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Order Preserving Transformations and application

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Abstract. In this paper we study the effects of a linear transformation on partial order relations that are generated by a closed and convex cone in a finite dimensional space. Sufficient conditions are provided for a transformation to preserve a given order. They are then applied to derive the relationship between the efficient set of a set and its image under a linear transformation. Generalized convex vector functions are characterized by using order preserving transformations and some calculus rules for subdifferential of convex vector functions are obtained.

KeyWords. Partial order, efficient point, generalized convex functions, subdifferential.

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1 Introduction

Throughout this paper C is a closed and convex cone in $\Re^n$ and L is a linear transformation from $\Re^n$ to $\Re^m$. The biggest linear subspace of C is denoted by $l(C)$, that is $l(C) = C \cap -C$. As usual the interior (resp. the relative interior) of a convex set $A \subseteq \Re^n$ is denoted by $intA$ (resp. riA). The cone C specifies several order relations in $\Re^n$. We shall mainly deal with the following ones:

$$x \geq=_C y \text{ if } x - y \in C$$

$$x \geq_C y \text{ if } x - y \in C \setminus \{0\}$$

$$x >_C y \text{ if } x - y \in C \setminus l(C)$$

$$x \gg_C y \text{ if } x - y \in riC.$$

These order relations are crucial in the study of vector optimization problems and related topics (see for instance Refs. 1, 2, 3). Note that if C is pointed, there is no distinction between $(\geq_C)$ and $(>_C)$. The main relationship between these order relations is as follows: $(\gg_C) \Rightarrow (>_C) \Rightarrow (\geq_C) \Rightarrow (\geq=_C)$. The converse is not true in general. With the help of the above order relations one defines several "optimal" solutions of a vector problem. Namely, let A be a nonempty set in $\Re^n$. We say that

i) $x \in A$ is *an ideal (minimal) point* of A if $y \geq=_C x$ for all $y \in A$;

ii) $x \in A$ is *a strictly efficient point* of A if there is no $y \in A$ such that $x \geq_C y$;

iii) $x \in A$ is *an efficient point* of A if there is no $y \in A$ such that $x >_C y$;

iv) $x \in A$ is *a weakly efficient point* of A if there is no $y \in A$ such that $x \gg_C y$;

iv) $x \in A$ is *a properly efficient point* of A if there is another convex and closed cone $K \subseteq \Re^n$ not identical to $\Re^n$ such that $C \setminus l(C) \subseteq intK$ and there is no $y \in A$ with $x - y \in intK$.

The sets of the points defined in i)-iv) are denoted respectively by $IMin\ (A|C)$, $SMin(A|C)$, $Min(A|C)$, $WMin(A|C)$ and $PrMin(A|C)$. The definition of properly efficient points given above is due to Ref. 4. We refer the interested reader to Refs. 5-8 for other kinds of properly efficient points and their properties.

The notion of efficient points was originally introduced by Pareto in the beginning of the last century, when he used the positive orthant

$$\Re^n_+ := \{x = (x_1, ..., x_n) \in \Re^n : x_i \geq 0, i = 1, ..., n\}$$

to generate the order. For this reason, efficient points are often called Pareto efficient points or Pareto points for short. Since $\Re^n$ equipped with the order defined by the positive orthant is a Banach lattice (every two elements have a supremum), the calculus of efficient points is much simplified. For istance, given a nonempty set $A \subseteq \Re^n$, in order to find a Pareto ideal point of A or to show that it does not exist, it suffices to compute

$\bar{x} = (\bar{x}_1, ..., \bar{x}_n)$ with $\bar{x}_i = inf\{x_i : (x_1, ..., x_n) \in A\}$. If $\bar{x}$ is finite and $\bar{x} \in A$, then it is a unique ideal point of A; otherwise A has no ideal points. Similarly, in order to find the set of all dominating points of A i.e. the set $A^D := \{x \in \Re^n : x \geq=_{\Re^n_+} a, \ \forall a \in A\}$, one finds $\bar{x} = (\bar{x}_1, ..., \bar{x}_n)$ with $\bar{x}_i = sup\{x_i : (x_1, ..., x_n) \in A\}$. If $\bar{x}$ is finite then $A^D = \bar{x} + \Re^n_+$; otherwise $A^D = \emptyset$.

Note that if the order is generated by an arbitrary convex and closed cone C, then the above problems are not simple. For instance, one sees that $A^D = \bigcap_{a \in A}(a + C)$ which is quite difficult to compute.

