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Coercivity Concepts and Recession Functions in Constrained Problems

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Abstract. The existence of minimum points for a real function f over a closed and unbounded set D is analyzed, focusing on the behavior of f along the so called recession directions of D. With this regard several new coercivity concepts are introduced together with an extension of the recession function. Relationships between coercivity and the behavior of the introduced recession function are studied, giving particular attention to their fundamental role in deriving optimality conditions. Necessary and sufficient conditions guaranteeing the existence of the minimum points are given as well as results related to the boundedness of the set of optimal solutions.

Key words. global optimization – existence of optimal solutions – coercivity – recession function
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1. Introduction

The existence of the minimum for a real function f is widely studied not only in optimization theory but also in several applied sciences, such as economics, mechanics, tomography, network theory and so on; just consider for example the classic cost minimization problem in microeconomic theory (Mas Colell et Al. [9] and Avriel et Al. [5]) or the unilateral problems in linear and nonlinear elasticity (Baiocchi et Al. [6]). Owing to the widespread interest in this kind of results, many studies have recently appeared, both from a theoretical and an algorithmic point of view, and several optimality conditions are given by means of the recession function and some related coercivity concepts. While we can find many results dealing with unconstrained problems or with constrained ones whose feasible region is polyhedral or asymptotically multipolyhedral or even asymptotically-linear (Auselender [2–4], Zălinescu [11]), there are few optimality conditions for a feasible region D which is just closed and unbounded. In order to handle this latter case, we are interested in investigating the behavior of f along the unbounded feasible sequences so that the recession cone of D and the recession function play a key role in our analysis. Starting from the characterization by Baiocchi et Al. [6] of the recession function, we provide a new characterization of f_∞ which allows us to introduce a recession function for

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constrained problems, called f_∞^D, coinciding with the known one when the set D is the whole space.

Beside that, coercivity of function f is the other key tool of our analysis. In the literature there are several different concepts, which are often referred to with the same name; sometimes they are defined by means of the recession function, so that coercivity and recession function wrongly seem to be necessarily related. In this paper we provide a generalization of these coercivity concepts for constrained problem, trying to present them in a unified framework. A new characterization of k-supercoercive function is given as well as the relationships among coercivity concepts, recession functions and optimality conditions.

Furthermore we provide a new characterization of the existence of minimum points which improves the one given by Auselender [2,3]. Some other conditions, which can be easily verified in applicative problems, are stated and the condition given by Auselender [4] for an asymptotically-linear region D is also improved. Finally some necessary and sufficient conditions guaranteeing the unboundedness of optimal solutions set are stated.

2. Preliminaries

In this paper we denote $X = \Re^m$ and $\|\cdot\|$ is the usual associated norm on X; our aim is to study conditions ensuring the existence of minimum points for the following constrained problem:

$$P_D : \begin{cases} \min \ f(x) \\ x \in D \subseteq X \end{cases}$$

where $f : X \to \Re \cup \{+\infty\}$ is a proper scalar function, that is $\mathrm{dom}(f) = \{x \in X : f(x) < +\infty\} \neq \emptyset$, and $D \subseteq X$ is a closed set. We also denote with $S_D = \{x_0 \in D : f(x_0) \leq f(x) \ \forall x \in D\}$ the set of minimum points for problem P_D. Note that the unconstrained problem associated with function f is also covered just assuming $D = X$.

Let us now recall some concepts regarding to the notions of recession cone and recession function which have a central role throughout the paper.

Given a nonempty set $D \subseteq X$, the *asymptotic cone* D_∞ of D (often referred to as *recession cone*) is defined by ([1]):

$$\begin{aligned} D_\infty &= \{y \in X : \exists \{x_n\} \subset D, \exists \{t_n\} \subset \Re, t_n \to +\infty, \frac{x_n}{t_n} \to y\} \\ &= \{0\} \cup \{y \in X : \exists \{x_n\} \subset D, \|x_n\| \to +\infty, \frac{x_n}{\|x_n\|} \to v, y = \lambda v, \lambda \geq 0\} \end{aligned}$$

[1] Remind that the asymptotic cone is a closed one and that a set $D \subseteq \Re^m$ is bounded if and only if $D_\infty = \{0\}$. Remind also that when D is a closed convex set then the asymptotic cone is convex and can be rewritten as follows:

$$\begin{aligned} D_\infty &= \{y \in X : \exists x \in D \text{ such that } x + \lambda y \in D \ \ \forall \lambda > 0\} \\ &= \{y \in X : x + \lambda y \in D \ \ \forall x \in D \ \ \forall \lambda > 0\} \end{aligned}$$

By means of the asymptotic cone it is possible to define the recession function. Let $f : X \to \Re \cup \{+\infty\}$ be a proper function; the corresponding *recession function* $f_\infty : X \to \Re \cup \{-\infty, +\infty\}$ is defined by:

$$\mathrm{epi}(f_\infty) = (\mathrm{epi}(f))_\infty$$

where, as usual, $\mathrm{epi}(f) = \{(x, \mu) \in X \times \Re : f(x) \leq \mu\}$.

It is worth noticing (see Auselender [2,3]) that f_∞ is positively homogeneous (i.e. $0 \in \mathrm{dom}(f_\infty)$ and $f_\infty(\lambda x) = \lambda f_\infty(x)\ \forall \lambda > 0$) and that if f is lower semicontinuous, that is $\liminf_{x \to x_0} f(x) \geq f(x_0)\ \forall x_0 \in X$, then f_∞ is lower semicontinuous too. Note also that the following useful properties hold:

i) $f_\infty(0) = 0$ or $f_\infty(0) = -\infty$ (2) ,
ii) $f_\infty(0) = 0 \Longrightarrow f_\infty(y) > -\infty\ \forall y \neq 0$ (3) .

In studying the existence of minimum points for f a key role is played by the behavior of f with respect to the directions y such that $f_\infty(y) = 0$. With this aim let us recall the following widely used notation:

$$\ker(f_\infty) = \{y \in X : f_\infty(y) = 0\}$$

Note that, by means of the positive homogeneity of f_∞, the set $\ker(f_\infty)$ comes out to be a cone, with or without the origin (in the case $f_\infty(0) = -\infty$); if f is lower semicontinuous then $\ker(f_\infty)$ is a cone with the origin, but it does not necessarily contains lines.

In Baiocchi et Al. [6] the following equivalent expression for f_∞ is given:

$$f_\infty(y) = \inf_{\{v_n\} \subset X, \{t_n\} \subset \Re} \left[\liminf_{n \to +\infty} \frac{f(t_n v_n)}{t_n} : t_n \to +\infty, v_n \to y \right] \tag{1}$$

Let us now state a new equivalent formula for the recession function, which will be useful in the rest of the paper. With this aim let us firstly introduce the following definition.

Definition 1. *Let us consider problem P_D. Function f is said to be lower unbounded on compacts over D if $\exists \mu \in \Re$, $0 < \mu < +\infty$, $\exists \{x_n\} \subset D$, $\|x_n\| \leq \mu$, such that $\liminf_{n \to +\infty} f(x_n) = -\infty$. Function f is said to be lower bounded on compacts over D if it is not lower unbounded on compacts. In the case $D = X$ function f will be simply said to be respectively lower unbounded on compacts or lower bounded on compacts.*

Note that if at least one of the following conditions hold:

i) f is locally Lipschitz
ii) f is lower semicontinuous

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2 By means of the definition $\mathrm{epi}(f_\infty(0))$ is a cone, so that $f_\infty(0) \in \{-\infty, 0, +\infty\}$; being f proper $\exists y \in X$ such that $f(y) < +\infty$ so that the halfline $\{(0, \mu), \mu \geq 0\} \subset (\mathrm{epi}(f))_\infty$ and hence $f_\infty(0) \neq +\infty$.

