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Asset Prices under Bounded Rationality and Noise Trading

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Abstract

We study an asset prices model under bounded rationality. In the economy there are rational traders and noise traders. When the noise traders' demand is modeled as pure noise (random walk) and rational traders compute the expected price as a geometric average of the observed prices (bounded rationality), we show that in the limit, as the trade interval goes to zero, the asset price is described by a mean reverting process with a drift given by the agents' expctations. The autocorrelation of the price increments is negative. When noise trading is persistent (autoregressive process), the autocorrelation of the price increments is positive for small intervals of time and negative for large intervals.

Keywords: Asset Prices, Bounded Rationality, Noise Trading, Diffusion Processes.
Classification: (JEL 1995) C61, C62, D83, D84, E32

1 Introduction

The classical asset pricing theory, based on the absence of arbitrage opportunities in the market, provides us with some testable implications. Assuming a stationary economy, excess returns are not predictable and no sign of autocorrelation should be observed (the so called random walk hypothesis). Many studies have tested empirically these implications. An evaluation of the literature allows us to establish that returns are predictable. They are characterized by mean reversion when they are computed over long horizons (excess returns are negatively correlated, see [11]), and they are positively correlated when the horizon is a week or a month, see [16]. Moreover, some phenomena such as booms, crashes and excess volatility are difficult to be interpreted inside the classical asset prices theory. On the empirical literature about asset prices see also [6]. Two well established schools of thinking can be identified in the literature on the interpretation of these facts: the classical asset pricing school, and the so called *behavioural finance* school. Those belonging to the first school explain these phenomena by relaxing the stationarity assumption for the assets' dividends-returns. In this setting the equity premia are not costant over time and excess returns turn out to be autocorrelated, e.g. see [11]. The partizans of the behavioural school argue that the nonstationarity of the model is not enough to explain the phenomena observed in financial markets and invoke the presence of some elements of irrationality in the market, e.g. see [8, 15]. On this debate see also [9, 10].

This paper aims to contribute to this debate, by providing a microfoundation for asset prices in continuous time under bounded rationality and noise trading.

In the last ten years, a large literature on asset prices with heterogeneous agents has grown up, e.g. see [7, 8]. In [12], a microeconomic approach has been developed to determine the stochastic differential equation for stock prices as the equilibrium outcome in a market populated by *rational* traders, *fundamentalist* traders and *noise* or *liquidity* traders. The agents of the first class aim at exploiting the arbitrage opportunities in the market, the traders of the second class base their decisions on the comparison between the stock price and the *fundamentals* about the security. Noise traders are pure noise in the market demand, noise due to traders' buying and selling stock for liquidity needs. In order to derive a diffusion process, it is assumed that rational agents are myiopic, i.e. they foresee the future price as the price one period before.

In this paper, we also look for a continuous time microfoundation of the evolution of stock prices in an equilibrium perspective with heterogeneous agents. The main feature of our analysis is that we assume the agents not to be fully rational, i.e. they are characterized by bounded rationality. There are two classes of agents: (boundedly) rational traders and noise traders. Bounded rationality is modeled by assuming that traders forecast the future price by updating their expectations through the first order autoregressive learning mechanism (*adaptive expectations*): the today expectation for the price tomorrow is a convex combination of the yesterday expectation for the price today and of the yesterday price. Agents are neither fully rational as it is usually assumed in financial markets models, nor myopic as in [12]. This learning rule can also be interpreted as an extrapolative technical analysis trading strategy. The noise traders' demand is described by an autonomous stochastic process. Two different specifications of the stochastic process are considered: a random walk and an autoregressive process modelling persistence in the noise trading demand. The diffusion process for the asset price, obtained in the standard weak limit by means of a suitable time rescaling of the discrete time equations (see [17]), performs a mean reverting process around the agent's expectation, which in turn is modeled by a recurrent Ornstein-Ulenbeck process. For the first specification of the noise traders demand, the price increments are negatively correlated. When the noise traders' demand is described by an autoregressive process we have that the price increments over a long horizon are negatively correlated, whereas they are positively correlated when the horizon is short enough.

This result calls for a discussion with those obtained in the financial markets literature under bounded rationality. The analysis of financial markets under bounded rationality has been developed in several papers, e.g. see [1, 5, 22, 23, 24, 4]. In those papers, agents do not know perfectly the dividend process, they learn some of its parameters as time goes, then the equilibrium asset price is computed rationally according to the no arbitrage condition. In this framework, mean reversion and high volatility are obtained. In this paper we do not rely upon the agents' ignorance of the dividend process, pure noise in the market not affecting the fundamental of the asset, which is equal to zero, generates mean reversion under bounded rationality. Our learning rule is more sound from a behavioural point of view, instead of estimating the dividend process parameters and computing rationally the price as the expectation of future dividends, the agents directly compute the expected price through an adaptive scheme. The asset price performs a mean reverting around the level given by agent's expectation process. If agents use an extrapolative learning mechanism-technical analysis trading rule, then the price's drift is determined by the agents' expectation.

The paper is organized as follows. In Section 2 we present the discrete time financial market model with *white* noise trading. In Section 3 we study the convergence of the asset price to a diffusion process. In Section 4 we consider the case of an autoregressive process describing the noise traders' demand.

2 Bounded Rationality in a Financial Market with Noise Trading

In [20] a simple model has been proposed to explain some anomalies encountered in testing the classical asset pricing theory. In the model there are rational agents aiming at exploiting all arbitrage opportunities offered in the market, and noise traders, whose nominal demand is pure noise with no economic *rationale*. The model gives us the following forward looking equation for the asset price:

$$S_k = v_k \hat{S}_k + \sigma_k Z_k, \quad k = 1, 2, \ldots, \tag{1}$$

where S_k is the asset price at time k and $\hat{S}_k$ denotes the agent's expectation at time k of the asset price at time $k+1$, the coefficient v_k is a suitable discount factor, the sequence $(Z_k)_{k\geq 1}$, which models the noise in the market, is a sequence of independent and normally distributed real random variables such that $\mathbf{E}[Z_k] = 0$ and $\mathbf{D}^2[Z_k] = 1$, and the coefficient σ_k is the noise traders variance component.

To simplify the analysis, we have assumed that the asset does not deliver dividends. The asset can be interpreted as a future contract. The noise component does not affect the fundamentals of the contract, therefore if the agents are fully rational the fair price of the contract should be constant over time and equal to zero. In fact, under rational expectations there is a unique fundamental value for the asset price which is constant and equal to zero. Our choice of not considering dividends is motivated by the fact that we want to isolate the effect of pure non fundamental noise on the asset price when the agents are not fully rational. The full rationality no correlation result for return and price increments provides us with a benchmark.

Equation (1) is the classical *no arbitrage equation* plus a noise component. In a market with two assets, a risky asset and a risk-free asset characterized by the interest rate r, setting $v_k \equiv (1+r)^{-1}$, $\hat{S}_k \equiv \mathbf{E}_k[S_{k+1}]$, where $\mathbf{E}_k$ denotes the conditional expectation at time k given the available information, and choosing $\sigma_k \equiv 0$, we end up with the classical no arbitrage equation with respect to the risk neutral probability measure.

Following [20], the random variable S_k in (1) can be interpreted as the equilibrium asset price in a market where there are two classes of traders: *rational traders* and *noise traders*.

