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Asset Pricing with a Backward-Forward Stochastic Differential Utility

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Abstract

In an intertemporal setting we model the anticipation-disappoinment effect through a habit formation process which is a function of past expected utility-consumption. A disappointment effect is captured when the agent's instantaneous utility is a decreasing function of past expected utility, anticipation is modeled by assuming an increasing function. Assuming a linear model, we show that the anticipation effect reduces the risk premium, whereas the disappointment effect induces a higher risk premium.

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1 Introduction

In [Antonelli, et al., 1999] we proposed a utility functional to represent agent's preferences under uncertainty, that we called the *Backward-Forward Stochastic Differential Utility* (BFSDU). The main feature of this utility functional is that it may capture the *disappointment* or *anticipation* effect, depending on the specification of the instantaneous utility.

Indeed, in the decision theory literature, it has been recognized that agents' tastes may be affected by what they expect for the future, see [Machina, 1989]. In particular, the expectation about future consumption or expected utility may affect the agent's tastes in two different and opposite directions: *disappointment* or *anticipation.* In [Bell, 1982, Loomes and Sugden, 1986], the authors point out that agents may experience disappointment-elation comparing an outcome with the expectation they had in the past about it. The utility that an agent gathers from consuming $c(t)$ at time t is affected by what he expected in the past for the future: if the agent's expectation was high in terms of utility or consumption-standard of living, then he will be disappointed when the outcome is not as good, the opposite happens when the outcome is better than expected. Instead, in [Lowenstein,1987, Lowenstein e Prelec, 1991] the authors point out that agents anticipate future utility and therefore the expectation of future utility positively affects the utility the agent gathers from current consumption.

In these papers the setting was either deterministic or stochastic, but simple (two periods, finite state economy). In [Antonelli, et al., 1999] we proposed a utility functional which captures these effects in a setting well suited for continuous time asset pricing applications. Disappointment and anticipation are modeled by relaxing the time separability of the utility functional and in particular by including in the instantaneous utility (u) a habit process defined through an integral of past consumption and past expected utility. In a sense, the agent's habit is backward and forward looking. Depending on the function u, we describe either an anticipation or a disappointment effect. For example, when the habit is only a (positive) function of past expected utility and the instantaneous utility is increasing in the habit process, we model the anticipation effect; if the instantaneous utility is decreasing in the habit, then we model the disappointment effect. In the first case, since the agent savored in the past a high level of expected utility, he gets a high level of satisfaction-utility from the actual consumption rate. In the second case, a high level of expected utility in the past induces the agent to ask a higher consumption rate today.

In this paper we obtain asset pricing results with a linear model. We provide explicit formulas for the interest rate and the risk premia in equilibrium. The interesting point is that the anticipation effect generates a risk premium smaller than the one obtained with an Additive Expected Utility (AEU), whereas the disappointment effect leads to a higher risk premium. Therefore a disappointment effect provides us with an interesting perspective to solve the equity premium puzzle, see [Mehra and Prescott, 1985].

2 The Economy

We consider a standard pure exchange one consumer economy with complete markets. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, on which a standard Brownian motion in $\mathbb{R}^d$, W, is defined. The economy has a finite time horizon $[0, T]$ and W determines the flow of information through its natural filtration, augmented of the P-null sets and made right continuous, that we indicate by $\{\mathcal{F}_t : t \geq 0\}$. Let $\mathcal{F}_0$ be trivial.

We denote by $\mathcal{L}^2 = \{X : X \text{ is a predictable process such that } E(\int_0^T |X_s|^2 ds) < +\infty\}$, and by $\mathcal{L}^2_+$, the space of $\mathcal{L}^2$ processes with values in $\mathbb{R}_+$

There are $d+1$ financial securities, which are continuously traded in frictionless markets and their equilibrium prices are denoted by S^i $(i = 0, \ldots, d)$. The 0-th security is the risk-free asset, its price is given by

$$S^0_t = s^0_0 \exp\{\int_0^t r_u du\},$$

where r_t is a strictly positive, progressively measurable bounded process and $s^0_0 > 0$. The d-dimensional vector of the security prices $S^\top = (S^1, \ldots, S^d)$ (where $\top$ denotes transpose) instead satisfies

$$dS_t = \overline{S}_t \cdot [\mu^S_t dt + \sigma^S_t \cdot dW_t], \quad S_0 = s_0, \quad \overline{S} = \begin{pmatrix} S^1 & & 0 \\ & \ddots & \\ 0 & & S^d \end{pmatrix},$$

where the d-dimensional vector of mean returns μ^S and the $d \times d$ volatility matrix σ^S are bounded and progressively measurable and $s^i_0 > 0$ for all $i = 1, \ldots, d$.