The above examples suggest us an idea of replacing the ordering cone C by another ordering cone of simpler structure, like $\Re^n_+$. Such a replacement must, of course, preserve order relations so that efficient points remain efficient with respect to the new ordering cone. In this paper, we shall use linear transformations for this purpose. More precisely, we are going to study the effects of linear transformations on order relations. Our attention will particularly be made on projections and on those transformations that allow to use positive orthant as an ordering cone. Then, we shall derive the relationship between efficient points of a set and their images under a linear transformation. Another application is devoted to characterizing generalized convex vector functions by using order preserving transformations and to deduce some calculus rules for subdifferential of convex vector functions.

2 Linear transformations

Let C be a closed and convex cone in $\Re^n$ and let L be a linear transformation from $\Re^n$ to $\Re^m$. It is evident that $L(C)$ is a closed and convex cone in $\Re^m$. In this section we are going to establish the relationships between the order relations determined by C and those determined by $L(C)$.

Lemma 2.1 *Let L be a linear transformation from $\Re^n$ to $\Re^m$. Then one has:*

i) $l(L(C)) = L(C \cap [KerL - C])$;

ii) $l(L(C)) \supseteq L(l(C))$ and equality holds provided either $KerL \subseteq C$ or $KerL \cap C = \{0\}$;

iii) $L(riC) = ri(L(C))$.

Proof. For i), let $v \in L(C) \cap -L(C)$. There exist x and $y \in C$ such that $v = L(x) = -L(y)$. Then $L(x+y) = 0$. Hence $x \in C \cap [KerL - C]$ and $v \in L(C \cap [KerL - C])$. Conversely, let $x \in C \cap [KerL - C])$ and $v = L(x)$. Then there are $u \in KerL$ and $y \in C$ such that $x = u - y$. We have $-v = L(-x) = L(y-u) = L(y)$. Consequently, $-v \in L(C)$ and $v \in l(L(C))$.

The second assertion is derived from the first one, and the last assertion is already known (see Ref. 9).

The following example points out that the relation $l(L(C)) = L(l(C))$ can happen even if $KerL \not\subseteq C$.

Example 2.1 Consider the cone $C = \{(x, y) \in \Re^2 : y \geq 0\}$ and the linear transformation L defined as $L(x, y) = (x, 0)$. It results $l(C) = \{(x, y) : y = 0\}$, $L(C) = \{(x, 0) : x \in \Re\}$, l(L(C))=L(l(C)) but $KerL$ is not contained in C.

Theorem 2.1 *Let L be a linear transformation from $\Re^n$ to $\Re^m$ and let C be a closed and convex cone in $\Re^n$. Then we have*

i) $x \geq=_C y \Rightarrow L(x) \geq=_{L(C)} L(y)$. The converse is also true if $KerL \subseteq C$;

ii) $x \geq_C y \Rightarrow L(x) \geq_{L(C)} L(y)$ provided $KerL \cap C = \{0\}$. The converse is also true if $KerL \subseteq C$;

iii) $x >_C y \Rightarrow L(x) >_{L(C)} L(y)$ provided either $KerL \subseteq C$ or $KerL \cap C = \{0\}$. The converse is also true if $KerL \subseteq C$;

iv) $x \gg_C y \Rightarrow L(x) \gg_{L(C)} L(y)$. The converse is also true if $KerL \subseteq C$.

Proof. We prove i) first. The implication ($\Rightarrow$) is straightforward. For the converse, $L(x) \geq=_{L(C)} L(y)$ means that $L(x-y) \in L(C)$. Hence $x - y \in C + KerL \subseteq C$ because $KerL \subseteq C$, which implies $x \geq=_C y$.

For ii), let $x \geq_C y$ and $KerL \cap C = \{0\}$. Then $L(x-y) \in L(C \setminus \{0\}) = L(C \setminus KerL)$ $= L(C) \setminus \{0\}$, which shows that $L(x) \geq_{L(C)} L(y)$. Conversely, if $L(x) \geq_{L(C)} L(y)$ and $KerL \subseteq C$, then $x - y \neq 0$ and $x - y \in C + KerL \subseteq C$. Hence $x - y \in C \setminus \{0\}$, which means that $x \geq_C y$.