3 This property follows directly from the definition being $(\mathrm{epi}(f))_\infty$ a closed cone.

then f is lower bounded on compacts; note also that these conditions are sufficient but not necessary, as it is pointed out in the next examples.

Example 1. Let us consider the following scalar functions $f : \Re \to \Re$:

i) $f(x) = \sqrt[3]{x}$ is lower bounded on compacts but it is not lipschitz in any neighborhood $(-\epsilon, \epsilon)$ of $x_0 = 0$;

ii) $f(x) = \begin{cases} x^2 & \text{for } x \leq 0 \\ x^2 - 1 & \text{for } x > 0 \end{cases}$ is lower bounded on compacts but it is not lower semicontinuous.

By means of the following notation:

$$\liminf_{\|x\| \to +\infty} f(x) = \inf_{\{x_n\} \subset X} \left[\liminf_{n \to +\infty} f(x_n) : \|x_n\| \to +\infty \right]$$

we are now able to state the following characterization of the recession function.

Theorem 1. *Let $f : X \to \Re \cup \{+\infty\}$ be a proper function, then*

i) $f_\infty(0) = 0$ if and only if f is lower bounded on compacts and

$$\liminf_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} > -\infty,$$

$f_\infty(0) = -\infty$ otherwise;

ii) for any $y \neq 0$ it results

$$f_\infty(y) = \|y\| \left[\liminf_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} : \frac{x}{\|x\|} \to \frac{y}{\|y\|} \right]$$

Proof. i) $\Leftarrow$) Suppose on the contrary that $f_\infty(0) = -\infty$; then, by means of the definition, $(0, -1) \in (\text{epi}(f))_\infty$ so that, being $\|(0, -1)\| = 1$:

$$\exists \{(x_n, \mu_n)\} \subset \text{epi}(f), \|(x_n, \mu_n)\| \to +\infty, \frac{(x_n, \mu_n)}{\|(x_n, \mu_n)\|} \to (0, -1).$$

Let $d = \lim_{n \to +\infty} \frac{x_n}{\|x_n\|}$ (this limit exists eventually substituting $\{x_n\}$ with a subsequence); we then have $\frac{\mu_n}{\|(x_n, \mu_n)\|} \to -1$, so that $\mu_n < 0$ definitively, and:

$$\frac{x_n}{\|(x_n, \mu_n)\|} = \frac{\frac{x_n}{\|x_n\|}}{\left\| \left(\frac{x_n}{\|x_n\|}, \frac{\mu_n}{\|x_n\|} \right) \right\|} \to 0$$

so that $\frac{\mu_n}{\|x_n\|} \to -\infty$; being $f(x_n) \leq \mu_n < 0$ it then follows that $\frac{f(x_n)}{\|x_n\|} \to -\infty$ which implies $\lim_{n \to +\infty} \|x_n\| < +\infty$ provided that $\liminf_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} > -\infty$; hence $\frac{f(x_n)}{\|x_n\|} \to -\infty$ only if $\lim_{n \to +\infty} \|x_n\| = 0$, being f lower bounded on compacts. Since $\|(x_n, \mu_n)\| \to +\infty$, we have $\mu_n \to -\infty$ and hence $f(x_n) \to -\infty$, since $f(x_n) \leq \mu_n$, which contradicts the lower boundedness on compacts of f.

$\Rightarrow$) Let us suppose on the contrary that f is not lower bounded on compacts or that $\liminf_{\|x\|\to+\infty} \frac{f(x)}{\|x\|} = -\infty$, then the following condition holds:

$$\exists \{x_n\} \subset X, f(x_n) \to -\infty, \text{ such that } \frac{f(x_n)}{\|x_n\|} \to -\infty$$

Let $(y,\beta) = \lim_{n\to+\infty} \frac{(x_n, f(x_n))}{\|(x_n, f(x_n))\|}$ and $d = \lim_{n\to+\infty} \frac{x_n}{\|x_n\|}$. Being $\frac{f(x_n)}{\|x_n\|} \to -\infty$ we have that $f(x_n) < 0$ definitively and:

$$y = \lim_{n\to+\infty} \frac{x_n}{\|(x_n, f(x_n))\|} = \lim_{n\to+\infty} \frac{\frac{x_n}{\|x_n\|}}{\|(\frac{x_n}{\|x_n\|}, \frac{f(x_n)}{\|x_n\|})\|} = 0$$

$$\beta = \lim_{n\to+\infty} \frac{f(x_n)}{\|(x_n, f(x_n))\|} = \lim_{n\to+\infty} \frac{\frac{f(x_n)}{|f(x_n)|}}{\|(\frac{x_n}{\|x_n\|} \frac{\|x_n\|}{|f(x_n)|}, \frac{f(x_n)}{|f(x_n)|})\|} = -1$$

Being $\{(x_n, f(x_n))\} \subset \text{epi}(f)$ and $\|(x_n, f(x_n))\| \to +\infty$ we have that $(0,-1) \in (\text{epi}(f))_\infty$; hence $\{(0,\mu) : \mu < 0\} \subset (\text{epi}(f))_\infty = \text{epi}(f_\infty)$ implying $f_\infty(0) = -\infty$ which is a contradiction.

ii) We prove that if $y \neq 0$ then $f_\infty(y)$ given by (1) is equivalent to the following:

$$f_\infty(y) = \|y\| \inf_{\{x_n\}\subset\Re^m} \left[\liminf_{n\to+\infty} \frac{f(x_n)}{\|x_n\|} : \|x_n\| \to +\infty, \frac{x_n}{\|x_n\|} \to \frac{y}{\|y\|}\right]$$

Let $x_n = t_n v_n$; being $t_n \to +\infty$ we can assume $t_n > 0\ \forall n$ so that $\|x_n\| = t_n \|v_n\|$. Firstly note that condition $v_n \to y$ is equivalent to the two conditions $\|v_n\| = \frac{\|x_n\|}{t_n} \to \|y\|$ and $\frac{v_n}{\|v_n\|} = \frac{x_n}{\|x_n\|} \to \frac{y}{\|y\|}$; observe also that the couple of conditions $t_n \to +\infty$ and $\|v_n\| = \frac{\|x_n\|}{t_n} \to \|y\|$ are equivalent to the two following ones: $\|x_n\| \to +\infty$ and $\|v_n\| = \frac{\|x_n\|}{t_n} \to \|y\|$. Formula (1) can then be rewritten as:

$$f_\infty(y) = \inf_{\{v_n\},\{x_n\}\subset\Re^m} \left[\liminf_{n\to+\infty} \frac{f(x_n)}{\|x_n\|} \|v_n\| : \|x_n\| \to +\infty, \|v_n\| \to \|y\|, \frac{x_n}{\|x_n\|} \to \frac{y}{\|y\|}\right]$$

It results also:

$$\|y\| = \inf_{\{v_n\}\subset\Re^m} \left[\liminf_{n\to+\infty} \|v_n\| : \|v_n\| \to \|y\|\right]$$

so that the thesis is proved.