Each security pays dividends and the cumulative dividends process of security i is denoted by D^i. The vector of cumulative dividends satisfies

$$dD_t = \mu^D_t dt + \sigma^D_t \cdot dW_t,$$

where $\mu^D \in \mathbb{R}^{d \times 1}$ and $\sigma^D \in \mathbb{R}^{d \times d}$ are again bounded and progressively measurable. Lastly, the gain process is defined as $G = S + D$, where the sum is done component by component and therefore it is an Itô process ($dG_t = [\mu_t^G dt + \sigma_t^G \cdot dW_t]$). The gain process can be written in returns rates as $\overline{S}_t \cdot [\mu_t dt + \sigma_t \cdot dW_t]$, where we assume the $d \times d$ matrix σ_t be invertible.

In a complete market economy there exists a unique equivalent martingale measure, called the *risk-neutral probability measure*, given by

$$(1) \qquad Q(A) = E[\psi_T 1_A], \qquad A \in \mathcal{F}_T,$$

$$(2) \qquad \psi_t = \exp\{-\int_0^t < \lambda_s, dW_s > -\frac{1}{2}\int_0^t ||\lambda_s||^2 ds\}$$

where $< \cdot, \cdot >$ denotes scalar product and the vector λ is given by

$$\lambda_t = \sigma_t^{-1} \cdot [\mu_t - r_t \mathbf{1}], \quad \mathbf{1} = (\underbrace{1, \ldots, 1}_{d})$$

and it denotes the market price of risk. Assuming no arbitrage, the discounted gain from trade is a martingale under Q.

The density ψ can be interpreted as the equilibrium price density of a *one consumer economy.* The agent is described by a pair (U, e), where $U : \mathcal{L}_+^2 \to \mathbb{R}$ is a utility functional and $e \in \mathcal{L}_+^2$ is an endowment process. Finally, by $c \in \mathcal{L}_+^2$ we denote the consumption process and by $C_t = \gamma_0 + \int_0^t c_s ds$, $\gamma_0 > 0$, the cumulative consumption.

A portfolio process or trading strategy, $\pi \equiv (\pi^0, \overline{\pi}) = (\pi^0, \pi^1, \ldots, \pi^d)$, is a measurable, square integrable adapted process, where the i–th component represents the amount of money invested by the agent in the i-th asset.

Definition 2.1 : *A pair of consumption and portfolio policies (c, π) for the representative agent is* **admissible**, *if it satisfies the budget constraint*

$$dX_t = (r_t X_t + e_t - c_t)dt + < \overline{\pi}_t, (\mu_t - r_t \mathbf{1}) > dt + < \overline{\pi}_t, \sigma_t \cdot dW_t >, \quad X_0 = 0, \quad X_T \geq 0,$$

where X represents the agent's wealth and $X_T \geq 0$ is the no-bankruptcy condition. An admissible pair (c, π) is **optimal** *if there is no other admissible pair (c', π') such that $U(c') > U(c)$.*

Definition 2.2 : *A triple (S, c, π) is called an* **equilibrium** *if $(c_t, \overline{\pi}_t) = (e_t, 0)$, $t \in [0, t]$, is optimal given the price processes S.*

To keep the notation simple, we assume $d = 1$. The results can be easily extended to the multidimensional case.

3 Backward-Forward Stochastic Differential Utility

The BFSDU, $U(c)$, is the initial state of the first component of the solution (V, Y) to the following system

$$(3) \qquad V_t = E(\Gamma + \int_t^T (u(c_s, Y_s) - \beta_s V_s) ds | \mathcal{F}_t)$$

$$(4) \qquad Y_t = y_0 e^{-\int_0^t \alpha_u du} + \delta \int_0^t e^{-\int_s^t \alpha_u du} [\mu V_s + (1-\mu) c_s] ds,$$

where β, α are bounded and positive adapted processes, $\mu \in [0,1]$, y_0, δ are constants and Γ is a square integrable $\mathcal{F}_T$–measurable random variable. The random variable Γ represents the utility at time T.

The process Y describes the agent's habit, y_0 is the standard of living at time 0. The constant μ is the weight describing the forward/backward characterization of Y. If $\mu = 0$, then Y is independent of the utility process V and we obtain the classical backward habit formation process, as in [Constantinides, 1990, Detemple and Zapatero, 1991]. If $\mu = 1$, then we have the other extremal case, when the habit is affected only by past expected utility. The processes α and δ measure the persistence of past habit and the effect of the instantaneous consumption on the habit.

