x_2 \geq 0 \\ \max(x_1, x_2) & \text{if } x_1 < 0, x_2 < 0 \end{cases}$$

It results:

$$\frac{\partial h}{\partial v}(x_0) = \begin{cases} 0 & \text{if } x_1 x_2 \leq 0 \\ h(v) & \text{if } x_1 x_2 > 0 \end{cases}$$

so that $Im_{\partial h} = \Re$, $Ker_{\partial h} = \{(x_1, x_2) : x_1 x_2 \leq 0\}$, $T(S, x_0) = S$ where:

$$S = \{(x_1, x_2) : x_1 = 0 \text{ or } x_2 = 0\} \cup \{(x_1, x_2) : x_1 > 0, x_2 < 0\},$$

and $I(S, x_0) = \{(x_1, x_2) : x_1 > 0, x_2 < 0\}$. Even if $\text{Co}(Im_{\partial h}) = Im_{\partial h} = \Re$, we then have $I(S, x_0) \neq \emptyset$ and $T(S, x_0) \subset Ker_{\partial h}$ but $T(S, x_0) \neq Ker_{\partial h}$, since for example $d = (-1, 1)^T \in Ker_{\partial h}$ but $d \notin T(S, x_0)$.

Remark 5.2 Note that in Lemma 5.1 no particular properties at all are required for the sets X and S. Note also the difficulty of stating a subcone of $T(X \cap S, x_0)$ greater than $\text{Cl}(I_X \cap T_S) \cup \text{Cl}(T_X \cap I_S) \cup \text{Cl}(F_X \cap F_S)$ since in general it results (see Examples 5.2 and 5.3):

$$\begin{aligned} \text{Int}(F(X, x_0)) \cap T(S, x_0) &\not\subseteq T(X \cap S, x_0), \\ \text{Cl}(I(X, x_0)) \cap T(S, x_0) &\not\subseteq T(X \cap S, x_0), \\ \text{Cl}(F(X, x_0)) \cap \text{Cl}(F(S, x_0)) &\not\subseteq T(X \cap S, x_0), \end{aligned}$$

Note finally that in general it is also (see Example 5.3):

$$\text{Cl}(I(X, x_0)) \not\subseteq (T(X \cap S, x_0) \cup Ker_{\partial h}^C),$$

Example 5.2 Let $X = X_1 \cup X_2 \subset \Re^2$, $X_1 = \{(x_1, x_2) : 0 \leq x_2 \leq \sqrt{|x_1|}\}$ and $X_2 = \{(x_1, x_2) : x_1 = 0\}$, let $x_0 = (0,0)$ and let $S = \{(x_1, x_2) : h(x_1, x_2) = x_2^2 - 4x_1 = 0\}$ so that $X \cap S = \{x_0\}$ and $\frac{\partial h}{\partial v}(x_0) = \nabla h(x_0)^T v = -4v_1$. It then results $T(X \cap S, x_0) = \{0\}$ and $T(S, x_0) = X_2$ so that:

$$\text{Int}(F_X) \cap T_S = \text{Cl}(I_X) \cap T_S = T_S = X_2 \not\subseteq \{0\} = T(X \cap S, x_0)$$

Example 5.3 Let $X = \{(x_1, x_2) : 0 \leq x_2 \leq \sqrt{|x_1|}\} \subset \Re^2$, let $x_0 = (0,0)$ and let $S = \{(x_1, x_2) : h(x_1, x_2) = x_1 = 0\}$, so that $X \cap S = \{x_0\}$ and $\frac{\partial h}{\partial v}(x_0) = \nabla h(x_0)^T v = v_1$. It then results $T(X \cap S, x_0) = \{0\}$ so that:

$$\mathrm{Cl}(I_X) \cap T_S = \mathrm{Cl}(F_X) \cap \mathrm{Cl}(F_S) = T_S = S \not\subseteq \{0\} = T(X \cap S, x_0)$$

Note also that $d = (0,1) \in \mathrm{Cl}(I_X)$ while $d \notin T(X \cap S, x_0)$ and $d \notin Ker^C_{\partial h}$ being $\frac{\partial h}{\partial d}(x_0) = d_1 = 0$.