3. Coercivity concepts

In the literature, the existence of minimum points for unconstrained problems is strictly related to the coercivity property of the objective function. Several definitions of coercivity can be found in the literature and different concepts are often referred to with the same name.

In this section we aim at defining some coercivity concepts related to constrained problems, generalizing the ones given in the literature and trying to study them in a unified framework. From now on we use the following notation:

$$\liminf_{\|x\|\to+\infty_D} f(x) = \inf_{\{x_n\}\subset D} \left[\liminf_{n\to+\infty} f(x_n) : \|x_n\| \to +\infty\right]$$

Definition 2. *Let us consider problem P_D. Function f is said to be* [4]*:*

i) semicoercive on D if $\exists x_0 \in D$ such that $\liminf_{\|x\| \to +\infty_D} f(x) \geq f(x_0)$
ii) coercive on D if $\liminf_{\|x\| \to +\infty_D} f(x) = +\infty$
iii) strictly coercive on D if $\liminf_{\|x\| \to +\infty_D} \frac{f(x)}{\|x\|} > 0$
iv) supercoercive on D if $\liminf_{\|x\| \to +\infty_D} \frac{f(x)}{\|x\|} = +\infty$
v) k-supercoercive on D, with $k \geq 1$, if $\liminf_{\|x\| \to +\infty_D} \frac{f(x)}{\|x\|^k} > 0$

In the case $D = X$, function f will be simply said to be semicoercive, coercive, strictly coercive, supercoercive or k-supercoercive.

Directly from the given definitions, we have that on the set D a coercive function is also semicoercive, a strictly coercive function is also coercive, a supercoercive function is also strictly coercive, a k-supercoercive function is also supercoercive for any $k > 1$, a strictly coercive function is nothing but a k-supercoercive function with $k = 1$.

Let us now point out that these introduced concepts are not sufficient to guarantee neither the existence of minimum points for problem P_D nor the lower boundedness on compacts of the function, as it is shown in the next Example 2 where function f is k-supercoercive but not lower bounded on compacts.

Example 2. Let us consider the following function $f : \Re \to \Re \cup \{+\infty\}$:

$$f(x) = \begin{cases} |x|^k + \log(x^2) & \text{for } x \neq 0 \\ +\infty & \text{for } x = 0 \end{cases}$$

with $k > 1$. Function f is k-supercoercive on $X = \Re$ even if $S_D = \emptyset$ since $\inf_{x \in \Re} f(x) = \lim_{x \to 0} f(x) = -\infty$.

The following results provide a characterization of k-supercoercive and strictly coercive functions.

Theorem 2. *Let us consider problem P_D. Then:*

i) f is k-supercoercive on D if and only if the following condition holds:

$$\exists a, b, c \in \Re, \ a > 0, \ \text{such that } f(x) \geq a \|x\|^k + b \ \ \forall x \in D, \ \|x\| > c$$

ii) f is k-supercoercive on D and lower bounded on compacts if and only if the following condition holds:

$$\exists a, b \in \Re, \ a > 0, \ \text{such that } f(x) \geq a \|x\|^k + b \ \ \forall x \in D$$

[4] Note that, in the case $D = X$, in [8] the supercoercive functions are referred to as *1-coercive*, while in [2,3] the strictly coercive functions are called *coercive*.

Proof. *i)* $\Leftarrow$) From the hypothesis it is:

$$\liminf_{\|x\|\to+\infty_D} \frac{f(x)}{\|x\|^k} \geq \liminf_{\|x\|\to+\infty_D} \left(a + \frac{b}{\|x\|^k}\right) = a > 0$$

so that f is k-supercoercive.

i) $\Rightarrow$) Suppose on the contrary that $\forall a, b, c \in \Re$, $a > 0$, $\exists x \in D$, $\|x\| > c$, such that $f(x) < a\|x\|^k + b$. Choose $a = \frac{1}{n}$, $b = 0$ and $c = n$, $n = 1, 2, 3, \ldots$; then $\exists x_n \in D$, $\|x_n\| > n$, such that $\frac{f(x_n)}{\|x_n\|^k} < \frac{1}{n}$. This implies:

$$\liminf_{\|x\|\to+\infty_D} \frac{f(x)}{\|x\|^k} \leq \lim_{n\to+\infty} \frac{f(x_n)}{\|x_n\|^k} \leq 0$$

which is a contradiction.

ii) $\Leftarrow$) Analogously to *i)* we have that f is k-supercoercive; f is also lower bounded, since $f(x) \geq a\|x\|^k + b \geq b \ \forall x \in D$.

ii) $\Rightarrow$) Suppose on the contrary that $\forall a, b \in \Re$, $a > 0$, $\exists x \in D$ such that $f(x) < a\|x\|^k + b$. Choose $a = \frac{1}{n}$ and $b = -n$; then $\exists x_n \in D$ such that $f(x_n) < \frac{1}{n}\|x_n\|^k - n$. If the sequence $\{x_n\}$ is bounded then $f(x_n) \to -\infty$ so that f is lower unbounded on compacts, which is a contradiction. If the sequence $\{x_n\}$ is unbounded then:

$$\liminf_{\|x\|\to+\infty_D} \frac{f(x)}{\|x\|^k} \leq \lim_{n\to+\infty} \frac{f(x_n)}{\|x_n\|^k} < \lim_{n\to+\infty} \left(\frac{1}{n} - \frac{n}{\|x_n\|^k}\right) \leq 0$$

which is a contradiction.

Since a strictly coercive function is nothing but a k-supercoercive function with $k = 1$, the following corollary trivially holds.

Corollary 1. *Let us consider problem P_D. Then:*

i) f is strictly coercive on D if and only if the following condition holds:

$$\exists a, b, c \in \Re, \ a > 0, \ \text{such that } f(x) \geq a\|x\| + b \ \ \forall x \in D, \ \ \|x\| > c$$

ii) f is strictly coercive on D and lower bounded on compacts if and only if the following condition holds:

$$\exists a, b \in \Re, \ a > 0, \ \text{such that } f(x) \geq a\|x\| + b \ \ \forall x \in D$$

Let us now introduce the following definition of recession function for constrained problems.

Definition 3. *Let us consider problem P_D. We say recession function over D the function $f_\infty^D : D_\infty \to \Re \cup \{-\infty, +\infty\}$ defined as follows:*

i) $f_\infty^D(0) = 0$ *if and only if* f *is lower bounded on compacts over* D *and:*

$$\liminf_{\|x\|\to+\infty_D} \frac{f(x)}{\|x\|} > -\infty,$$

$f_\infty^D(0) = -\infty$ *otherwise;*
ii) for any $y \neq 0$ *it is:*

$$f_\infty^D(y) = \|y\| \left[\liminf_{\|x\|\to+\infty_D} \frac{f(x)}{\|x\|} : \frac{x}{\|x\|} \to \frac{y}{\|y\|} \right]$$

Analogously to the set $\ker(f_\infty)$ we will use the following notation:

$$\ker(f_\infty^D) = \{y \in D_\infty : f_\infty^D(y) = 0\}.$$

Let us note that, directly from the definition ([5]), it is:

$$f_\infty(y) \le f_\infty^D(y) \ \forall y \in D_\infty$$

and that the equality may not hold, as it is shown in the next example.