Agents belonging to the first class behave according to the no arbitrage principle looking at the expected rate of the return of the asset, when the expected return is larger or lower than the risk free rate they buy or sell short the risky asset. Agents belonging to the second class act for pure liquidity needs and therefore their effect on the market price is purely idiosyncratic and is described by the sequence of random variables $(Z_k)_{k\geq 1}$. In what follows, rational agents are not characaterized by rational expectations. The rational expectations assumption is a mile stone in modern economic and finance theory, every other behavioural assumption is named *bounded rationality.* The rational expectations assumption is based on two main hypotheses: agents know the model and use all the available information in the best way. Bounded rationality requires to weaken these two assumptions. In our analysis, following among the others [2], we assume that *rational traders* do not compute the expected price according to a probability measure, but update their expectation according to the *first order autoregressive learning mechanism:*

$$\hat{S}_k = \hat{S}_{k-1} + \alpha_k(S_{k-1} - \hat{S}_{k-1}), \quad k = 1, 2, \ldots \tag{2}$$

for a suitable learning coefficient α_k $(0 \leq \alpha_k \leq 1)$, the today expectation for the tomorrow price is a convex combination of the yesterday expectation for the today price and of the yesterday price. Note that, to avoid simultaneity problems between the expectation formation and the determination of the equilibrium price, the asset price is not compared to the contemporaneous expectation, as it is done in the classical adaptive expectation framework. (2) says that the expected price is a geometric average of the observed prices.

3 Convergence to a Diffusion Process

The system of stochastic difference equations (1)-(2) can be rewritten in the following canonical innovation form:

$$\begin{aligned} S_k &= v_k \hat{S}_k + \sigma_k Z_k, \\ \hat{S}_{k+1} &= \hat{S}_k + \alpha_{k+1}\left(v_k - 1\right)\hat{S}_k + \alpha_{k+1}\sigma_k Z_k, \end{aligned} \tag{3}$$

where

$$\hat{S}_1 = \hat{S}_0 + \alpha_1(S_0 - \hat{S}_0).$$

Since S_0 is the datum asset price at time $k = 0$, if we make the natural assumption that the random variables of the sequence $\left(\hat{S}_0, Z_1, \ldots Z_n, \ldots\right)$ are independent, then it is well known that the solution $\left(S_k, \hat{S}_k\right)_{k\geq 0}$ of (3) is a Markov chain with respect to the filtration $(\mathcal{F}_k)_{k\geq 0}$ generated by the sequence $\left(\hat{S}_0, Z_1, \dots Z_n, \dots\right)$ itself.

Following [17], we can show, by means of a standard stepwise time-rescaling and under suitable hypotheses on the coefficients, that it is possible to obtain the weak convergence of the solutions of the rescaled systems to the solution of a system of diffusive stochastic differential equations. To this end, first, we rewrite (3) in the following equivalent form

$$\begin{aligned} S_k - S_{k-1} &= -d_k \hat{S}_k + \hat{S}_k - S_{k-1} + \sigma_k Z_k, \\ \hat{S}_{k+1} - \hat{S}_k &= -\alpha_{k+1} d_k \hat{S}_k + \alpha_{k+1}\sigma_k Z_k, \end{aligned} \tag{4}$$

where we have introduced the discount rate $d_k \equiv 1 - v_k$. Then, for each $n \geq 1$, we consider the partition of the interval $[k-1, k[$ $(k \geq 1)$ by means of the n points $k - 1 \equiv t_{n(k-1)} < t_{n(k-1)+1} < \dots < t_{nk-1} < t_{nk} \equiv k$, where $t_j - t_{j-1} \equiv \Delta t = 1/n$ for every $j \geq 1$, and we rescale the system accordingly by writing

$$\begin{aligned} S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)} &= -d_{t_j} \hat{S}_{t_j}^{(n)} + \left(\hat{S}_{t_j}^{(n)} - S_{t_{j-1}}^{(n)}\right)\Delta t + \sigma_{t_j} Z_{t_j}^{(n)}, \\ \hat{S}_{t_{j+1}}^{(n)} - \hat{S}_{t_j}^{(n)} &= -\alpha_{t_{j+1}} d_{t_j} \hat{S}_{t_j}^{(n)} + \alpha_{t_{j+1}}\sigma_{t_j} Z_{t_j}^{(n)}. \end{aligned} \tag{5}$$

Notice that, since we want to make both the drift terms and the variance of the noise terms of the rescaled system (5) proportional to Δt, we are led to introduce the term $\left(\hat{S}_{t_j}^{(n)} - S_{t_{j-1}}^{(n)}\right)\Delta t$, and we require $\left(Z_{t_j}^{(n)}\right)_{j\geq 1}$ to be a sequence of independent and normally distributed real random variables having mean 0 and variance Δt. On the other hand, the discount rate d_{t_j} depends on Δt owing to its own nature. Actually $d_{t_j} \equiv d(t_j, t_{j+1}) \equiv d(t_j, \Delta t)$, and we assume

$$d_{t_j} = \delta_{t_j}\Delta t + o(\Delta t), \tag{6}$$

where δ_{t_j} is the instantaneous interest rate at time t_j, for every $j \geq 0$.

Likewise the solution of (3), the solution $\left(S_{t_j}^{(n)}, \hat{S}_{t_j}^{(n)}\right)_{j\geq 0}$ of (5) is a Markov chain with respect to the filtration $\left(\mathcal{F}_{t_j}^{(n)}\right)_{j\geq 0}$ generated by the sequence $\left(\hat{S}_0^{(n)}, Z_{t_1}^{(n)}, \dots, Z_{t_n}^{(n)}, \dots\right)$.

Now, we introduce the sequence $\left(W_{t_j}^{(n)}\right)_{j\geq 0}$ given by

$$W_{t_j}^{(n)} \stackrel{def}{=} \begin{cases} \sum_{i=1}^{j} Z_{t_i}^{(n)} & \text{if } j \geq 1 \\ 0 & \text{if } j = 0 \end{cases},$$

and we write

$$S_t^{(n)} \stackrel{def}{=} S_{t_j}^{(n)}, \quad \hat{S}_t^{(n)} \stackrel{def}{=} \hat{S}_{t_j}^{(n)}, \quad W_t^{(n)} \stackrel{def}{=} W_{t_j}^{(n)}, \quad \text{for } t_j \leq t < t_{j+1}.$$

The processess $\left(S_t^{(n)}\right)_{t\geq 0} \equiv S^{(n)}$, $\left(\hat{S}_t^{(n)}\right)_{t\geq 0} \equiv \hat{S}^{(n)}$, and $\left(W_t^{(n)}\right)_{t\geq 0} \equiv W^{(n)}$ have right continuous paths with finite left-hand limits (RCLL paths). Moreover, given the Polish space $D\left([0,+\infty[\,;\mathbb{R}\right)$ of all RCLL paths endowed with the Skorohod distance, it is well known that the $D\left([0,+\infty[\,;\mathbb{R}\right)$-valued sequence of random variables $\left(W^{(n)}\right)_{n\geq 1}$ converges weakly to the Wiener process starting at 0.

We want to show how, applying Nelson's criteria, it is possible to check the weak convergence of the sequence $\left(S^{(n)}, \hat{S}^{(n)}\right)_{n\geq 0}$, as n goes to infinity, to the solution of a system of stochastic differential equation. To simplify the analysis, we assume a constant learning rate, market volatility and instanteanous interest rate:

$$\alpha_{t_j} \equiv \alpha, \quad \sigma_{t_j} \equiv \sigma, \quad \delta_{t_j} \equiv \delta, \quad \text{for } j \geq 1. \tag{7}$$

The results obtained below can be easily extended to time varying parameters. Our main result is the following.