6 Optimality conditions in the Decision Space: the differentiable case

In this section we are going to furthermore specialize the previously stated necessary optimality conditions under the following differentiability hypothesis (see [15]), which will be assumed in the rest of the paper:

Differentiability Hypothesis
i) functions f and g are Gâteaux differentiable at $x_0 \in X$ ([7]) ,
ii) function h is locally Fréchet differentiable on a neighbourhood of x_0,
iii) the Jacobian matrix $J_h(x)$ is continuous at x_0,
iv) the Jacobian matrix $J_h(x_0)$ is surjective.

First of all, let us point out the consequences of these strong hypothesis with respect to problem P. We can easily see that, being h Gâteaux differentiable at x_0, it results:
- $Ker_{\partial h} = \mathrm{Ker}(J_h(x_0))$ and $Im_{\partial h} = \mathrm{Img}(J_h(x_0)) = \mathrm{Co}(Im_{\partial h})$,
- $\mathrm{Co}(Im_{\partial h}) = \Re^p \Longleftrightarrow J_h(x_0)$ is surjective.

Anyway, what is more interesting is that the Differentiability Hypothesis i)-iv) imply that:

$$T(S, x_0) = Ker_{\partial h} \quad \text{and} \quad I(S, x_0) = \emptyset \tag{6.1}$$

The first condition is stated in the following Theorem 6.1, which is a generalization of the well known Lyusternik theorem (see [20, 21]), while the second one is proved in Theorem 6.2.

[7] Let $F : A \rightsquigarrow \Re^m$, with $A \subseteq \Re^n$ open set, and let $J_F(x_0)$ be the Jacobian matrix of F at x_0. $F(x)$ is called *Gâteaux differentiable at* $x_0 \in A$ if for all directions v it yields $\lim_{\lambda \to 0^+} \frac{F(x_0+\lambda v) - F(x_0)}{\lambda} = J_F(x_0)^T v$. $F(x)$ is called *Fréchet differentiable at* $x_0 \in A$ if for all directions v it yields $\lim_{\|v\| \to 0^+} \frac{F(x_0+v) - F(x_0) - J_F(x_0)^T v}{\|v\|} = 0$.

Theorem 6.1 [20] *Let $h : X \to \Re^p$, $X \subseteq \Re^n$, be a given mapping and let $x_0 \in S = \{x \in \Re^n : h(x) = 0\}$. Let also $h(x)$ be locally Fréchet differentiable on a neighbourhood of x_0, let $J_h(x)$ be continuous at x_0 and let $J_h(x_0)$ be surjective. Then it follows:*

$$T(S,x_0) = \mathrm{Ker}(J_h(x_0)) = \{d \in \Re^n : J_h(x_0)^T d = 0\} = Ker_{\partial h}$$

Theorem 6.2 *Let $h : X \to \Re^p$, $X \subseteq \Re^n$, be a given mapping, let $x_0 \in S = \{x \in \Re^n : h(x) = 0\}$ and let $h(x)$ be Gâteaux differentiable at x_0.*
i) If $\exists \overline{d} \in \Re^n$ such that $J_h(x_0)\overline{d} \neq 0$ then $I(S,x_0) = \emptyset$,
ii) if $I(S,x_0) \neq \emptyset$ then $\mathrm{Img}(J_h(x_0)) = \{0\}$ and $\mathrm{Ker}(J_h(x_0)) = \Re^n$,
iii) if $J_h(x_0)$ is surjective then $I(S,x_0) = \emptyset$.

Proof *i)* Let $d \in I(S,x_0) \neq \emptyset$; if $d = 0$ then $x_0 \in \mathrm{Int}(S)$ ([8]), so that there exists a suitable neighbourhood of x_0, say I_{x_0}, such that $h(x) = 0\ \forall x \in I_{x_0}$ and this implies that $J_h(x_0) = 0$ which contradicts $J_h(x_0)\overline{d} \neq 0$. Suppose now $d \neq 0$; then there exists a suitable neighbourhood of d, say I_d, such that all the directions $v \in I_d$ are feasible for the set S, this implies that $h(x_0 + tv) = 0$ in a neighbourhood of $t = 0\ \forall v \in I_d$ and hence $J_h(x_0)v = 0$ $\forall v \in I_d$; since n linearly independent directions d_i exist in I_d we then have that $J_h(x_0)v = 0\ \forall v \in \Re^n$ which is a contradiction.
ii),iii) Follow directly from the previous result i). □

Note that Example 5.1 points out that conditions (6.1) may not hold under subdifferentiability hypothesis.