Example 3. Let us consider $X = \Re^2$, the function $f : \Re^2 \to \Re$ such that $f(x,y) = xy$, and let $D = D_\infty = \Re^2_+ = \{(x,y) \in \Re^2 : x \ge 0, y \ge 0\}$. It results $f_\infty(y) = f_\infty^D(y)\ \forall y \in \mathrm{Int}(D_\infty)$ while $-\infty = f_\infty(y) < f_\infty^D(y) = 0\ \forall y \in \mathrm{Fr}(D_\infty)$.

It is now possible to study the relationships existing among the coercivity concepts and the recession function. With this aim it is worth noticing that the coercivity concepts are not based on the recession function associated with f, but just on the behavior of f when $\|x\| \to +\infty$; actually in the literature the coercivity concepts are often defined by means of the recession function ([6]), so that they wrongly seem to be necessarily related. Note that in the case $D = X$ statement *ii)* of the following theorem has already been proved by Zălinescu [11]; for the sake of completeness we provide an independent proof of the whole theorem.

Theorem 3. *Let us consider problem* P_D.

i) if f *is semicoercive on* D *then* $f_\infty^D(y) \ge 0\ \forall y \in D_\infty,\ y \neq 0$
ii) f *is strictly coercive on* D *if and only if* $f_\infty^D(y) > 0\ \forall y \in D_\infty,\ y \neq 0$
iii) f *is supercoercive on* D *if and only if* $f_\infty^D(y) = +\infty\ \forall y \in D_\infty,\ y \neq 0$

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[5] Note also that the following implication holds:

$$g(x) = \begin{cases} f(x) & \text{for } x \in D \\ +\infty & \text{for } x \notin D \end{cases} \quad \Longrightarrow \quad g_\infty(y) = \begin{cases} f_\infty^D(y) & \text{for } x \in D_\infty \\ +\infty & \text{for } x \notin D_\infty \end{cases}$$

[6] In [2–4], for example, in the case $D = X$ a function f is said to be coercive (strictly coercive with our notation) if $f_\infty(y) > 0\ \forall y \neq 0$ and is said to be *weakly coercive* if $f_\infty(y) \ge 0\ \forall y \neq 0$ and f is constant on each line with direction $y \in \ker(f_\infty)$.

Proof. i) By means of the semicoercivity of f it results:

$$\liminf_{\|x\|\to+\infty_D} \frac{f(x)}{\|x\|} \geq \liminf_{\|x\|\to+\infty_D} \frac{f(x_0)}{\|x\|} = 0$$

so that $f^D_\infty(y) \geq 0 \ \forall y \in D_\infty, \ y \neq 0$.

ii) $\Rightarrow$) The thesis follows directly from the definitions.

$\Leftarrow$) From Theorem 2 it is sufficient to prove that $\exists a, b, c \in \Re$, $a > 0$, such that $f(x) \geq a\|x\| + b \ \forall x \in D$, $\|x\| > c$. Suppose on the contrary that $\forall a, b, c \in \Re$, $a > 0$, $\exists x \in D$, $\|x\| > c$, such that $f(x) < a\|x\| + b$. Choose $a = \frac{1}{n}$, $b = 0$ and $c = n$, $n = 1, 2, 3, \ldots$; then $\exists x_n \in D$, $\|x_n\| > n$, such that $\frac{f(x_n)}{\|x_n\|} < \frac{1}{n}$. Defining $y = \lim_{n\to+\infty} \frac{x_n}{\|x_n\|} \in D_\infty$ we then have $f^D_\infty(y) \leq \lim_{n\to+\infty} \frac{f(x_n)}{\|x_n\|} \leq 0$ which is a contradiction.

iii) $\Rightarrow$) The thesis follows directly from the definitions.

$\Leftarrow$) Suppose on the contrary that:

$$\liminf_{\|x\|\to+\infty_D} \frac{f(x)}{\|x\|} = \inf_{\{x_n\}\subset D} \left[\liminf_{n\to+\infty} \frac{f(x_n)}{\|x_n\|} : \|x_n\| \to +\infty\right] < +\infty$$

then $\exists\{x_n\} \subset D$ such that $\|x_n\| \to +\infty$ and $\liminf_{n\to+\infty} \frac{f(x_n)}{\|x_n\|} < +\infty$; denoting with $y = \lim_{n\to+\infty} \frac{x_n}{\|x_n\|} \in D_\infty$ it results $f^D_\infty(y) < +\infty$ and this is a contradiction.

Remark 1. Note that having $f^D_\infty(y) = +\infty \ \forall y \in D_\infty$, $y \neq 0$, that is having a supercoercive function f, does not imply that $f^D_\infty(0) = 0$, as it is shown in Example 2 where f is k-supercoercive and lower unbounded on compacts so that $f^D_\infty(0) = -\infty$.

In the next section we are going to deep on conditions characterizing the existence of minimum points for a proper function f.

As a preliminary result, let us point out the behavior of coercivity with respect of the existence of minimum points.

Theorem 4. *Let us consider problem P_D as well as the following properties:*

i) f *is lower semicontinuous and* $f^D_\infty(y) > 0 \ \forall y \in D_\infty, \ y \neq 0$
ii) f *is strictly coercive on* D *and lower semicontinuous*
iii) f *is semicoercive on* D *and lower semicontinuous*
iv) $S_D \neq \emptyset$
v) f *is semicoercive on* D *and* $f^D_\infty(0) = 0$
vi) f *is semicoercive on* D *and lower bounded on compacts over* D
vii) $\inf_{x\in D} f(x) > -\infty$
viii) f *is lower bounded on compacts over* D *and* $f^D_\infty(y) \geq 0 \ \forall y \in D_\infty, \ y \neq 0$
ix) $f^D_\infty(y) \geq 0 \ \forall y \in D_\infty$

Then the following sequence of implications holds:
i) $\Leftrightarrow$ *ii)* $\Rightarrow$ *iii)* $\Rightarrow$ *iv)* $\Rightarrow$ *v)* $\Leftrightarrow$ *vi)* $\Rightarrow$ *vii)* $\Rightarrow$ *viii)* $\Leftrightarrow$ *ix)*

Proof. *i)* $\Leftrightarrow$ *ii)* Follows directly from *ii)* of Theorem 3.

ii) $\Rightarrow$ *iii)* From definition 2 trivially follows that any strictly coercive function is also semicoercive. Hence the result follows directly from *i)* of Theorem 3.

iii) $\Rightarrow$ *iv)* Let $\{x_n\} \subset D$ be a sequence such that $f(x_n) \to \inf_{x \in D} f(x)$. If $\{x_n\}$ is unbounded ($\|x_n\| \to +\infty$) then by means of the semicoercivity on D of f we have that $\exists x_0 \in D$ such that $f(x_0) \le \liminf_{\|x\| \to +\infty_D} f(x) \le \inf_{x \in D} f(x)$ so that $x_0 \in S_D$ and hence $S_D \neq \emptyset$. If $\{x_n\}$ is bounded, that is $\exists \mu > 0$ such that $\|x_n\| \le \mu$, then it is possible to extract a converging subsequence $\{x_j\} \subseteq \{x_n\}$ such that, being D closed, $x_j \to \overline{x} \in D$; from the lower semicontinuity of f we then have $\inf_{x \in D} f(x) = \liminf_{j \to +\infty} f(x_j) \ge f(\overline{x})$ so that $\overline{x} \in S_D$ and hence $S_D \neq \emptyset$.