Proposition 1 *As n goes to infinity, the sequence $\left(S^{(n)}, \hat{S}^{(n)}\right)_{n\geq 0}$ converges weakly to the solution of the system of stochastic differential equations*

$$\begin{cases} dS_t = \left((1-\delta)\,\hat{S}_t - S_t\right)\,dt + \sigma\,dW_t \\ d\hat{S}_t = -\alpha\delta\hat{S}_t\,dt + \alpha\sigma\,dW_t, \end{cases} \tag{8}$$

where $(W_t)_{t\geq 0}$ is a standard Wiener process.

Proof. The proof follows the guideline outlined in [17], and it is based on a classical existence result for stochastic differential equations (see [13, Chap. 5, Theor. 2.9]).

Let us consider first the matrix field

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^2 \otimes \mathbb{R}^2, \quad \sigma(x_1, x_2) \stackrel{def}{=} \begin{pmatrix} \sigma & 0 \\ \alpha\sigma & 0 \end{pmatrix},$$

and the vector field

$$b : \mathbb{R}^2 \to \mathbb{R}^2, \quad b(x_1, x_2) \equiv \left(b_1(x_1, x_2), b_2(x_1, x_2)\right),$$

where

$$b_1(x_1, x_2) \stackrel{def}{=} (1-\delta)\,x_2 - x_1, \quad b_2(x_1, x_2) \stackrel{def}{=} -\alpha\delta x_2.$$

Since it is easily seen that the conditions given in [13, Chap. 5, Theor. 2.9] hold true, we can conclude that (8) has a unique non-exploding strong solution for every given initial price S_0 and for every distribution of the expected price $\hat{S}_0$.

Now, following [17], to prove the weak convergence of $\left(S^{(n)}, \hat{S}^{(n)}\right)_{n\geq 0}$ to the solution of (8), as n goes to infinity, we want to show that the conditional variance-covariance matrix and the conditional expectation vector per unit of time of $\left(S^{(n)}, \hat{S}^{(n)}\right)_{n\geq 0}$ converge uniformly on compact sets to the components of the symmetric non-negative definite matrix field

$$a : \mathrm{IR}^2 \to \mathrm{IR}^2, \quad a(x_1, x_2) \equiv \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix},$$

given by

$$a \stackrel{def}{=} \sigma\sigma^{\intercal},$$

and to the components of the vector field $b : \mathrm{IR}^2 \to \mathrm{IR}^2$ respectively.

To this task, observe that the hypotheses on the noise sequence $\left(Z_{t_j}^{(n)}\right)_{j\geq 1}$ and on the filtration $\left(\mathcal{F}_{t_j}^{(n)}\right)_{j\geq 0}$ give

$$\begin{aligned}
\mathbf{E}\left[S_{t_j}^{(n)}|\mathcal{F}_{t_j}^{(n)}\right] &= S_{t_j}^{(n)}, \\
\mathbf{E}\left[\hat{S}_{t_{j+1}}^{(n)}|\mathcal{F}_{t_j}^{(n)}\right] &= \hat{S}_{t_{j+1}}^{(n)}, \\
\mathbf{E}\left[Z_{t_{j+1}}^{(n)}|\mathcal{F}_{t_j}^{(n)}\right] &= \mathbf{E}\left[Z_{t_{j+1}}^{(n)}\right] = 0, \\
\mathbf{E}\left[\left(Z_{t_{j+1}}^{(n)}\right)^2|\mathcal{F}_{t_j}^{(n)}\right] &= \mathbf{E}\left[\left(Z_{t_{j+1}}^{(n)}\right)^2\right] = \Delta t, \\
\mathbf{E}\left[\left(Z_{t_{j+1}}^{(n)}\right)^3|\mathcal{F}_{t_j}^{(n)}\right] &= \mathbf{E}\left[\left(Z_{t_{j+1}}^{(n)}\right)^3\right] = 0, \\
\mathbf{E}\left[\left(Z_{t_{j+1}}^{(n)}\right)^3|\mathcal{F}_{t_j}^{(n)}\right] &= \mathbf{E}\left[\left(Z_{t_{j+1}}^{(n)}\right)^3\right] = 3\Delta t^2.
\end{aligned}$$

Then, taking into account of (6) and (7), by straightforward computations, we obtain

$$\mathbf{E}\left[\left(S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)}\right)|\mathcal{F}_{t_{j-1}}^{(n)}\right] = -d\hat{S}_{t_j}^{(n)} + \Delta t\left(\hat{S}_{t_j}^{(n)} - S_{t_{j-1}}^{(n)}\right), \tag{9}$$

$$\mathbf{E}\left[\left(\hat{S}_{t_{j+1}}^{(n)} - \hat{S}_{t_j}^{(n)}\right)|\mathcal{F}_{t_{j-1}}^{(n)}\right] = -\alpha d\hat{S}_{t_j}^{(n)}, \tag{10}$$

$$\begin{aligned}
\mathbf{E}\left[\left(S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)}\right)^2|\mathcal{F}_{t_{j-1}}^{(n)}\right] &= d^2\left(\hat{S}_{t_j}^{(n)}\right)^2 + \Delta t^2\left(\hat{S}_{t_j}^{(n)} - S_{t_{j-1}}^{(n)}\right)^2 + \Delta t\sigma^2 \\
&\quad -2\Delta t d\hat{S}_{t_j}^{(n)}\left(\hat{S}_{t_j}^{(n)} - S_{t_{j-1}}^{(n)}\right),
\end{aligned} \tag{11}$$

$$\begin{aligned}
\mathbf{E}\left[\left(S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)}\right)\left(\hat{S}_{t_{j+1}}^{(n)} - \hat{S}_{t_j}^{(n)}\right)|\mathcal{F}_{t_{j-1}}^{(n)}\right] &= \alpha d^2\left(\hat{S}_{t_j}^{(n)}\right)^2 - \Delta t\alpha d\left(\hat{S}_{t_j}^{(n)} - S_{t_{j-1}}^{(n)}\right)\hat{S}_{t_j}^{(n)} \\
&\quad +\Delta t\alpha\sigma^2,
\end{aligned} \tag{12}$$

$$\mathbf{E}\left[\left(\hat{S}_{t_{j+1}}^{(n)} - \hat{S}_{t_j}^{(n)}\right)^2|\mathcal{F}_{t_{j-1}}^{(n)}\right] = \alpha^2 d^2\left(\hat{S}_{t_j}^{(n)}\right)^2 + \Delta t\alpha^2\sigma^2. \tag{13}$$

Now, writing $P^{(n)}_{t_{j-1},t_j} : \mathbb{R}^2 \times \mathfrak{B}(\mathbb{R}^2) \to \mathbb{R}_+$ for the j-th transition probability of the Markov chain $\left(S^{(n)}_{t_j}, \hat{S}^{(n)}_{t_j}\right)_{j\geq 0}$, where $\mathfrak{B}(\mathbb{R}^2)$ denotes the Borel σ-algebra on $\mathbb{R}^2$, for $k,l = 1,2$, we define

$$\hat{a}^{(n)}_{k,l}(x_1,x_2) \stackrel{def}{=} \int_{\mathbb{R}^2} (y_k - x_k)(y_l - x_l) P^{(n)}_{t_{j-1},t_j}(x_1,x_2,dy_1,dy_2),$$

and

$$\hat{b}^{(n)}_{k}(x_1,x_2) \stackrel{def}{=} \int_{\mathbb{R}^2} (y_k - x_k) P^{(n)}_{t_{j-1},t_j}(x_1,x_2,dy_1,dy_2),$$

where we claim that the integrals on the right hand side of the above equalities exist and are finite.