For iii), assume $x >_C y$, that is $x - y \in C \setminus l(C)$. Then by i), $L(x) \geq=_{L(C)} L(y)$. If $L(x) - L(y) \in l(L(C))$, then in view of Lemma 2.1, $L(x-y) \in L(l(C))$. This and the assumption of iii) imply $x - y \in l(C)$, which is a contradiction. Conversely, assume $L(x) >_{L(C)} L(y)$, that is $L(x-y) \in L(C) \setminus l(L(C))$. By i) one has $x \geq=_C y$. If $x - y \in l(C)$, then by Lemma 2.1 we obtain a contradiction $L(x-y) \in l(L(C))$. Thus, $x - y \notin l(C)$ and $x >_C y$ follows.

Finally, by Lemma 2.1, $x \gg_C y$ implies $L(x-y) \in L(riC) = riL(C)$ which shows that $L(x) \gg_{L(C)} L(y)$. For the converse, one observes that $L(x-y) \in riL(C)$ implies $x - y \in riC + KerL \subseteq riC$, which shows that $x \gg_C y$.

The following example points out the role of the condition $KerL \subseteq C$ in Theorem 2.1.

Example 2.2 Let $C = \Re^2_+$ and let L be the linear transformation defined as $L(x,y) = (x,0)$. We have $KerL \not\subseteq C$ and $KerL \cap C \neq \{0\}$.

Setting $a = (4,2), b = (3,4)$, it results $L(a) >_{L(C)} L(b)$, so that $L(a) \geq_{L(C)} L(b)$ and $L(a) \geq=_{L(C)} L(b)$. On the other hand $a \not\geq=_C b$ and thus $a \not\geq_C b$ and $a \not>_C b$.

Furthermore setting $a = (2,4), b = (2,3)$, it results $a >_C b$ and $a \geq_C b$, while $L(a) \not>_{L(C)} L(b)$ and $L(a) \not\geq_{L(C)} L(b)$.

At last consider the linear transformation $L(x,y) = x$. Setting $a = (2,3), b = (1,4)$, we have $L(a) \gg_{L(C)} L(b)$ while $a \not\gg_C b$.

Particular case 1: If $l(C)$ is not trivial, we may decompose $\Re^n$ into a direct sum of $l(C)$ and its orthogonal space $[l(C)]^\perp$. Let q denote the canonical projection of $\Re^n$ onto $[l(C)]^\perp$. Then it is a linear transformation from $\Re^n$ to $[l(C)]^\perp$.

Corollary 2.1 *Let q be the canonical projection of $\Re^n$ onto $[l(C)]^\perp$. Then we have $a \geq=_C b$ if and only if $q(a) \geq=_{q(C)} q(b)$. The conclusion remains true if the order relation $(\geq=)$ is substituted by $(>)$ and $(\gg)$.*

Proof. Since $Ker q = l(C) \subseteq C$, by applying Theorem 2.1 we obtain at once the conclusion of the corollary.

Observe that the cone $q(C)$ is convex, closed and pointed. Therefore, the order relation $a >_{q(C)} b$ means that $a \geq=_{q(C)} b$ and $a \neq b$.

Particular case 2: Now assume that C is given by the following system of inequalities:

$$C = \{x \in \Re^n :< \xi_i, x > \geq 0, i = 1, ..., k\}.$$

The positive polar cone C^+ of C is defined by

$$C^+ = \{v \in \Re^n :< v, x > \geq 0, \forall x \in C\}.$$

It is known (by Farkas' lemma) that C^+ coincides with the convex cone generated by the vectors $\xi_1, ..., \xi_k$, that is

$$C^+ = \{\sum_{i=1}^{k} \lambda_i \xi_i : \lambda_i \geq 0, i = 1, ..., k\}$$

We recall that C^+ distinguishes the points of $\Re^n$ if for every $x, y, \in \Re^n$, $x \neq y$, there is some $\xi \in C^+$ such that $< \xi, x > \neq < \xi, y >$.

Below there are some criteria for C^+ to distinguish the points of $\Re^n$.

Lemma 2.2 *The following statements are equivalent:*

i) C^+ *distinguishes the points of* $\Re^n$*;*

ii) C^+ *contains* n *linearly independent vectors;*

iii) $intC^+ \neq \emptyset$*;*

iv) C *is pointed.*

Proof. It is easy to see that the three first statements are equivalent. The equivalence between iii) and iv) has been proven in Ref. 9.

Now, let us define a transformation T from $\Re^n$ to $\Re^k$ by

$$T(x) = (< \xi_1, x >, ..., < \xi_k, x >), \forall x \in \Re^n. \tag{2.1}$$

This transformation has useful properties , as shown next.