iv) $\Rightarrow$ *v)* The semicoercivity of f follows directly from the definition choosing x_0 among the points of S_D; being $S_D \neq \emptyset$ then f is also lower bounded so that $f_\infty^D(0) = 0$ by means of i) of Definition 3.

v) $\Rightarrow$ *vi)* Follows directly from i) of Definition 3.

vi) $\Rightarrow$ *vii)* Suppose by contradiction that $\inf_{x \in D} f(x) = -\infty$ and let $\{x_n\} \subset D$ be a sequence such that $f(x_n) \to \inf_{x \in D} f(x) = -\infty$. Being f semicoercive on D the sequence $\{x_n\}$ is bounded and this contradicts the fact that f is lower bounded on compacts over D.

vii) $\Rightarrow$ *viii)* Function f has a finite infimum on D, hence it is trivially lower bounded on compacts over D; suppose now on the contrary that $\exists y \in D_\infty$, $y \neq 0$, such that $f_\infty^D(y) < 0$. Let $\{x_n\} \subset D$, $\|x_n\| \to +\infty$, be the sequence such that:

$$f_\infty^D(y) = \|y\| \lim_{n \to +\infty} \frac{f(x_n)}{\|x_n\|} \quad \text{with} \quad \frac{x_n}{\|x_n\|} \to \frac{y}{\|y\|}$$

It then results:

$$\lim_{n \to +\infty} f(x_n) = \lim_{n \to +\infty} \frac{f(x_n)}{\|x_n\|} \lim_{n \to +\infty} \|x_n\| = \frac{f_\infty^D(y)}{\|y\|} \lim_{n \to +\infty} \|x_n\| = -\infty$$

and this is a contradiction.

viii) $\Rightarrow$ *ix)* Suppose on the contrary that $f_\infty^D(0) < 0$; being f lower bounded on compacts over D it follows from i) of Definition 3 that:

$$\liminf_{\|x\| \to +\infty_D} \frac{f(x)}{\|x\|} = -\infty$$

so that $\exists y \in D_\infty$, $y \neq 0$, such that $f_\infty^D(y) = -\infty$ and this is a contradiction.

ix) $\Rightarrow$ *viii)* Just note that, by means of the definition, $f_\infty^D(0) = 0$ implies that f is lower bounded on compacts over D.

vi) $\Rightarrow$ *v)* By means of the previously proved implications, we have that *vi)* implies *ix)* so that $f_\infty^D(0) = 0$.

The following examples point out that the implications given in the previous theorem are proper.

Example 4. Let us consider the following functions $f : \Re \to \Re \cup \{+\infty\}$ and let $D = X = \Re$:

i) $f(x) = 0$ is a continuous semicoercive function which is not strictly coercive; hence in Theorem 4 *iii)* $\not\Rightarrow$ *ii)*.

ii) $f(x) = \begin{cases} 0 & \text{for } x < 0 \\ x+1 & \text{for } x \geq 0 \end{cases}$ is semicoercive and $S_D = \{x : x < 0\} \neq \emptyset$ even if f is not lower semicontinuous; hence in Theorem 4 *iv)* $\not\Rightarrow$ *iii)*.

iii) $f(x) = \begin{cases} x^2 & \text{for } x \leq 0 \\ x^2 - 1 & \text{for } x > 0 \end{cases}$ is supercoercive with $f^D_\infty(0) = f_\infty(0) = 0$ but it is not lower semicontinuous and $S_D = \emptyset$ with $\inf_{x \in \Re} f(x) = -1 > -\infty$; hence in Theorem 4 *v)* $\not\Rightarrow$ *iv)*. Note that this example shows the importance of the lower semicontinuity of f in order to have minimum points, providing that:

$$f \text{ is supercoercive, } f^D_\infty(0) = 0 \text{ and } \inf f(x) > -\infty \quad \not\Rightarrow \quad S_D \neq \emptyset.$$

iv) $f(x) = \begin{cases} \frac{1}{x^2} & \text{for } x \neq 0 \\ +\infty & \text{for } x = 0 \end{cases}$ is lower bounded on compacts with finite infimum, even if it is not semicoercive; hence in Theorem 4 *vii)* $\not\Rightarrow$ *vi)*.

v) $f(x) = \begin{cases} -\log(x^2) & \text{for } x \neq 0 \\ +\infty & \text{for } x = 0 \end{cases}$ is lower semicontinuous, hence lower bounded on compacts, and $f^D_\infty(y) = f_\infty(y) = 0 \ \forall y \in \Re$, even if f is not semicoercive and $\inf_{x \in \Re} f(x) = -\infty$; hence in Theorem 4 *viii)* $\not\Rightarrow$ *vii)*. Moreover, the converse of *i)* in Theorem 3 does not hold and $f^D_\infty(y) \geq 0 \ \forall y \in \Re^m$ is a necessary but not sufficient condition for the existence of a finite infimum.

4. Existence of Optimal Solutions

Theorem 4 underlines the importance of the recession function and coercivity properties for optimality conditions. It comes immediately out that strictly coercivity on D together with the lower semincontinuity is a sufficient optimality condition while $f^D_\infty(y) \geq 0$ for every $y \in D_\infty$ is a necessary one. When the feasible region D is the whole space, these conditions coincide with the standard ones existing in the literature, but is worthy noting that in general we cannot replace f^D_∞ with f_∞ as it is shown by Example 3 where the function $f(x, y)$ has minimum points over $D = \Re^2_+$ and $f^D_\infty(y) \geq 0 \ \forall y \in D_\infty$ while $f_\infty(y) = -\infty$ $\forall y \in \text{Fr}(D_\infty)$. Analogously to the unconstrained case, condition $f^D_\infty(y) \geq 0$ is necessary but not sufficient for the existence of the minimum points and this suggests to investigate the behavior of f along the direction $y \neq 0$ such that $f^D_\infty(y) = 0$.

The goal of this section is to state results characterizing the existence of minimum points analyzing the behavior of the objective function f along the directions $y \in \ker(f^D_\infty)$. In particular we will generalize in different ways the following result.

Proposition 1 (Auslender [2]). *Let us consider problem P_D with $D = X$ and suppose function f to be lower semicontinuous. $S_D \neq \emptyset$ if and only if both the two following conditions hold:*

i) $f_\infty(d) \geq 0 \ \forall d \in \Re^m$,
ii) $\forall\{x_n\} \subset \Re^m$, *such that* $\|x_n\| \to +\infty$, $\frac{x_n}{\|x_n\|} \to d \in \ker(f_\infty)$ *and the sequence* $\{f(x_n)\}$ *is bounded above, there exist* $\rho_n \in]0,1]$, $z_n \to z$, *with* $\|d - z\| < 1$, *such that for n sufficiently large:*

$$f(x_n - \|x_n\| \rho_n z_n) \leq f(x_n)$$

The generalization of the previous Proposition 1 is based on the following lemma (Attouch [1] gives a result similar to statements *i)* and *ii)* in reflexive Banach spaces, while Zălinescu [11] proves an analogous theorem under stronger assumptions).