Indeed, setting

$$c^{(n)}_{k}(x_1,x_2) \stackrel{def}{=} \Delta t^{-1} \int_{\mathbb{R}^2} (y_k - x_k)^4 P^{(n)}_{t_{j-1},t_j}(x_1,x_2,dy_1,dy_2)$$

for $k = 1,2$, and recalling that the Markov property gives

$$\mathbf{E}\left[\left(X^{(n)}_{t_j} - X^{(n)}_{t_{j-1}}\right)^4 |\mathcal{F}^{(n)}_{t_{j-1}}\right] = \int_{\mathbb{R}^2} \left(y_k - X^{(n)}_{t_{j-1}}\right)^4 P^{(n)}_{t_{j-1},t_j}\left(\hat{S}^{(n)}_{t_{j-1}}, S^{(n)}_{t_{j-1}}, dy_1, dy_2\right),$$

for both $X \equiv S$ and $X \equiv \hat{S}$, from

$$\begin{aligned}
\mathbf{E}\left[\left(S^{(n)}_{t_j} - S^{(n)}_{t_{j-1}}\right)^4 |\mathcal{F}^{(n)}_{t_{j-1}}\right] &= d^4\left(\hat{S}^{(n)}_{t_j}\right)^4 - 4\Delta t d^3 \left(\hat{S}^{(n)}_{t_j}\right)^3 \left(\hat{S}^{(n)}_{t_j} - S^{(n)}_{t_{j-1}}\right) \\
&+ 6\Delta t^2 d^2 \left(\hat{S}^{(n)}_{t_j}\right)^2 \left(\hat{S}^{(n)}_{t_j} - S^{(n)}_{t_{j-1}}\right)^2 + 6\Delta t d^2\sigma^2 \left(\hat{S}^{(n)}_{t_j}\right)^2 \\
&- 4\Delta t^3 d \hat{S}^{(n)}_{t_j}\left(\hat{S}^{(n)}_{t_j} - S^{(n)}_{t_{j-1}}\right)^3 - 12\Delta t^2 d\sigma^2 \hat{S}^{(n)}_{t_j}\left(\hat{S}^{(n)}_{t_j} - S^{(n)}_{t_{j-1}}\right) \\
&+ \Delta t^3 \left(\hat{S}^{(n)}_{t_j} - S^{(n)}_{t_{j-1}}\right)^4 + 6\Delta t^3 \sigma^2 \left(\hat{S}^{(n)}_{t_j} - S^{(n)}_{t_{j-1}}\right)^2 + 3\Delta t^2\sigma^4,
\end{aligned}$$

and

$$\mathbf{E}\left[\left(\hat{S}^{(n)}_{t_j} - \hat{S}^{(n)}_{t_{j-1}}\right)^4 |\mathcal{F}^{(n)}_{t_{j-1}}\right] = \alpha^4 d^4 \left(\hat{S}^{(n)}_{t_j}\right)^4 + 6\Delta t \alpha^4 d^2 \left(\hat{S}^{(n)}_{t_j}\right)^2 + 3\Delta t^2 \alpha^4 \sigma^4,$$

we obtain

$$\begin{aligned}
c^{(n)}_1(x_1,x_2) &= \Delta t^{-1} d^4 x_2^4 - 4 d^3 x_2^3 (x_2 - x_1) + 6\Delta t d^2 x_2^2 (x_2 - x_1)^2 + 6 d^2 \sigma^2 x_2^2 \\
&- 4\Delta t^2 d x_2 (x_2 - x_1)^3 - 12\Delta t d\sigma^2 x_2 (x_2 - x_1) + \Delta t^3 (x_2 - x_1)^3 \\
&+ 6\Delta t^2 \sigma^2 (x_2 - x_1)^2 + 3\Delta t \sigma^4,
\end{aligned}$$

and

$$c^{(n)}_2(x_1,x_2) = \Delta t^{-1}\alpha^4 d^4 x_2^4 + 6\alpha^4 d^2 x_2^2 + 3\Delta t \alpha^4 \sigma^4.$$

Therefore, taking again into account of (6) and (7), it follows that for $k = 1, 2$

$$\lim_{n\to\infty} c_k^{(n)}(x_1, x_2) = 0,$$

uniformly on compact sets of $\mathbb{R}^2$, which gives our claim (see [17, sect. 2.2]).

This existence and finiteness result allows us to combine the relations

$$\mathbf{E}\left[\left(S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)}\right)|\mathcal{F}_{t_{j-1}}^{(n)}\right] = \int_{\mathbb{R}^2}\left(y_1 - S_{t_{j-1}}^{(n)}\right) P_{t_{j-1},t_j}^{(n)}\left(S_{t_{j-1}}^{(n)}, \hat{S}_{t_j}^{(n)}, dy_1, dy_2\right),$$

$$\mathbf{E}\left[\left(\hat{S}_{t_{j+1}}^{(n)} - \hat{S}_{t_j}^{(n)}\right)|\mathcal{F}_{t_{j-1}}^{(n)}\right] = \int_{\mathbb{R}^2}\left(y_2 - \hat{S}_{t_{j+1}}^{(n)}\right) P_{t_{j-1},t_j}^{(n)}\left(S_{t_{j-1}}^{(n)}, \hat{S}_{t_j}^{(n)}, dy_1, dy_2\right),$$

$$\mathbf{E}\left[\left(S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)}\right)^2|\mathcal{F}_{t_{j-1}}^{(n)}\right] = \int_{\mathbb{R}^2}\left(y_1 - S_{t_{j-1}}^{(n)}\right)^2 P_{t_{j-1},t_j}^{(n)}\left(S_{t_{j-1}}^{(n)}, \hat{S}_{t_j}^{(n)}, dy_1, dy_2\right),$$

$$\begin{aligned}\mathbf{E}&\left[\left(S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)}\right)\left(\hat{S}_{t_{j+1}}^{(n)} - \hat{S}_{t_j}^{(n)}\right)|\mathcal{F}_{t_{j-1}}^{(n)}\right] \\ &= \int_{\mathbb{R}^2}\left(y_1 - S_{t_{j-1}}^{(n)}\right)\left(y_2 - \hat{S}_{t_j}^{(n)}\right) P_{t_{j-1},t_j}^{(n)}\left(S_{t_{j-1}}^{(n)}, \hat{S}_{t_j}^{(n)}, dy_1, dy_2\right),\end{aligned}$$

$$\mathbf{E}\left[\left(\hat{S}_{t_{j+1}}^{(n)} - \hat{S}_{t_j}^{(n)}\right)^2|\mathcal{F}_{t_{j-1}}^{(n)}\right] = \int_{\mathbb{R}^2}\left(y_2 - \hat{S}_{t_j}^{(n)}\right)^2 P_{t_{j-1},t_j}^{(n)}\left(S_{t_{j-1}}^{(n)}, \hat{S}_{t_j}^{(n)}, dy_1, dy_2\right),$$

with (9)-(13), to obtain

$$\begin{aligned}\hat{a}_{1,1}^{(n)}(x_1, x_2) &= d^2x_2^2 + \Delta t^2 (x_2 - x_1)^2 + \sigma^2\Delta t - 2\Delta t d x_2 (x_2 - x_1),\\ \hat{a}_{1,2}^{(n)}(x_1, x_2) &= \hat{a}_{2,1}^{(n)}(x_1, x_2) = \alpha d^2 x_2^2 - \Delta t\alpha d (x_2 - x_1) x_2 + \Delta t\alpha\sigma^2,\\ \hat{a}_{2,2}^{(n)}(x_1, x_2) &= \alpha^2 d^2 x_2^2 + \Delta t\alpha^2\sigma^2,\\ \hat{b}_1^{(n)}(x_1, x_2) &= -dx_2 + \Delta t (x_2 - x_1),\\ \hat{b}_2^{(n)}(x_1, x_2) &= -\alpha d x_2.\end{aligned}$$