Lemma 2.3 *The transformation* T *has the following properties:*

i) T *is linear and* $KerT = l(C)$*;*

ii) T *is injective if and only if* C *is pointed;*

iii) T *is an isomorphism if and only if* C *is pointed and* $k = n$*.*

Proof. It is evident that T is a linear transformation. Let $x \in KerT$. Then $x \in C$ and $-x \in C$ at the same time. This shows that $x \in l(C)$ and therefore $KerT \subseteq$

$l(C)$. Conversely, let $x \in l(C)$. Then $< \xi_i, x > \geq 0$ and $< \xi_i, -x > \geq 0$ for i=1,...,k. Consequently, $< \xi_i, x >= 0$, i=1,...,k, which implies $x \in KerT$. Thus, $KerT = l(C)$.

For the second assertion, it is sufficient to note that T is injective if and only if $KerT = \{0\}$, so that this assertion follows from i).

To prove the last assertion, it suffices to note that if T is injective, then $dimT(\Re^n) = n$, and to apply the second assertion.

Corollary 2.2 *Let T be the transformation defined by (2.1). Then one has*

i) $x \geq=_C y$ if and only if $T(x) \geq=_{\Re^k_+} T(y)$;

ii) $x >_C y$ if and only if $T(x) >_{\Re^k_+} T(y)$;

iii) $x \gg_C y$ if and only if $T(x) \gg_{\Re^k_+} T(y)$.

Proof. By Lemma 2.3, $KerT = l(C)$. In view of Theorem 2.1, $x \succ_C y$ if and only if $T(x) \succ_{T(C)} T(y)$, where " $\succ$ " may be " $\geq=$ " or " $>$ " or " $\gg$ ". Moreover, as $T(C) = T(\Re^n) \cap \Re^k_+$, we deduce that $T(x) \succ_{T(C)} T(y)$ if and only if $T(x) \succ_{\Re^k_+} T(y)$. This completes the proof.

3 Efficient solutions under linear transformations

In this section we shall apply the results of the preceding section to study the relationship between efficient points of a set and their images under linear transformations.

Lemma 3.1 *Let M be a linear subspace of $\Re^m$, $C \subseteq M$ a convex and closed cone, and $A \subseteq M$. Then a point $x \in A$ is a properly efficient point of A with respect to C in the space M if and only if it is so in the space $\Re^m$.*

Proof. Let $x \in A$ be a properly efficient point of A with respect to C in the space M. There is a closed convex cone $K \neq M$ such that $C \setminus l(C) \subseteq int_M K$ and $x \in WMin(A|K)$. Let N be the orthogonal complement to M in $\Re^m$. Then the cone $K_0 := K + N$ is a closed convex cone, different from $\Re^m$ and verifies the property that $C \setminus l(C) \subseteq intK_0$ and $x \in WMin(A|K_0)$. Hence x is a properly efficient point of A in $\Re^m$. For the converse, the case $l(C) = C$ being trivial, we may assume that $C \setminus l(C) \neq \emptyset$. Given K_0 in $\Re^m$ with the above property, let us define $K := K_0 \cap M$. Then $K \neq M$, otherwise K_0 should coincide with $\Re^m$. Using this cone K, one sees that x is a properly efficient point of A in M.

Theorem 3.1 *Let A be a nonempty set and C a closed and convex cone in $\Re^n$. Let L be a linear transformation from $\Re^n$ to $\Re^m$. Then the following assertions hold:*

i) If $x \in IMin(A \mid C)$, then $L(x) \in IMin(L(A) \mid L(C))$. The converse is also true if $KerL \subseteq C$;

ii) If $x \in SMin(A \mid C)$, then $L(x) \in SMin(L(A) \mid L(C))$ provided $KerL \subseteq C$. The converse is also true if $KerL \cap C = \{0\}$;

iii) If $x \in Min(A \mid C)$, then $L(x) \in Min(L(A) \mid L(C))$ provided $KerL \subseteq C$. The *converse is also true if either* $KerL \subseteq C$ *or* $KerL \cap C = \{0\}$*;*

iv) If $x \in WMin(A \mid C)$*, then* $L(x) \in WMin(L(A) \mid L(C))$ *provided* $KerL \subseteq C$*. The converse is always true;*

v) $x \in PrMin(A \mid C)$ *if and only if* $L(x) \in PrMin(L(A) \mid L(C))$ *provided* $KerL \subseteq C$*.*

Proof. The four first assertions follow directly from Theorem 2.1.