By assumption, we take u increasing in c, but it can be either decreasing or increasing in Y; in the first case we intend to model a disappointment effect, in the second an anticipation effect. This interpretation is clarified by the example given below. We assume u strictly concave in c.

In [Antonelli, et al., 1999], it is proved existence of the utility functional, its continuity and concavity. To provide asset pricing results with this utility functional we would need an explicit solution of the system, this cannot be attained in general, but in the linear case it is possible to recover an explicit formula for $U(c) = V_0$.

We consider the system (3)-(4) with constant α and β for $u(c,y) = u(c_s) - \gamma y$ ($\gamma \in \mathbb{R}$). Differently from [Constantinides, 1990], where $u(c,y) = v(c-y)$, we assume that the habit affects u linearly. To simplify the notation, we take $\Gamma \equiv 0$, $\nu = \delta\mu$ and $\eta = \delta(1-\mu)$, so we have

$$(5) \qquad V_t = E(\int_t^T [u(c_s) - \gamma Y_s - \beta V_s] ds | \mathcal{F}_t)$$

$$(6) \qquad Y_t = y_0 + \int_0^t [\nu V_s + \eta c_s - \alpha Y_s] ds.$$

If $K = \max(\alpha, \beta, \nu, |\gamma|)$ and T is such that $KT < 1$, then we have existence and uniqueness of the utility process in $\underline{\underline{S}}^2_{[0,T]}$, for any fixed $c \in \mathcal{L}^2$. From (5)-(6), it is possible to find an explicit expression of V_0 in the following manner. If we treat Y_T as given, we may rewrite the above system in backward form

$$\begin{pmatrix} V_t \\ Y_t \end{pmatrix} = E\left(\int_t^T \left[A\begin{pmatrix} V_s \\ Y_s \end{pmatrix} + \begin{pmatrix} u(c_s) \\ -\eta c_s \end{pmatrix}\right] ds + \begin{pmatrix} 0 \\ Y_T \end{pmatrix} \middle| \mathcal{F}_t\right),$$

where the matrix $A = \begin{pmatrix} -\beta & -\gamma \\ -\nu & \alpha \end{pmatrix}$ is made up of constants. The solution (V, Y) can be explicitly written in terms of c, Y_T as

$$\begin{pmatrix} V_t \\ Y_t \end{pmatrix} = E\left(\int_t^T e^{A(s-t)} \begin{pmatrix} u(c_s) \\ -\eta c_s \end{pmatrix} ds + e^{A(T-t)} \begin{pmatrix} 0 \\ Y_T \end{pmatrix} \middle| \mathcal{F}_t\right),$$

where the matrix $e^{A(s-t)}$ is intended to be $e^{A(s-t)} = \sum_{n=0}^{\infty} \frac{((s-t)A)}{n!}$. Therefore we have

$$\begin{aligned} V_t &= E(\int_t^T (e_{11}^{A(s-t)} u(c_s) - e_{12}^{A(s-t)} \eta c_s) ds + e_{12}^{A(T-t)} Y_T | \mathcal{F}_t) \\ Y_t &= E(\int_t^T (e_{21}^{A(s-t)} u(c_s) - e_{22}^{A(s-t)} \eta c_s) ds + e_{22}^{A(T-t)} Y_T | \mathcal{F}_t), \end{aligned}$$

where e_{ij}^{At} denotes the $ij-$th element $(i, j = 1, 2)$ of the matrix e^{At}, see Appendix A for the computation of the coefficients. Solving the last equation in $t = 0$ and recalling that $Y_0 = y_0$, we obtain

$$(7) \qquad E(Y_T) = \frac{y_0}{e_{22}^{AT}} - E\left(\int_0^T \frac{e_{21}^{As} u(c_s) - e_{22}^{As} \eta c_s}{e_{22}^{AT}} ds\right).$$

Hence, substituting $E(Y_T)$ in V_0 we get

$$(8) \qquad U(c) = E\left(\int_0^T (e_{11}^{As} u(c_s) - e_{12}^{As} \eta c_s - e_{12}^{AT} \frac{e_{21}^{As} u(c_s) - e_{22}^{As} \eta c_s}{e_{22}^{AT}}) ds + \frac{e_{12}^{AT}}{e_{22}^{AT}} y_0\right).$$

It is easy to show that $U(c)$ is concave in c and that the agent is risk averse.