Thererefore, writing

$$a_{k,l}^{(n)}(x_1, x_2) \stackrel{def}{=} \Delta t^{-1}\left(\hat{a}_{k,l}^{(n)}(x_1, x_2) - \hat{b}_k^{(n)}(x_1, x_2)\,\hat{b}_l^{(n)}(x_1, x_2)\right),$$

and

$$b_k^{(n)}(x_1, x_2) \stackrel{def}{=} \Delta t^{-1}\hat{b}_k^{(n)}(x_1, x_2)$$

for $k, l = 1, 2$, we have

$$a_{1,1}^{(n)}(x_1, x_2) = \sigma^2, \qquad a_{1,2}^{(n)}(x_1, x_2) = a_{2,1}^{(n)}(x_1, x_2) = \alpha\sigma^2, \qquad a_{2,2}^{(n)}(x_1, x_2) = \alpha^2\sigma^2$$

and

$$b_1^{(n)}(x_1,x_2) = -\Delta t^{-1} d x_2 + x_2 - x_1, \qquad b_2^{(n)}(x_1,x_2) = -\Delta t^{-1}\alpha d x_2.$$

Hence, it is immediately seen that, for all $k,l=1,2$, we have

$$\lim_{n\to\infty} a_{k,l}^{(n)}(x_1,x_2) = a_{k,l} \quad \text{and} \quad \lim_{n\to\infty} b_k^{(n)}(x_1,x_2) = b_k(x_1,x_2)$$

uniformly on compact sets of $\mathbb{R}^2$.

What shown above implies that we are in a position to apply Nelson's criteria (see [17, 2.2 - 2.3]) and the desired result easily follows. □

System (8) can be integrated by means of a standard procedure (see [18, 5.1.3, p. 64]) and the solution $\left(S_t, \hat{S}_t\right)_{t\geq 0}$ is given by

$$S_t = \frac{1-\delta}{1-\alpha\delta}\hat{S}_t + \left(S_0 - \frac{1-\delta}{1-\alpha\delta}\hat{S}_0\right)e^{-t} + \frac{1-\alpha}{1-\alpha\delta}\sigma e^{-t}\int_0^t e^s\, dW_s \tag{14}$$

$$\hat{S}_t = \hat{S}_0 e^{-\alpha\delta t} + \alpha\sigma e^{-\alpha\delta t}\int_0^t e^{\alpha\delta s}\, dW_s. \tag{15}$$

Therefore, the limiting price process looks like a mean-reverting Ornstein-Ulenbeck process around the level given by agent's expectation process.

Having obtained an explicit form (14) for the limiting price process, we can apply Itô calculus to compute the main features of $(S_t)_{t\geq 0}$. In particular, it is matter of straightforward computations to prove the following result:

Proposition 2 *For all $t, \Delta t \geq 0$ we have:*

$$\begin{aligned}
(16)\ & Cov\left(S_{t+\Delta t} - S_t, S_t - S_{t-\Delta t}\right) \\
&= \frac{(1-\delta)^2}{(1-\alpha\delta)^2}\left(\mathbf{D}^2\left[\hat{S}_0\right] - \frac{1}{2}\sigma^2\frac{\alpha}{\delta}\right)e^{-\alpha\delta(2t-\Delta t)}\left(1-e^{-\alpha\delta\Delta t}\right)^2 \\
&+ \frac{(1-\delta)^2}{(1-\alpha\delta)^2}\left(\mathbf{D}^2\left[\hat{S}_0\right] - \frac{1}{2}\sigma^2\frac{(1-\alpha)^2}{(1-\delta)^2}\right)e^{-(2t-\Delta t)}\left(1-e^{-\Delta t}\right)^2 \\
&- \frac{(1-\delta)^2}{(1-\alpha\delta)^2}\mathbf{D}^2\left[\hat{S}_0\right]e^{-(1+\alpha\delta)t}\left(\left(e^{\Delta t}-1\right)\left(1-e^{-\alpha\delta\Delta t}\right) + \left(e^{\alpha\delta\Delta t}-1\right)\left(1-e^{-\Delta t}\right)\right) \\
&- \frac{1}{2}\frac{\sigma^2\alpha(1-\delta)^2}{\delta(1-\alpha\delta)^2}\left(1-e^{-\alpha\delta\Delta t}\right)^2 \\
&- \frac{\sigma^2\alpha(1-\alpha)(1-\delta)}{(1+\alpha\delta)(1-\alpha\delta)^2}\Big(\left(1-e^{\alpha\delta\Delta t}\right)^2 + \left(1-e^{-\Delta t}\right)^2 \\
&\qquad + e^{-(1+\alpha\delta)t}\left(\left(e^{\Delta t}-1\right)\left(1-e^{-\alpha\delta\Delta t}\right) + \left(e^{\alpha\delta\Delta t}-1\right)\left(1-e^{-\Delta t}\right)\right)\Big)
\end{aligned}$$

$$-\frac{1}{2}\frac{\sigma^2(1-\alpha)^2}{(1-\alpha\delta)^2}\left(1-e^{-\Delta t}\right)^2.$$

Proof. From (14) it follows that for all $0 \leq s \leq t$ we have:

$$\mathbf{E}[S_t] = \frac{1-\delta}{1-\alpha\delta}\mathbf{E}\left[\hat{S}_0\right]\left(e^{-\alpha\delta t}-e^{-t}\right)+S_0e^{-t} \tag{17}$$

and, thanks to the formula

$$\mathbf{E}\left[\int_0^s e^{p\,r}\,dW_r\int_0^t e^{q\,r}\,dW_r\right] = \frac{1}{p+q}\left(e^{(p+q)s}-1\right),$$

that holds true for all $p, q \in \mathbb{R}$, we have

$$\begin{aligned} (18)\quad & Cov\left(S_s, S_t\right) \\ &= \frac{(1-\delta)^2}{(1-\alpha\delta)^2}\mathbf{D}^2\left[\hat{S}_0\right]\left(e^{-\alpha\delta s}-e^{-s}\right)\left(e^{-\alpha\delta t}-e^{-t}\right) \\ &\quad+\frac{1}{2}\frac{\sigma^2\alpha(1-\delta)^2}{\delta(1-\alpha\delta)^2}e^{-\alpha\delta(s+t)}\left(e^{2\alpha\delta s}-1\right). \\ &\quad+\frac{\sigma^2\alpha(1-\alpha)(1-\delta)}{(1+\alpha\delta)(1-\alpha\delta)^2}\left(e^{-(s+\alpha\delta t)}\left(e^{(1+\alpha\delta)s}-1\right)+e^{-(\alpha\delta s+t)}\left(e^{(1+\alpha\delta)s}-1\right)\right) \\ &\quad+\frac{1}{2}\frac{\sigma^2(1-\alpha)^2}{(1-\alpha\delta)^2}e^{-(s+t)}\left(e^{2s}-1\right). \end{aligned}$$

From the latter, thanks to the bilinearity property of the covariance functional, we obtain the stated result. □

Equation (16) shows clearly that if the variance of the expected initial price $\hat{S}_0$ is small enough, then the price process increments are negatively correlated. Moreover, (16) shows that for any value of $\mathbf{D}^2\left[\hat{S}_0\right]$ and for any time step Δt, the price process increments become negatively correlated as times flows. In particular, if the expected price $\hat{S}_0$ is a datum, then the price process increments are always negatively correlated. This result shows that a pure noise in a bounded rationality economy produces a mean reverting effect and negative correlation in the price increments as observed in large part of the empirical literature (negative correlation of returns and price reversals).