For the last statement, in view of Lemma 3.1, by considering $L(\Re^n)$ instead of $\Re^m$ if necessary, we may assume that L is surjective. Let $x \in PrMin(A \mid C)$, then there exists a closed convex cone K, different from $\Re^n$, such that $C \setminus l(C) \subset intK$ and $x \in WMin(A \mid K)$. Consider the closed convex cone $L(K)$. First, observe that $L(K) \neq \Re^m$. In fact, suppose to the contrary that it is not the case. Let $v \in \Re^n \setminus K$. Then $L(v) \in L(K)$, so that $v = k + c$ for some $c \in KerL \subseteq C \subseteq K$ and $k \in K$. Since K is a convex cone , we arrive at a contradiction $v \in K$. Second, $L(C) \setminus l(L(C)) \subseteq intL(K)$. In fact, by the assumption of v) and by the surjectivity of L, one has $L(C) \setminus l(L(C)) = L(C) \setminus L(l(C)) = L(C \setminus l(C)) \subseteq L(intK) = intL(K)$ as requested. Third, we obtain $L(x) \in WMin(L(A) \mid L(K))$ in view of Theorem 3.1 iii). Consequently $L(x) \in PrMin(L(A) \mid L(C))$.

Conversely, let $L(x) \in PrMin(L(A) \mid L(C))$. Then there exists a closed convex cone H, different from $\Re^m$, such that $L(C) \setminus l(L(C)) \subset intH$ and $L(x) \in WMin(L(A) \mid H)$. Consider the closed convex cone $L^{-1}(H)$. We easily see that $L^{-1}(H) \neq \Re^n$. Moreover,

$C \setminus l(C) \subset intL^{-1}(H)$ because $L^{-1}(intH) \neq \emptyset$ and $intH \supseteq L(C) \setminus l(L(C)) = L(C \setminus l(C))$, so that $intL^{-1}(H) = L^{-1}(intH) \supseteq L^{-1}(L(C \setminus l(C))) = C \setminus l(C) + KerL = C \setminus l(C)$. Finally, we show $x \in WMin(A \mid L^{-1}(H))$. Indeed, if this is not true, there exists $y \in A$ such that $x - y \in intL^{-1}(H) = L^{-1}(intH)$. It follows that $L(x - y) \in intH$, hence $L(x) - L(y) \in intH$, a contradiction. In this way, $x \in PrMin(A \mid C)$ and the proof is complete.

Note that a key condition for the transformation L to preserve efficient point is that $KerL \subseteq C$. The following elementary example shows that, without this condition, the conclusions of Theorem 3.1 may fail.

Let $A = \{(x, y) \in \Re^2 :$ either $x \geq 0, y \geq 1$, or $x \geq 1, y \geq 0\}$, $C = \Re^2_+$ and let L be the projection $L(x, y) = x$. It is easy to see that $(0,1) \notin IMin(A \mid \Re^2_+)$ with $L(0,1) \in IMinL((A) \mid L(\Re^2_+))$; $(1,0) \in IMin(A \mid \Re^2_+)$ with $L(1,0) \notin MinL((A) \mid L(\Re^2_+))$; $(0,2) \notin Min(A \mid \Re^2_+)$ with $L(0,2) \in MinL((A) \mid L(\Re^2_+))$. In this example $KerL \nsubseteq C$.

Corollary 3.1 *Assume that $l(C)$ is non-trivial and let q be the canonical projection of $\Re^n$ onto $[l(C)]^\perp$. Then $x \in Min(A \mid C)$ if and only if $q(x) \in Min(q(A) \mid q(C))$. The conclusion remains true if Min is substituted by IMin, WMin or PrMin.*

Proof. We know that $Ker q = l(C) \subseteq C$ and q is surjective. Therefore the conclusion of this corollary follows from Theorem 3.1.

Corollary 3.2 *Let $C \subseteq \Re^n$ be a convex polyhedral cone defined by the system $< \xi_i, x > \geq 0$, i=1,...,k and let T be the linear transformation from $\Re^n$ to $\Re^k$ defined by*

$$T(x) = (< \xi_1, x >, .., < \xi_k, x >)$$

Then $x \in Min(A \mid C)$ if and only if $T(x) \in Min(T(A) \mid \Re^k_+)$. The conclusion remains true if Min is substituted by IMin, WMin or PrMin.