The interpretation of the utility functional is that we model a disappointment effect with $\gamma > 0$ and an anticipation effect with $\gamma < 0$. This is made clear through the following example provided in [Antonelli, et al., 1999]. Consider the classical additive expected utility $E\big(\int_0^T e^{-\beta s} u(c_s) ds\big)$ with instantaneous utility u and the following binary choice

$$c_s^a = \underline{c} \quad \forall s \in [0, T], \qquad c_s^b = \begin{cases} 0 & s < t \\ \bar{c} & t \le s \le T \text{ with probability } \pi \\ 0 & t \le s \le T \text{ with probability } 1 - \pi. \end{cases}$$

for some fixed t and constants $\overline{c}$ and $\underline{c}$. Without loss of generality we assume $u(0) = 0$ and we set $\underline{c}$ and $\overline{c}$ so that the two consumption processes are ordinally equivalent for the additive expected utility, that is

$$\int_0^T e^{-\beta s} u(\underline{c}) ds = \pi \int_t^T e^{-\beta s} u(\overline{c}) ds. \tag{9}$$

A disappointment effect would say that the first process is better than the second one, an anticipation effect the opposite. It is possible to capture these effects by considering the linear BFSDU (5)-(6) with the same instantaneous utility function, $y_0 = 0$ and $\eta = 0$. From (8) we obtain

$$U(c) = E\left(\int_0^T H_s u(c_s) ds\right), \quad \text{where} \quad H_s = \frac{e_{11}^{As} e_{22}^{AT} - e_{12}^{AT} e_{21}^{As}}{e_{22}^{AT}}.$$

Without loss of generality, we set $\alpha = \beta = \nu = 1$ and we recall $\gamma \in (-1, 1)$, then we can compute

$$H_s = \frac{\sqrt{1+\gamma} \cosh(\sqrt{1+\gamma}(T-s)) + \sinh(\sqrt{1+\gamma}(T-s))}{\sqrt{1+\gamma} \cosh(\sqrt{1+\gamma} T) + \sinh(\sqrt{1+\gamma} T)}.$$

It is easy to show that $H_s > e^{-\beta s} \iff \gamma < 0$. We would like to prove that

$$\int_0^T H_s u(\underline{c}) ds > \pi \int_t^T H_s u(\overline{c}) ds \qquad \iff \qquad \gamma > 0, \tag{10}$$

that is when there is disappointment the first process is better than the second and $\gamma > 0$. Dividing (10) by(9) we have that this is equivalent to prove

$$\int_0^T H_s ds (e^{-\beta t} - e^{-\beta T}) > \int_t^T H_s ds (1 - e^{-\beta T}),$$

Which can be verified for every $\gamma > 0$. Note that the preference order associated with a disappointment effect can not be obtained with the classical habit formation process.

4 Optimal Consumption and Equilibrium Analysis

The optimal consumption problem, that is to say the maximization of U over the set of the admissible consumption-portfolio policies of Definition 2.1, can be handled via dynamic optimization techniques or via the martingale method, see [Cox and Huang, 1989]. Here we follow the second approach which seems the more appropriate for the BFSDU.

The optimal consumption problem of the representative agent is equivalent to the following constrained static maximization problem:

$$\max_{c,\pi} \; U(c) \qquad \text{under the constraint} \tag{11}$$

(12) $$E^*\left(\int_0^T e^{-\int_o^t r_s ds} c_t dt\right) \leq E^*\left(\int_0^T e^{-\int_o^t r_s ds} e_t dt\right).$$

E^* denotes the expectation under the equivalent martingale measure nested in the complete financial market model, that is $E^*(\cdot) = E(\psi_T \cdot)$, where ψ is defined by (1), while $U(c)$ is given by (8). Moreover we assume that the endowment process is given by

(13) $$de_t = \mu^e(t, e_t)dt + \sigma^e(t, e_t)dW_t, \quad e_0 = x > 0$$

with Lipschitz and deterministic coefficients μ^e and σ^e. If we choose bounded functions $\mu^e(t)$ and $\sigma^e(t)$ and we take

(14) $$e_t = x + \varepsilon + \int_0^t (e_s - x)\mu^e(s)ds + \int_0^t (e_s - x)\sigma^e(s)dW_s,$$

then we obtain an endowment process that is always greater than $x + \varepsilon$. If we want to read (13) in d dimensions, then we have to consider $\sigma^e(t, e_t)$ a d dimensional vector that is taken in scalar product with W.