4 Noise Traders Persistence

Now, we consider the following forward-looking difference equation

$$S_k = v_k\hat{S}_k + N_k, \quad k = 1, 2, \ldots, \tag{19}$$

where S_k, $\hat{S}_k$ and v_k represent the same variables as in the last section and the random variable N_k, which models the noise in the market, satisfies the following equation

$$N_k = m_k + \beta_k N_{k-1} + \sigma_k Z_k, \quad k = 1, 2, \ldots, \tag{20}$$

where the coefficient β_k represents the persistence in the noise traders demand and, as in (1), the sequence $(Z_k)_{k\geq 1}$ is a sequence of independent and normally distributed real random variables such that $\mathbf{E}[Z_k] = 0$ and $\mathbf{D}^2[Z_k] = 1$. Also this model has been proposed in [20].

Similarly to the case studied in Section 3, the system of stochastic difference equations (19), (2), and (20) can be rewritten in the following canonical innovation form:

$$\begin{aligned} S_k &= v_k \hat{S}_k + N_k, \\ \hat{S}_{k+1} &= \hat{S}_k + \alpha_{k+1}(v_k - 1)\hat{S}_k + \alpha_{k+1} N_k, \\ N_k &= m_k + \beta_k N_{k-1} + \sigma_k Z_k, \end{aligned} \tag{21}$$

where

$$\hat{S}_1 = \hat{S}_0 + \alpha_1(S_0 - \hat{S}_0).$$

Also in this case, S_0 is the datum asset price at time $t = 0$, and we assume that N_0 is the noise traders' component in the market at time $t = 0$. Therefore, under the hypothesis that the random variables of the sequence $\left(\hat{S}_0, Z_1, \ldots Z_n, \ldots\right)$ are independent, the solution $\left(S_k, \hat{S}_k, N_k\right)_{k\geq 0}$ of (21) is again a Markov chain with respect to the filtration $(\mathcal{F}_k)_{k\geq 0}$ generated by the sequence $\left(\hat{S}_0, Z_1, \ldots Z_n, \ldots\right)$ itself. Hence, we rewrite (21) in the equivalent form

$$\begin{aligned} S_k - S_{k-1} &= -d_k \hat{S}_k + \hat{S}_k - S_{k-1} + N_k, \\ \hat{S}_{k+1} - \hat{S}_k &= -\alpha_{k+1} d_k \hat{S}_k + \alpha_{k+1} N_k, \\ N_k - N_{k-1} &= m_k - \gamma_k N_{k-1} + \sigma_k Z_k, \end{aligned} \tag{22}$$

where we set $\gamma_k \equiv 1 - \beta_k$ and, with the same arguments and notation of those in Section 3, we rescale (22) obtaining

$$\begin{aligned} S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)} &= -d_{t_j} \hat{S}_{t_j}^{(n)} + \left(\hat{S}_{t_j}^{(n)} - S_{t_{j-1}}^{(n)} + N_{t_j}^{(n)}\right)\Delta t, \\ \hat{S}_{t_{j+1}}^{(n)} - \hat{S}_{t_j}^{(n)} &= -\alpha_{t_{j+1}} d_{t_j} \hat{S}_{t_j}^{(n)} + \alpha_{t_{j+1}} N_{t_j}^{(n)} \Delta t, \\ N_{t_j}^{(n)} - N_{t_{j-1}}^{(n)} &= \left(m_{t_j} - \gamma_{t_j} N_{t_{j-1}}^{(n)}\right)\Delta t + \sigma_{t_{j+1}} Z_{t_j k}^{(n)}, \end{aligned} \tag{23}$$

for every $n \geq 1$. Here we are making again both the drift terms and the variance of the noise terms of the rescaled system proportional to Δt, and we are assuming again that (6) holds true. Then for every $n \geq 1$ the solution $\left(S_{t_j}^{(n)}, \hat{S}_{t_j}^{(n)}, N_{t_j}^{(n)}\right)_{j\geq 0}$ of (23) is a Markov chain with respect to the filtration $\left(\mathcal{F}_{t_j}^{(n)}\right)_{j\geq 0}$ generated by the sequence $\left(\hat{S}_0^{(n)}, Z_{t_1}^{(n)}, \ldots, Z_{t_n}^{(n)}, \ldots\right)$. For sake of simplicity, we assume again that (7) holds true, moreover we assume that

$$m_{t_j} \equiv m, \quad \gamma_{t_j} \equiv \gamma, \quad \text{for j} \geq 1. \tag{24}$$

Then, an analysis similar to that in the proof of Proposition 8 gives:

Proposition 1 *As n goes to infinity, the sequence of the triple of $D([0,+\infty[\,;\mathbb{R})$-valued random variables $\left(S^{(n)}, \hat{S}^{(n)}, N^{(n)}\right)_{n\geq 0}$ converges weakly to the solution of the system of stochastic differential equations*

$$\begin{cases} dS_t = \left((1-\delta)\,\hat{S}_t - S_t + N_t\right)\,dt, \\ d\hat{S}_t = -\alpha\delta\hat{S}_t\,dt + \alpha N_t\,dt, \\ dN_t = (m - \gamma N_t)\,dt + \sigma\,dW_t, \end{cases} \tag{25}$$

where $(W_t)_{t\geq 0}$ is a standard Wiener process.

Likewise (8), System (25) can be integrated by means of a standard procedure and the solution $\left(S_t, \hat{S}_t, N_t\right)_{t\geq 0}$ is given by

$$\begin{aligned} S_t &= \frac{1-\delta}{1-\alpha\delta}\hat{S}_t + \left(S_0 - \frac{1-\delta}{1-\alpha\delta}\hat{S}_0\right)e^{-t} \\ &+ \frac{1-\alpha}{(1-\alpha\delta)(1-\gamma)}\left(N_t - \left(N_0 e^{-t} + m\left(1-e^{-t}\right) + \sigma e^{-t}\int_0^t e^s\,dW_s\right)\right) \end{aligned} \tag{26}$$

$$\begin{aligned} \hat{S}_t &= \hat{S}_0 e^{-\alpha\delta t} + \frac{\alpha}{\gamma-\alpha\delta}\left(N_0 e^{-\alpha\delta t} + \frac{m}{\alpha\delta}\left(1-e^{-\alpha\delta t}\right) + \sigma e^{-\alpha\delta t}\int_0^t e^{\alpha\delta s}\,dW_s\right) \\ &- \frac{\alpha}{\gamma-\alpha\delta}N_t. \end{aligned} \tag{27}$$

$$N_t = N_0 e^{-\gamma t} + \frac{m}{\gamma}\left(1-e^{-\gamma t}\right) + \sigma e^{-\gamma t}\int_0^t e^{\gamma s}\,dW_s \tag{28}$$