Proof. Let us denote by $N := T(\Re^n)$. Then T is a surjective linear transformation from $\Re^n$ to N. By Lemma 2.3, $KerT = l(C)$. Applying Theorem 3.1 to this case we obtain that $x \in Min(A \mid C)$ if and only if $T(x) \in Min(T(A) \mid T(C)) \subseteq N$. Observe that $Min(T(A) \mid T(C)) = Min(T(A) \mid \Re^k_+)$; the conclusion of the corollary follows.

The same reasoning is available for IMin, WMin and PrMin.

Corollary 3.3 *Assume that C is a polyhedral, convex, closed and pointed cone defined as in Corollary 3.2 and A is a nonempty subset of $\Re^n$. Then $Min(A \mid C)$ is homeomorphic to $Min(T(A) \mid \Re^k_+)$. The conclusion remains valid if Min is substituted by WMin and PrMin.*

Proof. This follows from Corollary 3.2 and Lemma 2.3.

In the remaining of this section, let us apply Theorem 3.1 to the case where L is a linear function, which in fact leads to the linear scalarization method in vector optimization.

Corollary 3.4 *Assume that L is given by $L(x) = \langle a, x \rangle$ where a is some vector of $\Re^m$. Assume further that $L(C) = \Re_+$. Then the following assertions are true:*

i) Every minimum point of L on A is an ideal point (hence a properly efficient point) of A if $KerL \subseteq C$;

ii) Every strictly minimum point of L on A is a strictly efficient point of A if $KerL \cap C = \{0\}$;

iii) Every minimum point of L on A is a weakly efficient point of A.

Proof. This follows from Theorem 3.1.

We end up this section with noticing that the condition $L(C) = \Re_+$ is equivalent to the fact that $a \in C^+ \setminus \{0\}$. If in addition, $KerL \cap C = \{0\}$, then a must be in the strictly positive polar cone of C, that is $\langle a, x \rangle > 0$ for all $x \in C \setminus \{0\}$.

4 Generalized convex vector functions

Let X be a nonempty convex subset of $\Re^l$, C a convex and closed cone in $\Re^n$ and let L be a linear transformation from $\Re^n$ to $\Re^m$. Let f be a vector function from X to $\Re^n$.

With the orders generated by C one may define a class of convex vector functions and their generalizations. In this section we shall study the effect of linear transformations on convexity and generalized convexity properties of vector functions. We shall adress only to those generalized convex functions that are frequently used in applications and other kinds of generalized convex functions are left to the interested readers. In the sequel P and Q are among the sets $C, intC$ and $C \setminus l(C)$.

Let us recall that

i) f is said to be $C - convex$ (resp. $strictly\ C - convex$) if for each $x, y \in X, x \neq y$ and $\lambda \in (0,1)$, one has

$$f(\lambda x + (1-\lambda)y) \leq=_C \lambda f(x) + (1-\lambda) f(y) \tag{4.1}$$

$$(resp. f(\lambda x + (1-\lambda)y) \ll_C \lambda f(x) + (1-\lambda) f(y)) \tag{4.2}$$

ii) f is $C - quasiconvex$ if for each $x, y \in X$ and $\lambda \in (0,1)$, one has that

$$f(x), f(y) \leq=_C a, implies f(\lambda x + (1-\lambda)y) \leq=_C a \tag{4.3}$$

iii) f is $(P, Q) - quasiconvex$ if for $x, y \in X$ and $\lambda \in (0,1)$, one has that

$$f(x) \leq=_P f(y), implies f(\lambda x + (1-\lambda)y) \leq=_Q f(y) \tag{4.4}$$

We refer to Refs. 2, 8, 10-16 for the above definitions and other generalizations of convex vector functions. Note that a (C, C)-quasiconvex function is called C-quasiconvex by Refs. 1, 17.

Theorem 4.1 *Let C be a convex and closed cone in $\Re^n$ and L a linear transformation from $\Re^n$ to $\Re^m$. Let f be a vector function from a convex set $X \subseteq \Re^l$ to $\Re^n$. Then the following assertions hold:*

i) If f is $C-convex$, then $L \circ f$ is $L(C)-convex$. The converse is also true provided $KerL \subseteq C$;

ii) If f is strictly $C-convex$, then $L \circ f$ is strictly $L(C)-convex$. Conversely, if $L \circ f$ is strictly $L(C)-convex$ and $KerL \subseteq C$, then f is strictly $C-convex$;

iii) under the hypothesis that $KerL \subseteq C$, f is $(P, Q)-quasiconvex$ if and only if $L \circ f$ is $(L(P), L(Q))-quasiconvex$;

iv) under the hypothesis that $KerL \subseteq C$ and L is surjective, f is $C-quasiconvex$ if and only if $L \circ f$ is $L(C)-quasiconvex$.