The constrained maximization problem is solved by exploiting the first order necessary conditions for optimality and the concavity of U. For an AEU ($\alpha = \nu = \gamma = 0$)(see for example [Duffie, 1996, p. 205-208]), the problem is solved through the associated Lagrangean. The consumption plan obtained from the first order necessary conditions associated with the Lagrangean is parametrized with respect to the Lagrange multiplier, then the multiplier is determined by imposing the consumption plan to satisfy the budget constraint. Thanks to the Inada conditions on the utility function, a unique Lagrange multiplier is determined and therefore the optimal consumption plan is defined. Concavity of U assures that this solution is the optimal one.

Setting $\xi_t = \mathrm{e}^{\int_0^t (\beta - r_u)du}\psi_t$, the Arrow-Debreu price process adjusted by the preference discount factor, the first order necessary conditions for the Additive Expected Utility evaluated along the endowment process show that $\xi_s = u'(e_s)$. To ensure that the price process ξ belongs to $\mathcal{L}^2_+$, it is enough to assume that the endowment process is bounded away from zero and that the utility function satisfies the standard Inada conditions, see [Duffie and Zame, 1989]. We remark that, looking at equation (14), it is easy to obtain such a process by taking $x + \varepsilon > 0$. The price process ξ_t contains many interesting pieces of information about our economy. In particular, the equilibrium interest rate is the opposite of the expected growth rate of ξ_t, while the market prices of risk are the opposite of the volatility in the growth of ξ_t. Keeping in mind the equalities

$$d\psi_t = -\lambda_t dW_t, \quad \text{and} \quad \mathrm{e}^{-\beta t}u'(e_t) = \mathrm{e}^{-\beta t}\xi_t = \mathrm{e}^{-\int_0^t r_u du}\psi_t,$$

differentiating both sides one obtains the equilibrium interest rate and the market price of risk

$$\begin{aligned} r_t &= \beta - (\xi_t)^{-1}(u''(e_t)\mu^e(t,e_t) + \frac{1}{2}u'''(e_t)\sigma_t^{e2}), \\ \mu_t - r_t &= -\sigma_t \frac{u''(e_t)}{u'(e_t)}\sigma_t^e = -\beta_t^e Cov(dG_t/S_t, de_t), \end{aligned}$$

where $\beta_t^e = \frac{u''(e_t)}{u'(e_t)}$. Of course these formulas may be read in the multidimensional version.

When assuming a nonseparability in time and in particular a habit persistence effect, the optimal consumption process, of the equilibrium interest rate and of the risk premium can be obtained through the Gateaux derivative of the utility functional, we refer the reader to [Duffie and Skiadas, 1994, Schroder and Skiadas, 1999] for a stochastic differential utility and to [Detemple and Zapatero, 1991, Detemple and Zapatero, 1992] for a habit formation process. As explained in [Duffie and Skiadas, 1994], provided the optimal consumption exists, the Arrow-Debreu equilibrium price process is given by the Riesz representation of the Gateaux derivative of $U(c)$ evaluated along the endowment process.

Given a reference pair of cumulative consumption and trading strategy, $(\overline{\pi}, \overline{c})$, and a set F of feasible directions, the Gateaux derivative of $U(c)$ at $(\overline{\pi}, \overline{c})$ is defined as the functional

$$\nabla U(\overline{c}; c) = \lim_{\alpha \to 0} \frac{U(\overline{c} + \alpha c) - U(\overline{c})}{\alpha}, \quad c \in F.$$

We say that $\nabla U(\overline{c}; c)$ admits a Riesz representation if there exists a process γ_t such that

$$\nabla U(\overline{c}; c) = E(\int_0^T (c_t - \overline{c}_t)\gamma_t dt).$$

In [Duffie and Skiadas, 1994, Proposition 2] it is shown that γ_t represents the Arrow-Debreu price process if $\overline{c}$ is the optimal consumption policy and coincides with the endowment process of the economy.

We can adopt the same procedure in our linear setting. We can compute the Gateaux derivative of $U(c)$, defined as in (8), at a consumption process c_s along a feasible direction and find its Riesz representation γ_s, which is given by

$$\gamma_s = \frac{e_{11}^{As} e_{22}^{AT} - e_{12}^{AT} e_{21}^{As}}{e_{22}^{AT}} u'(c_s) + \frac{e_{12}^{AT} e_{22}^{As} - e_{12}^{As} e_{22}^{AT}}{e_{22}^{AT}} \eta.$$

We set $H_s = \frac{e_{11}^{As} e_{22}^{AT} - e_{12}^{AT} e_{21}^{As}}{e_{22}^{AT}}$ and $K_s = \frac{e_{12}^{AT} e_{22}^{As} - e_{12}^{As} e_{22}^{AT}}{e_{22}^{AT}}$. These functions are deterministic and differentiable in time, by h and k we denote their derivatives. In Appendix A we analyze the coefficients H, K and their derivatives.