Lemma 1. *Let us consider problem P_D and suppose function f to be lower semicontinuous. Suppose also that $f_\infty^D(y) \geq 0 \ \forall y \in D_\infty$ and define, for any integer $n \geq 1$, the following auxiliary function:*

$$g_n(x) = f(x) + \epsilon_n \|x\|^2, \quad n = 1, 2, \ldots$$

where $\epsilon_n > 0 \ \forall n$ and $\epsilon_n \to 0^+$. Then the following properties hold:

i) $\exists x_n \in \operatorname{argmin}\{g_n(x), x \in D\}$ *for any integer $n \geq 1$;*
ii) $\lim_{n\to+\infty} f(x_n) = \inf_{x\in D} f(x)$

If the sequence $\{\epsilon_n\}$ is also strictly decreasing then:

iii) $\|x_n\| \leq \|x_{n+1}\|$ *for any integer $n \geq 1$;*
iv) the sequence $\{f(x_n)\}$ is decreasing;

Proof. i) Being $f_\infty^D(y) \geq 0 \ \forall y \in D_\infty$ it comes out that function g_n is supercoercive on D for any integer n and hence for Theorem 4 there exists $x_n \in \operatorname{argmin}\{g_n(x), x \in D\}$.

ii) Let z be any element of D. By means of the definition, for any integer n we have:

$$f(x_n) \leq f(x_n) + \epsilon_n \|x_n\|^2 = g_n(x_n) \leq g_n(z) = f(z) + \epsilon_n \|z\|^2$$

so that:

$$\lim_{n\to+\infty} f(x_n) \leq \lim_{n\to+\infty} f(z) + \epsilon_n \|z\|^2 = f(z) \quad \forall z \in D$$

and hence $\lim_{n\to+\infty} f(x_n) = \inf_{x\in D} f(x)$.

iii) Let us consider, for any n, the following inequalities:

$$f(x_n) + \epsilon_n \|x_n\|^2 = g_n(x_n) \leq g_n(x_{n+1}) = f(x_{n+1}) + \epsilon_n \|x_{n+1}\|^2$$
$$f(x_{n+1}) + \epsilon_{n+1} \|x_{n+1}\|^2 = g_{n+1}(x_{n+1}) \leq g_{n+1}(x_n) = f(x_n) + \epsilon_{n+1} \|x_n\|^2$$

adding these inequalities it yields:

$$\epsilon_n \|x_n\|^2 + \epsilon_{n+1} \|x_{n+1}\|^2 \leq \epsilon_n \|x_{n+1}\|^2 + \epsilon_{n+1} \|x_n\|^2$$

so that:

$$(\epsilon_n - \epsilon_{n+1}) \|x_n\|^2 \leq (\epsilon_n - \epsilon_{n+1}) \|x_{n+1}\|^2$$

Being the sequence $\{\epsilon_n\}$ strictly decreasing it is $(\epsilon_n - \epsilon_{n+1}) > 0$ so that $\|x_n\|^2 \leq \|x_{n+1}\|^2$ and the thesis is proved.

iv) By means of the definition of $g_{n+1}(x)$ and being $\|x_n\|^2 \leq \|x_{n+1}\|^2$, we have that for any n:

$$\begin{aligned} f(x_{n+1}) + \epsilon_{n+1} \|x_{n+1}\|^2 &= g_{n+1}(x_{n+1}) \leq g_{n+1}(x_n) = \\ &= f(x_n) + \epsilon_{n+1} \|x_n\|^2 \leq f(x_n) + \epsilon_{n+1} \|x_{n+1}\|^2 \end{aligned}$$

so that $f(x_{n+1}) \leq f(x_n)$ and the thesis is proved.

By means of Lemma 1, we are able to state the following characterization for the existence of optimal points.

Theorem 5. *Let us consider problem P_D and suppose function f to be lower semicontinuous. $S_D \neq \emptyset$ if and only if both the two following conditions hold:*

i) $f^D_\infty(y) \geq 0 \ \forall y \in D_\infty$,

ii) $\forall \{x_n\} \subset D$, *such that* $\|x_n\| \to +\infty$, $\frac{x_n}{\|x_n\|} \to y \in \ker(f^D_\infty)$ *and the sequence* $\{f(x_n)\}$ *is strictly decreasing, there exist* $\{\rho_n\} \subset]0,1]$, $\{z_n\} \subset X$, $\{v_n\} \subset X$, *with* $\|v_n\| = 1$ *and* $(\|x_n\| (v_n - \rho_n z_n)) \in D$, *such that for n sufficiently large:*

$$\|v_n - z_n\| < 1 \quad \textit{and} \quad f(\|x_n\| (v_n - \rho_n z_n)) \leq f(x_n)$$

Proof. $\Rightarrow$) Condition *i)* follows directly from Theorem 4; as regards to condition *ii)* let $y_0 \in S_D \neq \emptyset$ and assume $\rho_n = 1$, $v_n = \frac{x_n}{\|x_n\|}$ and $z_n = \frac{x_n - y_0}{\|x_n\|}$; it results $(\|x_n\| (v_n - \rho_n z_n)) = y_0 \in D$, $\|v_n\| = 1$ and, for n sufficiently large, $\|v_n - z_n\| = \frac{\|y_0\|}{\|x_n\|} < 1$, so that the thesis is proved since $f(y_0) \leq f(x_n)$ being $y_0 \in S_D$.

$\Leftarrow$) Let $g_n = f(x) + \epsilon_n \|x\|^2$, $n = 1, 2, \ldots$, with $\{\epsilon_n\}$ strictly decreasing; by means of condition *i)* and Lemma 1 we have that $\exists x_n \in \operatorname{argmin}\{g_n(x), x \in D\}$ $\forall n$ and that $\{f(x_n)\}$ is decreasing with $\lim_{n \to +\infty} f(x_n) = \inf_{x \in D} f(x)$. Let us now prove that if the sequence $\{x_n\}$ is unbounded then we can not extract a strictly decreasing subsequence of $\{f(x_n)\}$. Suppose by contradiction that $\|x_n\| \to +\infty$ and that $\{f(x_n)\}$ is strictly decreasing (substituting $\{x_n\}$ with a subsequence if necessary): let $y = \lim_{n \to +\infty} \frac{x_n}{\|x_n\|}$, being $f(x_n) < f(x_1)$ we have $\lim_{n \to +\infty} \frac{f(x_n)}{\|x_n\|} \leq \lim_{n \to +\infty} \frac{f(x_1)}{\|x_n\|} = 0$ so that, by means of the definition, it results $f^D_\infty(y) \leq 0$ and hence, for *i)*, $y \in \ker(f_\infty)$. By means of *ii)* there exist $\rho_n \in]0,1]$, $z_n, v_n \in X$, with $\|v_n\| = 1$ and $(\|x_n\| (v_n - \rho_n z_n)) \in D$, such that for n sufficiently large $\|v_n - z_n\| < 1$ and $f(\|x_n\|(v_n - \rho_n z_n)) \leq f(x_n)$; it then results:

$$\begin{aligned} f(x_n) + \frac{1}{n}\|x_n\|^2 &= g_n(x_n) \leq g_n(\|x_n\|(v_n - \rho_n z_n)) \\ &= f(\|x_n\|(v_n - \rho_n z_n)) + \frac{1}{n}\|\|x_n\|(v_n - \rho_n z_n)\|^2 \\ &\leq f(x_n) + \frac{1}{n}\|x_n\|^2\|v_n - \rho_n z_n\|^2 \end{aligned}$$

and hence, being $0 < \rho_n \leq 1$, $\|v_n\| = 1$, and $\|v_n - z_n\| < 1$:

$$\begin{aligned} 1 \leq \|v_n - \rho_n z_n\| &= \|v_n - \rho_n v_n + \rho_n v_n - \rho_n z_n\| \\ = \|v_n(1-\rho_n) + \rho_n(v_n - z_n)\| &\leq \|v_n\|(1-\rho_n) + \rho_n\|v_n - z_n\| \\ &= 1 + \rho_n(\|v_n - z_n\| - 1) < 1 \end{aligned}$$

which is impossible.