Proof. Assume that f is C-convex, i.e. (4.1) holds. By Theorem 2.1, we have

$$L \circ f(\lambda x + (1-\lambda)y \leq=_{L(C)} \lambda L \circ f(x) + (1-\lambda) L \circ f(y) \tag{4.5}$$

which shows that $L \circ f$ is $L(C)$-convex. Conversely, if $L \circ f$ is $L(C)$-convex, then (4.5) holds and by Theorem 2.1, (4.1) holds as well. Hence f is C-convex. The second assertion is proven in a similar way.

Now assume that f is (P,Q)-quasiconvex. To show that $L\circ f$ is $(L(P),L(Q))$-quasiconvex, let $L\circ f(x) \leq=_{L(P)} L\circ f(y)$. By Theorem 2.1, $f(x) \leq=_{P} f(y)$, which implies $f(\lambda x + (1-\lambda)y \leq=_{Q} f(y)$ for $\lambda \in (0,1)$ because f is (P,Q)-quasiconvex. Again by Theorem 2.1, $L\circ f(\lambda x+(1-\lambda)y) \leq=_{L(Q)} L\circ f(y)$ which shows that $L\circ f$ is $(L(P),L(Q))$-quasiconvex. The converse is proven similarly.

For the last assertion one observes that for each $b \in \Re^m$, there exists $a \in \Re^n$ such that $L(a) = b$ because L is surjective. The argument of proving i) can be used to achieve the conclusion.

We now deduce two corollaries for two particular cases.

Corollary 4.1 *Let C be a convex and closed cone in $\Re^n$ and q the canonical projection of $\Re^n$ onto $[l(C)]^\perp$. Let f be a vector function from a convex set $X \subseteq \Re^\ell$ to $\Re^n$. Then f is $C-convex$ if and only if $q\circ f$ is $q(C)-convex$. The conclusion remains valid if "$C-$convex" is substituted by "strictly C-convex", "C-quasiconvex" and "(P,Q)-quasiconvex".*

Proof. This is immediate from Theorem 4.1 and the fact that $Ker q = l(C)$.

Corollary 4.2 *Let $C \subseteq \Re^n$ be a convex polyhedral cone defined by the system $<\xi_i, x>\geq 0$, i=1, ..., k and let T be the linear transformation from $\Re^n$ to $\Re^k$ as in the previous section. Then the following assertions are true:*

i) f is $C-convex$ (resp. strictly $C-convex$ or $(C,C)-quasiconvex$) if and only if $T \circ f$ is $\Re^k_+ - convex$ (resp. strictly $\Re^k_+ - convex$, $(\Re^k_+, \Re^k_+) - quasiconvex$);

ii) f is $C-quasiconvex$ if $T \circ f$ is $\Re^k_+ - quasiconvex$. The converse is also true provided T is surjective (or equivalently, ξ_1, ξ_2, ..., ξ_k, are linearly independent).

Proof. The first assertion is derived from Corollary 2.2 without any difficulty. To prove the second assertion we observe that $T(C) = \Re^k_+$ whenever T is surjective. Now, apply Theorem 4.1 to complete the proof.

By using the last corollary we recapture the following useful criteria for C-quasiconvex vector functions Ref.2.

Corollary 4.3 *Let C be a convex polyhedral cone as in the previous corollary. If the scalar functions, $< \xi_i, f(x) >$, i=1, ..., k are quasi-convex, then f is $C-quasiconvex$. The converse is also true provided k=n and the vectors ξ_1, ξ_2, ..., ξ_n, are linearly independent.*

Proof. It can be easily seen that the vector function $T \circ f$ is $\Re^k_+ - quasiconvex$ if and only if its components $< \xi_1, f(x) >$,..., $< \xi_k, f(x) >$, are quasiconvex. This observation and Corollary 4.2 produce Corollary 4.3.

5 Subdifferential of convex vector functions

Let f be a C-convex vector function from a convex set $X \subseteq \Re^\ell$ to $\Re^n$. We recall (Refs. 15-17) that the subdifferential of f at x is the set

$$\partial_C f(x) := \{A \in L(\Re^\ell, \Re^n) : f(y) - f(x) \geq=_C A(y - x), \forall y \in X\}$$

where $L(\Re^\ell, \Re^n)$ denotes the space of $n \times \ell$-matrices.