To ensure existence of the optimal consumption policy and of a well behaved Arrow-Debreu price process we impose the following conditions, see [Detemple and Zapatero, 1991, Detemple and Zapatero, 1992].

Assumption 4.1 *The following conditions are satisfed:*

- $u(\cdot) : [0, \infty) \to (0, \infty)$ *is three times continuously differentiable, strictly increasing and strictly concave,* $\lim_{c \to 0} u'(c) = \infty$ *and* $\lim_{c \to \infty} u'(c) = 0$;
- *In equilibrium* ($c_t = e_t$, $\forall t \in [0, T]$) *we have*

$$H_s u'(e_s) + \eta K_s > 0, \ \forall s \in [0, T];$$

- $e_s >> 0, \ \forall s \in [0, T]$.

The first condition is the standard concavity-Inada conditions. The second and the third conditions ensure that the Arrow-Debreu price process belongs to $\mathcal{L}^2_+$.

From (8) it is straightforward to verify that the concavity of u implies that also U is concave in c, moreover the first condition of Assumption 4.1 guarantees that the inverse of the marginal utility is well defined for every positive value, therefore we can apply a procedure similar to the one employed for the AEU to prove existence of the optimal solution.

Given a positive Lagrangean multiplier ρ, we consider the Lagrangean associated to (11)-(12)

$$E\left(\int_0^T [H_s u(c_s) + \eta K_s c_s] ds\right) + \frac{e_{12}^{AT}}{e_{22}^{AT}} y_0 - \rho E\left(\int_0^T e^{-\beta s} \xi_s c_s ds\right), \tag{15}$$

see [Duffie, 1996] for the rearrangement of the constraint in terms of the orginal probability measure. First order necessary conditions give

$$H_t u'(c_t) + \eta K_t = \rho e^{-\beta t} \xi_t. \tag{16}$$

We denote by $\tilde{I} : (0, +\infty) \longrightarrow (0, +\infty)$ the inverse of u'. As we show in the Appendix, $K_t > 0$ all t if and only if $\gamma < 0$, while H_t is always positive. The above equation cannot be solved in all the situations. We extend $\tilde{I}$ to the real line by defining

$$I(x) = \begin{cases} \tilde{I}(x) & \text{for} \quad x > 0 \\ +\infty & \text{for} \quad x \leq 0. \end{cases}$$

With this notation, by virtue of the concavity of U, we determine the candidate optimal consumption as

$$c_t^*(\rho) = I\left(\frac{\rho e^{-\beta t} \xi_t - \eta K_t}{H_t}\right).$$

Note that $c_t^* = +\infty$ when the argument is non positive. Let us define the function in ρ,

$$\mathcal{H}(\rho) = E\left(\int_0^T e^{-\int_0^s r_u du} c_s^*(\rho) ds\right) = E\left(\int_0^T e^{-\int_0^s r_u du} I\left(\frac{\rho e^{-\beta s}\xi_s - \eta K_s}{H_s}\right) ds\right).$$

By virtue of the continuity of the coefficients, $\mathcal{H}(\rho)$ is continuous in ρ. When $\rho \to 0$ we have to distinguish two cases. If $\gamma < 0$ the argument of the integrand I tends to zero and then becomes negative, this implies that I is going to infinity that means $\mathcal{H}(\rho) \longrightarrow +\infty$. When $\gamma > 0$, then $K_t < 0$, all t, therefore as $\rho \to 0$ we have, thanks to the integrability of ξ, that

$$\mathcal{H}(\rho) \to x_0 = E\left(\int_0^T e^{-\int_0^s r_u du} I\left(-\frac{\eta K_s}{H_s}\right) ds\right).$$

Viceversa if $\rho \to \infty$, $c \to 0$, which implies that $\mathcal{H}(\rho) \to 0$. Note that $\mathcal{H}(\rho)$ is a monotonic (decreasing) function. All these arguments imply that the constraint equation $\mathcal{H}(\rho) = x = E(\int_0^T e^{-\int_0^s r_u du} e_s ds)$ uniquely determines ρ when $\gamma > 0$; when $\gamma < 0$, being $\mathcal{H}$ bounded by x_0, the constraint is trivially satisfied for $x \geq x_0$ giving $\rho = 0$ and $c_s^*(0) = I(-\frac{\eta K_s}{H_s})$, otherwise the constraint equation again uniquely determines ρ.