We are now ready to state the following necessary optimality conditions in the image space and in the decision space. Note that these results extend and generalize to the vector case the result given by Jahn in [20] (Lemma 5.2 page 113), who proved the minimum principle necessary optimality condition for a scalar problem (that is having a scalar objective function $f(x)$) with just $U(X,S,x_0) = I(X,x_0)$.

Theorem 6.3 *Consider Problem P under the Differentiability Hypothesis i)-iv) and let:*

$$U(X,S,x_0) \subseteq I_X \cup \mathrm{Cl}(F_X \cap F_S) \cup Ker^C_{\partial h},$$

If the feasible point $x_0 \in X$ is a local efficient point then:

$$K_U \cap (\mathrm{Int}(C) \times \mathrm{Int}(V) \times 0) = \emptyset$$

[8]It is known that the following conditions i), ii) and iii) are equivalent for any set $S \in \Re^n$ (see [17]): i) $0 \in I(S,x_0)$ ii) $x_0 \in \mathrm{Int}(S)$ iii) $I(S,x_0) = \Re^n$

and if in addiction a first order U-regularity condition holds then $\exists \alpha_f \in C^+$, $\exists \alpha_g \in V^+$, $\exists \alpha_h \in \Re^p$, $(\alpha_f, \alpha_g, \alpha_h) \neq 0$, *such that:*

$$[\alpha_f^T J_f(x_0) + \alpha_g^T J_g(x_0) + \alpha_h^T J_h(x_0)]v \leq 0 \quad \forall v \in \mathrm{Cl}(U(X, S, x_0))$$

Proof By means of the differentiability hypothesis we have that conditions (6.1) hold, that is to say that $T(S, x_0) = Ker_{\partial h}$ and $I(S, x_0) = \emptyset$. The whole thesis then follows from Lemma 5.1 and Theorems 4.1 and 5.2. □

It is easy to prove the following additional first order U-regularity conditions related to Problem P, which come out to be stronger than the ones stated in Theorem 5.3 since they are based on the differentiability of functions f, g and h.

Theorem 6.4 *Consider problem P under the Differentiability Hypothesis i)-iv) and a cone:*

$$U(X, S, x_0) \subseteq I_X \cup \mathrm{Cl}(F_X \cap F_S) \cup Ker^C_{\partial h}.$$

The following conditions are first order U-regularity conditions:
i) $U(X, S, x_0)$ *is a convex cone,*
ii) $U(X, S, x_0) = I(X, x_0)$ *is a convex cone,*
iii) $U(X, S, x_0) = I(X, x_0) \neq \emptyset$ *and* X *is a locally convex set at* x_0 *([9]),*
iv) $U(X, S, x_0) = I(X, x_0)$ *and* X *is convex with* $\mathrm{Int}(X) \neq \emptyset$.

Remark 6.1 Note that Theorem 2.3 [3], related to a scalar optimization problem (that is that f is a scalar function), can be generalized to the vector case by means of Theorem 6.3 using the U-regularity condition ii) of Theorem 6.4.

Remark 6.2 Note that assuming the U-regularity condition iv) of Theorem 6.4 we have that the necessary optimality condition stated in Theorem 6.3 holds $\forall v \in T(X, x_0) = \mathrm{Cl}(I(X, x_0))$ ([10]), generalizing to the vector case Theorem 2.2 [20, 22], where the thesis is verified for a scalar optimization problem $\forall v \in F(X, x_0)$. Note also that this is the first time that we require $\mathrm{Int}(X) \neq \emptyset$.

[9] $X \subseteq \Re^n$ is a locally convex set at x_0 if $\exists I_{x_0}$, arbitrary open ball about x_0, such that $X \cap I_{x_0}$ is convex

[10] Note that if X is convex, with $\mathrm{Int}(X) \neq \emptyset$, then $T(X, x_0) = \mathrm{Cl}(I(X, x_0))$ (see [16, 17]).

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