We then have that $\{f(x_n)\}$ does not admit any strictly decreasing subsequence regardless the sequence corresponding $\{x_n\}$ is bounded or unbounded. In the latter case $\exists q$ such that $f(x_n) = f(x_q)$ $\forall n \geq q$, $x_q \in D$, and hence $\inf_{x\in D} f(x) = \lim_{n\to+\infty} f(x_n) = f(x_q)$ so that the infimum is reached as a minimum. If otherwise $\{x_n\}$ is bounded then it is possible to extract a subsequence $\tilde{x}_n \to y_0 \in D$ such that, by means of the lower semicontinuity of f, $\inf_{x\in D} f(x) = \lim_{n\to+\infty} f(\tilde{x}_n) = f(y_0)$, implying that the infimum is reached as a minimum. The proof is then complete.

Remark 2. It is worth noticing the differences existing between Theorem 5 and the known Proposition 1:

i) in Proposition 1 the unconstrained case is considered, while in Theorem 5 we study a constrained problem, covering the unconstrained case just assuming $D = X$;
ii) in Proposition 1 v_n is fixed to be equal to $\frac{x_n}{\|x_n\|}$, while in Theorem 5 v_n may be any vector with $\|v_n\| = 1$;
iii) in Proposition 1 it is required that $\|v_n - z_n\| \to \|v - z\| < 1$, while in Theorem 5 we simply assume that definitively $\|v_n - z_n\| < 1$, thus we cover also the case $\|v_n - z_n\| \to \|v - z\| = 1$;
iv) in Proposition 1 all the bounded above sequences $\{f(x_n)\}$ are considered, while in Theorem 5 just the strictly decreasing ones.

The following useful corollary can also be stated, providing a practical condition to determine the existence of optimal points. Note that this result can not be derived from Proposition 1.

Corollary 2. *Let us consider problem P_D and suppose function f to be lower semicontinuous. $S_D \neq \emptyset$ if and only if both the two following conditions hold:*

i) $f_\infty^D(y) \geq 0 \ \forall y \in D_\infty$,
ii) $\forall \{x_n\} \subset D$, *such that* $\|x_n\| \to +\infty$, $\frac{x_n}{\|x_n\|} \to y \in \ker(f_\infty^D)$ *and the sequence* $\{f(x_n)\}$ *is strictly decreasing, there exists* $\{y_n\} \subset D$ *such that for n sufficiently large:*

$$\|y_n\| < \|x_n\| \quad \textit{and} \quad f(y_n) \leq f(x_n)$$

Proof. $\Rightarrow$) Condition *i)* follows directly from Theorem 4; as regards to condition *ii)* we just have to choose $y_0 \in S_D \neq \emptyset$ and assume $y_n = y_0 \ \forall n$.

$\Leftarrow$) Assume $\rho_n = 1$, $v_n = \frac{x_n}{\|x_n\|}$ and $z_n = \frac{x_n - y_n}{\|x_n\|}$; it results $\{\rho_n\} \subset]0,1]$, $\|v_n\| = 1$, $(\|x_n\|(v_n - \rho_n z_n)) = y_n \in D$ so that, by means of the hypothesis, for n sufficiently large it is:

$$\|v_n - z_n\| = \frac{\|y_n\|}{\|x_n\|} < 1 \quad \text{and} \quad f(\|x_n\|(v_n - \rho_n z_n)) = f(y_n) \leq f(x_n)$$

so that the thesis follows directly from Theorem 5.

Another useful result can be stated when the set D is asymptotically-linear. Remind that a nonempty closed set $D \subseteq X$ is said to be an *asymptotically-linear set* if $\forall \rho > 0$, $\forall \{x_n\} \subset D$ such that $\|x_n\| \to +\infty$ and $\frac{x_n}{\|x_n\|} \to y$, for n sufficiently large it is $(x_n - \rho y) \in D$ (see Auselender [4]).

Note that directly from the definition it follows that the finite intersection or the finite union of asymptotically-linear sets is still an asymptotically-linear set. Note also that, as it has been proved in Auselender [4], an asymptotically polyhedral set D is also asymptotically-linear ([7]).

The following sufficient condition holds.

Corollary 3. *Let us consider problem P_D and suppose function f to be lower semicontinuous and the set $D \subseteq X$ to be asymptotically-linear. If the two following conditions hold:*

i) $f_\infty^D(y) \geq 0 \ \forall y \in D_\infty$,
ii) $\forall \{x_n\} \subset D$, *such that* $\|x_n\| \to +\infty$, $\frac{x_n}{\|x_n\|} \to y \in \ker(f_\infty^D)$ *and the sequence* $\{f(x_n)\}$ *is strictly decreasing, there exists* $\rho > 0$ *such that for n sufficiently large* $f(x_n - \rho y) \leq f(x_n)$,

then the optimal set S_D is nonempty.

[7] Remind, see [4], that a set $D \subset X$ is said to be a *simple asymptotically polyhedral set* if $\exists \mu \geq 0$ for which the set $D_\mu := D \cap \{x : \|x\| \geq \mu\}$ admits the decomposition $D_\mu = K + M$ with K compact and M a polyhedral cone. D is said to be *asymptotically polyhedral* if it is the intersection of a finite number of sets, each of them being the union of a finite number of simple asymptotically polyhedral sets.

Proof. We just have to verify condition *ii)* of Corollary 2. Assume $y_n = (x_n - \rho y)$; being D asymptotically-linear then for n sufficiently large it is $y_n \in D$, by means of the hypothesis it is also $f(y_n) \leq f(x_n)$. We are left to prove that for n sufficiently large it is $\|x_n - \rho y\| < \|x_n\|$; with this aim let us firstly note that:

$$\lim_{n\to+\infty} y^T x_n = \lim_{n\to+\infty} y^T \frac{x_n}{\|x_n\|} \lim_{n\to+\infty} \|x_n\| = \|y\|^2 \lim_{n\to+\infty} \|x_n\| = +\infty;$$

then for n sufficiently large it is $y^T x_n > \frac{1}{2}\rho \|y\|^2$ and hence:

$$\|x_n - \rho y\|^2 = \|x_n\|^2 - 2\rho y^T x_n + \rho^2 \|y\|^2 < \|x_n\|^2$$

so that definitively $\|x_n - \rho y\| < \|x_n\|$ and the thesis is proved.

It is now worth to compare Corollary 3 with the following result.

Proposition 2 (Auslender [4]). *Let us consider problem P_D and suppose function f to be lower semicontinuous and the set $D \subseteq X$ to be asymptotically-linear. If the two following conditions hold:*

i) $\inf_{x\in D} f(x) > -\infty$ *and* $f_\infty(y) \geq 0$ $\forall y \in D_\infty,\ y \neq 0$,
ii) $f \in \mathcal{F}$ ([8]), *that is to say that* $\forall \{x_n\} \subset X$, *such that* $\|x_n\| \to +\infty$ *and* $\frac{x_n}{\|x_n\|} \to y \in \ker(f_\infty)$, $\forall \rho > 0$, *for n sufficiently large it results* $f(x_n - \rho y) \leq f(x_n)$,

then the optimal set S_D is nonempty.