Assuming market equilibrium, the optimal consumption must coincide with the endowment of the economy, so we obtain the following characterization of the Arrow-Debreu price process for the one consumer economy:

$$(17) \qquad e^{-\int_0^s r_u du} \psi_s = e^{-\beta s}\xi_s = H_s u'(e_s) + \eta K_s,$$

where we set the Lagrange multiplier equal to 1, by rescaling the price process. The equilibrium price is made up of two components. The first one is related to the instantaneous marginal utility, the second to η. Differentiating both sides of (17) we obtain the following

Proposition 4.2 *Let Assumption 4.1 be satisfied then the interest rate of equilibrium has the following expression:*

$$\begin{aligned}(18)\ r_t &= -(e^{-\beta t}\xi_t)^{-1}[h_t u'(e_t) + H_t(\mu^e(t,e_t)u''(e_t) + \frac{1}{2}(\sigma^e(t,e_t))^2 u'''(e_t)) + \eta k_t] \\ &= -(H_t u'(e_t) + \eta K_t)^{-1}[h_t u'(e_t) + H_t(\mu^e(t,e_t)u''(e_t) + \frac{1}{2}(\sigma^e(t,e_t))^2 u'''(e_t)) + \eta k_t\end{aligned}$$

and the market price of risk is the following:

$$(19)\ \mu_t - r_t = -\sigma_t(e^{-\beta t}\xi_t)^{-1} H_t u''(e_t)\sigma^e(t,e_t) = -\sigma_t(H_t u'(e_t) + \eta K_t)^{-1} H_t u''(e_t)\sigma^e(t,e_t).$$

We remark that the prvious results remain true also in the multidimensional case. This procedure can be applied also when the matrix A is time varying, but still deterministic. We would like to remark the similarity of these results with those for the standard AEU, which is in fact included in our model when we take $\gamma = \nu = 0$.

A (single factor) Consumption CAPM similar to the one associated with the AEU is obtained:

$$\mu_t - r_t\mathbf{1} = -\beta_t^e Cov(dG_t/S_t, de_t), \quad \text{where} \quad \beta_t^e = \frac{H_t u''(e_t)}{e^{-\beta t}\xi_t}.$$

Given the instantaneous utility function, from (19), we find that the risk premium results higher than the one obtained with the AEU when $\gamma > 0$ and lower when $\gamma < 0$. This is easily verified. For a given endowment process e_s and instantaneous utility function u satisfying Assumption 4.1, being $H_s > 0$ and $u''(e_s) < 0$, we have

$$-\frac{H_s u''(e_s)}{H_s u'(e_s) + \eta K_s} \geq -\frac{u''(e_s)}{u'(e_s)}, \quad \forall s \in [0, T],$$

when $K_s < 0$, that is to say $\gamma > 0$, whereas for $-\frac{(\alpha+\beta)^2}{4\nu} < \gamma < 0$ we have $K_s > 0$ and the opposite inequality is derived.

Allowing the agent's tastes to change over time as a consequence of what he expected in the past in terms of expected utility for the future is an interesting way to solve the equity premium puzzle. The analysis gives us a striking result: a disappointment effect induces a larger risk premium than the one obtained with an AEU, an anticipation effect induces a smaller one.

Let us analyze the interest rate of equilibrium. As for the AEU, this consists of three components. The first one comes from the agent's discount factor, which is simply β in the AEU framework and $-e^{\beta t}(\frac{h_t}{\xi_t}u'(e_t) + \eta\frac{k_t}{\xi_t})$ assuming the presence of the backward-forward habit. The second component is related to the expected growth rate in consumption (the interest rate is positively related to it). Since $\frac{H_s}{H_s u'(e_s) + \eta K_s} > \frac{1}{u'(e_s)}$ if $K_s < 0$, we have that a disappointment effect amplifies this component, whereas an anticipation effect reduces it. The last component is related to the expected variance of consumption growth (the interest rate is negatively related to it if $u'''(e_t) > 0$).

5 Conclusions

In this paper we have modeled the anticipation-disappoinment effect through a habit formation process which is a function of past expected utility. Disappointment is captured when the agent's utility is a decreasing function of past expected utility, anticipation is modeled by assuming an increasing function. Assuming a linear model we have shown that the anticipation effect reduces the risk premium, whereas the disappointment effect induces a higher risk premium.