Notice that, since we can rewrite

$$\begin{aligned} S_t &= S_0 e^{-t} + \frac{1-\delta}{1-\alpha\delta}\left(\left(\hat{S}_0 + \frac{\alpha}{\gamma-\alpha\delta}N_0\right)\left(e^{-\alpha\delta t} - e^{-\gamma t}\right)\right. \\ &\left. + \frac{\alpha}{\gamma-\alpha\delta}m\left(\frac{1-e^{-\alpha\delta t}}{\alpha\delta} - \frac{1-e^{-\gamma t}}{\gamma}\right)\right) \\ &+ \frac{1-\alpha}{(1-\alpha\delta)(1-\gamma)}\left(N_0\left(e^{-\gamma t} - e^{-t}\right) + m\left(\frac{1-e^{-\gamma t}}{\gamma} - \left(1-e^{-t}\right)\right)\right) \end{aligned} \tag{29}$$

$$+\frac{\alpha(1-\delta)}{(1-\alpha\delta)(\gamma-\alpha\delta)}\sigma\left(e^{-\alpha\delta t}\int_0^t e^{\alpha\delta s}\,dW_s - e^{-\gamma t}\int_0^t e^{\gamma s}\,dW_s\right)$$
$$+\frac{1-\alpha}{(1-\alpha\delta)(1-\gamma)}\sigma\left(e^{-\gamma t}\int_0^t e^{\gamma s}\,dW_s - e^{-t}\int_0^t e^{s}\,dW_s\right)$$

and

$$\hat{S}_t = \hat{S}_0 e^{-\alpha\delta t} + \frac{\alpha}{\gamma-\alpha\delta}N_0\left(e^{-\alpha\delta t}-e^{-\gamma t}\right) + \frac{\alpha}{\gamma-\alpha\delta}m\left(\frac{1-e^{-\alpha\delta t}}{\alpha\delta}-\frac{1-e^{-\gamma t}}{\gamma}\right) + \frac{\alpha}{\gamma-\alpha\delta}\sigma\left(e^{-\alpha\delta t}\int_0^t e^{\alpha\delta s}\,dW_s - e^{-\gamma t}\int_0^t e^{\gamma s}\,dW_s\right). \tag{30}$$

Thanks to the inequality

$$\frac{1-e^{-\phi t}}{\phi} \geq \frac{1-e^{-\psi t}}{\psi},$$

that holds true for all $0<\phi\leq\psi$ and every $t\geq 0$, it is clearly seen that both the price process and the expectation process have a positive drift, for all $0<\alpha,\gamma,\delta<1$.

From (29) it follows

Proposition 2 *For all $t, \Delta t \geq 0$ we have:*

$$\begin{aligned}
(31)\quad & Cov\left(S_t - S_{t-\Delta t}, S_{t+\Delta t}-S_t\right) \\
= & \frac{(1-\delta)^2}{(1-\alpha\delta)^2}\left(\mathbf{D}^2\left[\hat{S}_0\right]-\frac{1}{2\delta}\frac{\alpha}{(\gamma-\alpha\delta)^2}\sigma^2\right)e^{-2\alpha\delta t}\left(e^{\alpha\delta\Delta t}-1\right)\left(1-e^{-\alpha\delta\Delta t}\right) \\
& +\left(\frac{(1-\delta)^2}{(1-\alpha\delta)^2}\mathbf{D}^2\left[\hat{S}_0\right]-\frac{1}{2\gamma}\frac{(\gamma-\alpha)^2}{(1-\gamma)^2(\gamma-\alpha\delta)^2}\sigma^2\right) \\
& \times e^{-2\gamma t}\left(e^{\gamma\Delta t}-1\right)\left(1-e^{-\gamma\Delta t}\right) \\
& -\frac{(1-\delta)^2}{(1-\alpha\delta)^2}\mathbf{D}^2\left[\hat{S}_0\right]e^{-(\gamma+\alpha\delta)t} \\
& \times\left(\left(e^{\gamma\Delta t}-1\right)\left(1-e^{-\alpha\delta\Delta t}\right)+\left(e^{\alpha\delta\Delta t}-1\right)\left(1-e^{-\gamma\Delta t}\right)\right) \\
& -\frac{1}{2\delta}\frac{\alpha(1-\delta)^2}{(1-\alpha\delta)^2(\gamma-\alpha\delta)^2}\sigma^2\left(1-e^{-\alpha\delta\Delta t}\right)^2 \\
& -\frac{1}{2\gamma}\frac{(\gamma-\alpha)^2}{(1-\gamma)^2(\gamma-\alpha\delta)^2}\sigma^2\left(1-e^{-\gamma\Delta t}\right)^2 \\
& -\frac{1}{\gamma+\alpha\delta}\frac{\alpha(1-\delta)(\gamma-\alpha)}{(1-\gamma)(1-\alpha\delta)(\gamma-\alpha\delta)^2}\sigma^2
\end{aligned}$$

$$\times\left(\left(1-e^{-\gamma\Delta t}\right)^2+e^{-(\gamma+\alpha\delta)t}\left(e^{\alpha\delta\Delta t}-1\right)\left(1-e^{-\gamma\Delta t}\right)\right)$$
$$+\frac{1}{1+\alpha\delta}\frac{\alpha(1-\alpha)(1-\delta)}{(1-\gamma)(1-\alpha\delta)(\gamma-\alpha\delta)^2}\sigma^2$$
$$\times\left(\left(1-e^{-\Delta t}\right)^2+e^{-(1+\alpha\delta)t}\left(e^{\alpha\delta\Delta t}-1\right)\left(1-e^{-\Delta t}\right)\right)$$
$$-\frac{1}{\gamma+\alpha\delta}\frac{\alpha(1-\delta)(\gamma-\alpha)}{(1-\gamma)(1-\alpha\delta)(\gamma-\alpha\delta)^2}\sigma^2$$
$$\times\left(\left(1-e^{-\alpha\delta\Delta t}\right)^2+e^{-(\gamma+\alpha\delta)t}\left(e^{\gamma\Delta t}-1\right)\left(1-e^{-\alpha\delta\Delta t}\right)\right)$$
$$+\frac{1}{1+\gamma}\frac{(1-\alpha)(\gamma-\alpha)}{(1-\gamma)^2(1-\alpha\delta)(\gamma-\alpha\delta)}\sigma^2$$
$$\times\left(\left(1-e^{-\Delta t}\right)^2+e^{-(1+\gamma)t}\left(e^{\gamma\Delta t}-1\right)\left(1-e^{-\Delta t}\right)\right)$$
$$+\frac{1}{1+\alpha\delta}\frac{\alpha(1-\alpha)(1-\delta)}{(1-\gamma)(1-\alpha\delta)(\gamma-\alpha\delta)^2}\sigma^2$$
$$\times\left(\left(1-e^{-\alpha\delta\Delta t}\right)^2+e^{-(1+\alpha\delta)t}\left(e^{\Delta t}-1\right)\left(1-e^{-\alpha\delta\Delta t}\right)\right)$$
$$+\frac{1}{1+\gamma}\frac{(1-\alpha)(\gamma-\alpha)}{(1-\gamma)^2(1-\alpha\delta)(\gamma-\alpha\delta)}\sigma^2$$
$$\times\left(\left(1-e^{-\gamma\Delta t}\right)^2+e^{-(1+\gamma)t}\left(e^{\Delta t}-1\right)\left(1-e^{-\gamma\Delta t}\right)\right)$$
$$-\frac{1}{2}\frac{(1-\alpha)^2}{(1-\gamma)^2(1-\alpha\delta)^2}\sigma^2\left(\left(1-e^{-\Delta t}\right)^2+e^{-2t}\left(e^{\Delta t}-1\right)\left(1-e^{-\Delta t}\right)\right).$$