Subdifferential of C-convex vector functions enjoys several useful properties (see Ref. 15). In this section we are going to establish some more calculus rules by using linear transformations.

Lemma 5.1 *Assume that $C = \Re^n_+$. Then we have*

$$\partial_C f(x) = \partial f_1(x) \times ... \times \partial f_n(x)$$

where $f_1, ..., f_n$ are components of f and $\partial f_1, ..., \partial f_n$ are their classical convex subdifferential.

Proof. Let $A \in \partial_C f(x)$ with n rows $A_1, ..., A_n$. The inequality

$$f(y) - f(x) \geq=_{\Re^n_+} A(y - x), y \in X \tag{5.1}$$

means that

$$f_i(y) - f_i(x) \geq= A_i(y - x), y \in X, i = 1, ..., n \tag{5.2}$$

Hence $A_i \in \partial f_i(x)$, i=1,...,n. Conversely, if $A_i \in \partial f_i(x)$, i=1,...,n, then (5.2) holds, and so does (5.1). This shows that $A \in \partial_C f(x)$.

Proposition 5.1 *Let L be a linear transformation from $\Re^n$ to $\Re^m$. Then one has the inclusion*

$$L \circ \partial_C f(x) \subseteq \partial_{L(C)}(L \circ f)(x).$$

Equality holds provided L is an isomorphism.

Proof. Let $A \in \partial_C f(x)$. By definition, one has

$$f(y) - f(x) \geq=_C A(y-x), \forall y \in X \tag{5.3}$$

Hence

$$L \circ f(y) - L \circ f(x) \geq=_{L(C)} L \circ A(y-x), \forall y \in X \tag{5.4}$$

which means that $L \circ A \in \partial_{L(C)}(L \circ f)(x)$.

Now, if L is an isomorphism, then by using the above inclusion for L^{-1}, we obtain

$$L^{-1} \circ \partial_{L(C)}(L \circ f)(x) \subseteq \partial_{L^{-1}(L(C))} L^{-1} \circ L \circ f(x) = \partial_C f(x)$$

Consequently,

$$\partial_{L(C)} L \circ f(x) \subseteq L \circ \partial_C f(x)$$

and equality follows.

We now apply this result to the case where C is a polyhedral convex cone and T is the linear transformation defined in Section 2.

Corollary 5.1 *With C and T as above, we have*

$$T \circ \partial_C f(x) \subseteq \partial_{\Re^k_+}(T \circ f)(x) = \partial(\xi_1 \circ f)(x) \times ... \times \partial(\xi_n \circ f)(x)$$

Proof. According to Proposition 5.1 we obtain

$$T \circ \partial_C f(x) \subseteq \partial_{T(C)} T \circ f)(x).$$

Moreover, as $T(C) \subseteq \Re^k_+$, we have

$$\partial_{T(C)}(T \circ f)(x) \subseteq \partial_{\Re^k_+}(T \circ f)(x).$$

Consequently,

$$T \circ \partial_C f(x) \subseteq \partial_{\Re^k_+}(T \circ f)(x).$$

It remains to apply Lemma 5.1 to achieve the proof.

Assume further that $intX \neq \emptyset$, and C is pointed. Then f is a locally Lipschitz vector function around $x \in intX$ (Theorem 3.1 of Ref. 15). In this case the generalized Jacobian $Jf(x)$ of f at x is the convex hull of all $m \times n$-matrices obtained as limits of sequences $\left\{f'(x_i)\right\}$, where $\{x_i\}$ converges to x and the derivative $f'(x_i)$ exists. By Theorem 2.6.6

of Ref. 18 and as for scalar functions the convex subdifferential and Clark's generalized subdifferential coincide, we deduce

$$\partial(\xi_i \circ f)(x) = J(\xi_i \circ f)(x) = \xi_i(Jf(x))$$

for every i= 1, ..., k. This and Corollary 5.1 implies

$$\xi_i \partial_C f(x) \subseteq \xi_i J(f(x), i = 1, , k$$

Actually, we have equality because by Theorem 4.4 of [14] , $Jf(x) \subseteq \partial_C f(x)$. It is worthwhile mentioning that even for $C = \Re^n_+$, $\partial_C f(x) \neq Jf(x)$ as pointed out in Ref. 15.

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