A Appendix

Given the matrix $A = \begin{pmatrix} -\beta & -\gamma \\ -\nu & \alpha \end{pmatrix}$, there are two real eigenvalues:

$$\lambda_1 = \frac{\alpha - \beta - \sqrt{(\alpha+\beta)^2 + 4\gamma\nu}}{2}, \quad \lambda_2 = \frac{\alpha - \beta + \sqrt{(\alpha+\beta)^2 + 4\gamma\nu}}{2}.$$

Then we have the following:

$$e^{As} = \begin{pmatrix} \frac{(\beta+\lambda_2)e^{\lambda_1 s} - (\beta+\lambda_1)e^{\lambda_2 s}}{\lambda_2 - \lambda_1} & \frac{\gamma(e^{\lambda_1 s} - e^{\lambda_2 s})}{\lambda_2 - \lambda_1} \\ \frac{\nu(e^{\lambda_1 s} - e^{\lambda_2 s})}{\lambda_2 - \lambda_1} & \frac{(\beta+\lambda_2)e^{\lambda_2 s} - (\beta+\lambda_1)e^{\lambda_1 s}}{\lambda_2 - \lambda_1} \end{pmatrix}.$$

Given the assumptions of our model, we have two real eigenvalues with $\lambda_1 < 0 < \beta < \lambda_2$, if we choose $\gamma > -\frac{(\alpha+\beta)^2}{4\nu}$.

It is straightforward to determine the sign of the elements of e^{As}

$$e_{21}^{As} < 0, \quad e_{12}^{As} < 0, \quad e_{11}^{As} > 0, \quad e_{22}^{As} > 0, \quad \forall s \in [0,T].$$

Besides we observe that

$$\begin{aligned} H_s = \frac{e_{11}^{As}e_{22}^{AT} - e_{12}^{AT}e_{21}^{As}}{e_{22}^{AT}} &= \frac{((\beta+\lambda_2)^2 + \gamma\nu)e^{\lambda_1 s + \lambda_2 T} + ((\beta+\lambda_1)^2 + \gamma\nu)e^{\lambda_1 T + \lambda_2 s}}{(\lambda_2 - \lambda_1)(((\beta+\lambda_2)e^{\lambda_2 T} - (\beta+\lambda_1)e^{\lambda_1 T})} \\ &= \frac{(\beta+\lambda_2)^2 + \gamma\nu)e^{\lambda_1 s + \lambda_2 T} + ((\beta+\lambda_1)^2 + \gamma\nu)e^{\lambda_1 T + \lambda_2 s}}{((\beta+\lambda_2)^2 + \gamma\nu)e^{\lambda_2 T} + ((\beta+\lambda_1)^2 + \gamma\nu)e^{\lambda_1 T}} > 0 \end{aligned}$$

and

$$K_s = \frac{e_{12}^{AT}e_{22}^{As} - e_{12}^{As}e_{22}^{AT}}{e_{22}^{AT}} = \gamma\frac{e^{\lambda_1 T}e^{\lambda_2 s} - e^{\lambda_2 T}e^{\lambda_1 s}}{(\beta+\lambda_2)e^{\lambda_2 T} - (\beta+\lambda_1)e^{\lambda_1 T}}$$

is negative for $\gamma > 0$ and positive for $-\frac{(\alpha+\beta)^2}{4\nu} < \gamma < 0$. About the time derivatives of K_s and H_s, k_s and h_s respectively, we have the following

$$\begin{aligned} h_s &= e^{\lambda_1(T+s)}\frac{\lambda_1((\beta+\lambda_2)^2 + \gamma\nu)e^{(\lambda_2-\lambda_1)T} + \lambda_2((\beta+\lambda_1)^2 + \gamma\nu)e^{(\lambda_2-\lambda_1)s}}{((\beta+\lambda_2)^2 + \gamma\nu)e^{\lambda_2 T} + ((\beta+\lambda_1)^2 + \gamma\nu)e^{\lambda_1 T}} < 0 \\ k_s &= \gamma\frac{\lambda_2 e^{\lambda_1 T}e^{\lambda_2 s} - \lambda_1 e^{\lambda_2 T}e^{\lambda_1 s}}{(\beta+\lambda_2)e^{\lambda_2 T} - (\beta+\lambda_1)e^{\lambda_1 T}}. \end{aligned}$$

The first implies that $0 < H_t \leq 1 = H_0$ and the second implies that K is decreasing for $\gamma < 0$ and increasing for $\gamma > 0$, in fact k_s has the same sign of K_s.

We restricted the analysis to real eigenvalues since considering complex eigenvalues is not really meaningful as it regards umplausible values of γ (negative state prices).

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