In particular, if $\hat{S}_0$ is a datum and Δt is small, then it is easily seen that (31) gives

$$(32)\qquad Cov\left(S_t-S_{t-\Delta t},S_{t+\Delta t}-S_t\right)$$
$$\simeq -\frac{1}{2}\frac{\alpha^3\delta(1-\delta)^2}{(1-\alpha\delta)^2(\gamma-\alpha\delta)^2}\sigma^2\Delta t^2\left(1+e^{-2\alpha\delta t}\right)$$
$$-\frac{1}{\gamma+\alpha\delta}\frac{\alpha\gamma^2(1-\delta)(\gamma-\alpha)}{(1-\gamma)(1-\alpha\delta)(\gamma-\alpha\delta)^2}\sigma^2\Delta t^2\left(1+\alpha\delta\gamma^{-1}e^{-(\gamma+\alpha\delta)t}\right)$$
$$+\frac{1}{1+\alpha\delta}\frac{\alpha(1-\alpha)(1-\delta)}{(1-\gamma)(1-\alpha\delta)(\gamma-\alpha\delta)^2}\sigma^2\Delta t^2\left(1+\alpha\delta e^{-(1+\alpha\delta)t}\right)$$
$$-\frac{1}{\gamma+\alpha\delta}\frac{\alpha^3\delta^2(1-\delta)(\gamma-\alpha)}{(1-\gamma)(1-\alpha\delta)(\gamma-\alpha\delta)^2}\sigma^2\Delta t^2\left(1+\alpha^{-1}\delta^{-1}\gamma e^{-(\gamma+\alpha\delta)t}\right)$$
$$-\frac{1}{2}\frac{\gamma(\gamma-\alpha)^2}{(1-\gamma)^2(\gamma-\alpha\delta)^2}\sigma^2\Delta t^2\left(1+e^{-2\gamma t}\right)$$
$$+\frac{1}{1+\gamma}\frac{(1-\alpha)(\gamma-\alpha)}{(1-\gamma)^2(1-\alpha\delta)(\gamma-\alpha\delta)}\sigma^2\Delta t^2\left(1+\gamma e^{-(1+\gamma)t}\right)$$

$$
\begin{aligned}
&+\frac{1}{1+\alpha\delta}\frac{\alpha^3\delta^2(1-\alpha)(1-\delta)}{(1-\gamma)(1-\alpha\delta)(\gamma-\alpha\delta)^2}\sigma^2\Delta t^2\left(1+\alpha^{-1}\delta^{-1}e^{-(1+\alpha\delta)t}\right)\\
&+\frac{1}{1+\gamma}\frac{\gamma^2(1-\alpha)(\gamma-\alpha)}{(1-\gamma)^2(1-\alpha\delta)(\gamma-\alpha\delta)}\sigma^2\Delta t^2\left(1+\gamma^{-1}e^{-(1+\gamma)t}\right)\\
&-\frac{1}{2}\frac{(1-\alpha)^2}{(1-\gamma)^2(1-\alpha\delta)^2}\sigma^2\Delta t^2\left(1+e^{-2t}\right).
\end{aligned}
$$

Now, it is matter of straightforward computation to show that for large t the right hand member of (32) becomes

$$2\alpha\gamma(1-\alpha)+2\alpha\gamma^2+2\alpha\delta(\alpha-\gamma)+\gamma^3+\text{“higher order terms”}.$$

We can conclude that for small values of the increment Δt, and large t, the price process increments are positively correlated. The result holds true for small t, as it can be proven easily by observing that the right hand member of (32) computed for $t=0$ reduces to

$$\frac{\alpha(1-\alpha)(1-\delta)(1+\alpha\delta)}{(1-\alpha\delta)^2(\beta-\alpha\delta)^2}\sigma^2\Delta t^2$$

and also observing that its derivative, computed for $t=0$, reduces to

$$\beta^2\sigma^2\Delta t^2+\text{“higher order terms”}.$$

On the contrary, keeping up the assumption that the expected price $\hat{S}_0$ is a datum, if we assume that Δt is big, then also t is necessarily big ($t\geq\Delta t$), and from (31) we obtain

$$
\begin{aligned}
&(33)\, Cov\left(S_t-S_{t-\Delta t},S_{t+\Delta t}-S_t\right)\\
&\simeq\ -\frac{1}{2\delta}\frac{\alpha(1-\delta)^2}{(1-\alpha\delta)^2(\gamma-\alpha\delta)^2}\sigma^2-\frac{1}{\gamma+\alpha\delta}\frac{\alpha(1-\delta)(\gamma-\alpha)}{(1-\gamma)(1-\alpha\delta)(\gamma-\alpha\delta)^2}\sigma^2\\
&+\frac{1}{1+\alpha\delta}\frac{\alpha(1-\alpha)(1-\delta)}{(1-\gamma)(1-\alpha\delta)(\gamma-\alpha\delta)^2}\sigma^2-\frac{1}{\gamma+\alpha\delta}\frac{\alpha(1\delta)(\gamma-\alpha)}{(1-\gamma)(1-\alpha\delta)(\gamma-\alpha\delta)^2}\sigma^2\\
&-\frac{1}{2\gamma}\frac{(\gamma-\alpha)^2}{(1-\gamma)^2(\gamma-\alpha\delta)^2}\sigma^2+\frac{1}{1+\gamma}\frac{(1-\alpha)(\gamma-\alpha)}{(1-\gamma)^2(1-\alpha\delta)(\gamma-\alpha\delta)}\sigma^2\\
&+\frac{1}{1+\alpha\delta}\frac{\alpha(1-\alpha)(1-\delta)}{(1-\gamma)(1-\alpha\delta)(\gamma-\alpha\delta)^2}\sigma^2+\frac{1}{1+\gamma}\frac{(1-\alpha)(\gamma-\alpha)}{(1-\gamma)^2(1-\alpha\delta)(\gamma-\alpha\delta)}\sigma^2\\
&-\frac{1}{2}\frac{(1-\alpha)^2}{(1-\gamma)^2(1-\alpha\delta)^2}\sigma^2.
\end{aligned}
$$

The right hand member of (33) reduces to

$$-\alpha\gamma^2(\gamma+1) + \text{“higher order terms”},$$

this shows that for big values of the increment Δt the price process increments are negatively correlated.

Despite the above results hold true for any triple of positive parameters α, γ, δ, they are strenghtened by choosing $\delta \ll 1$.

Differently from the non correlation result obtained under full rationality, we have been able to show that the price movements are positively correlated over short horizons and negatively correlated over long orizons, confirming the regularities observed in the literature for asset returns (price increment divided by the price). Unfortunately, no closed form solution is available for the returns, therefore we are not able to provide a complete explanation of the anomalies detected in the literature. However a Monte Carlo analysis performed in [19] shows that the same type of correlation is enconutered for the return process.

5 Conclusions

In this paper we have analyzed an asset price model under bounded rationality. The fundamental of the asset is constant and equal to zero being null the dividend process. A noise component not affecting the fundamental describing the noise traders' demand gives the null price under full rationality and a price characterized by mean reversion around the agents' expectation under bounded rationality. Price increments over short horizons are positively correlated, whereas over long horizons they are negatively correlated. These theoretical results provide a theoretical explanation of teh regularities observed in the empirical literature. Future research calls for an analysis of the bounded rationality economy assuming a non null dividend process.

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