Note that Corollary 3 is more general that the result by Auslender since:

i) in Proposition 2 $f_\infty(y) \geq 0$ is required, while in Corollary 3 $f_\infty^D(y) \geq 0$, with $f_\infty^D(y) \geq f_\infty(y)$, is just needed;
ii) as it has been proved in Theorem 4, condition *i)* of Proposition 2 implies condition *i)* of Corollary 3, while the converse is not true;
iii) in condition *ii)* of Proposition 2 all the sequences $\forall\{x_n\} \subset X$ are considered, while in *ii)* of Corollary 3 just the feasible sequences such that $\{f(x_n)\}$ is strictly decreasing are used;
iv) in condition *ii)* of Proposition 2 all $\rho > 0$ are considered, while in Corollary 3 just one of them is used.

Directly from Corollary 3 follows the next result which generalizes the sufficient condition, given in Auselender [2,3], related to asymptotically polyhedral sets and based on the hypothesis that the objective function f is constant along the directions $y \in \ker(f_\infty)$ (in other words, based on the hypothesis that f is weakly coercive).

[8] Note that this condition is equivalent to Definition 7 given in [4], page 51. To prove this note that from one side it represent the particular case of Def.7 with $\epsilon_n = f(x_n)$; on the other side it implies Def.7 since $S_f(\alpha) \subseteq S_f(\beta)$ $\forall \alpha \leq \beta$ so that for any ϵ_n if $x_n \in S_f(\epsilon_n)$ then $f(x_n) \leq \epsilon_n$ and $x_n - \rho y \in S_f(f(x_n)) \subseteq S_f(\epsilon_n)$.

Corollary 4. *Let us consider problem P_D and suppose function f to be lower semicontinuous and the set $D \subseteq X$ to be asymptotically-linear. If function f is nondecreasing along the directions $y \in \ker(f_\infty^D)$ then:*

$$S_D \neq \emptyset \iff f_\infty^D(y) \geq 0 \ \forall y \in D_\infty$$

We conclude this section looking for conditions related to the boundedness or unboundedness of the set S_D of minimum points.

Theorem 6. *Let us consider problem P_D and suppose function f to be lower semicontinuous and the set S_D to be nonempty. Then:*

i) if S_D is unbounded then $f_\infty^D(y) \geq 0 \ \forall y \in D_\infty$ and $\ker(f_\infty^D) \neq \{0\}$.

Suppose also D to be a closed convex set and f to be constant along the directions $y \in \ker(f_\infty^D)$; then:

ii) S_D is unbounded if and only if $f_\infty^D(y) \geq 0 \ \forall y \in D_\infty$ and $\ker(f_\infty^D) \neq \{0\}$.

Proof. i) By means of Theorem 4 we just have to prove that $\ker(f_\infty^D) \neq \{0\}$. Let $\{x_n\} \subset S_D$ such that $\|x_n\| \to +\infty$ and let $\frac{x_n}{\|x_n\|} \to y \in D_\infty$; it results $0 \leq f_\infty^D(y) \leq \lim_{n \to +\infty} \frac{f(x_n)}{\|x_n\|} = 0$ and hence $y \in \ker(f_\infty^D)$.

ii) $\Leftarrow$) Let $y \in \ker(f_\infty^D)$, $y \neq 0$, and let $x_0 \in S_D$; being D closed and convex then $x_0 + \lambda y \in D \ \forall \lambda > 0$, being f constant along the directions $y \in \ker(f_\infty^D)$, $y \neq 0$, we have also $f(x_0) = f(x_0 + \lambda y) \ \forall \lambda > 0$ and hence $x_0 + \lambda y \in S_D \ \forall \lambda > 0$.

The next result follows directly from Theorems 4 and 6.

Corollary 5. *Let us consider problem P_D and suppose function f to be lower semicontinuous. Then:*

i) if $f_\infty^D(y) > 0 \ \forall y \in D_\infty, \ y \neq 0$, then S_D is nonempty and bounded.

Suppose also D to be a closed convex set and f to be constant along the directions $y \in \ker(f_\infty^D)$; then:

ii) S_D is nonempty and bounded if and only if $f_\infty^D(y) > 0 \ \forall y \in D_\infty, \ y \neq 0$.

The following examples point out the importance in Theorems 6 and 5 of the convexity of D and the constant behavior of f along the directions $y \in \ker(f_\infty^D)$.

Example 5. Let us consider the following functions $f : \Re^2 \to \Re$ and sets $D \subset \Re^2$:

i) $f(x,y) = y$ and $D = \{x \geq 0, y \geq 1\} \cup \{x \geq 0, y \geq x\}$; D is nonconvex and $S_D = \{(0,0)^T\}$ is bounded even if $y = (1,0)^T \in \ker(f_\infty^D)$;
ii) $f(x,y) = y - {}^{-x^2}$ and $D = \Re^2_+$; D is convex, f is not constant along the direction $y = (1,0)^T \in \ker(f_\infty^D)$ and $S_D = \{(0,0)^T\}$ is bounded.

The following theorem is useful when the set D has empty interior, or when it is known "a priori" that there are no minimum points in the interior of D, such as in many mathematical programming problems.

Theorem 7. *Let us consider problem P_D and suppose function f to be lower semicontinuous, nondecreasing along the directions $y \in D_\infty$ and constant along the directions $y \in \ker(f_\infty^D)$, suppose also $D + D_\infty$, with $D \subsetneq D + D_\infty$, to be a closed convex set. If at least one of the following conditions hold:*

i) the set of minimum points for f over $(D + D_\infty) \setminus D$ is bounded,
ii) $D + D_\infty$ is strictly convex and the set of minimum points for f over $\mathrm{Int}(D + D_\infty) \setminus D$ is bounded,

then S_D is nonempty and bounded if and only if $f_\infty^D(y) > 0$ $\forall y \in D_\infty$, $y \neq 0$.

Proof. By means of Theorem 5 we just have to prove that if S_D is nonempty and bounded then $f_\infty^D(y) > 0$ $\forall y \in D_\infty$, $y \neq 0$. For Theorem 4 $f_\infty^D(y) \geq 0$ $\forall y \in D_\infty$; suppose now by contradiction that $\exists \overline{y} \in \ker(f_\infty^D)$, $\overline{y} \neq 0$ and let $x_0 \in S_D$. Being f nondecreasing along the directions $y \in D_\infty$ we have that $f(x_0) \leq f(x_0 + \lambda y)$ $\forall \lambda > 0$, $\forall y \in D_\infty$, so that x_0 is a minimum point for f over $D + D_\infty$; since $D + D_\infty$ is a closed convex set and $(D + D_\infty)_\infty = D_\infty$ then $x_0 + \lambda \overline{y} \in D + D_\infty$ $\forall \lambda > 0$; being f constant along the directions $y \in \ker(f_\infty^D)$ we then have $f(x_0) = f(x_0 + \lambda \overline{y})$ $\forall \lambda > 0$ and hence all the points $x_0 + \lambda \overline{y}$, $\forall \lambda > 0$, are minimum points for f over $D + D_\infty$; note also that being S_D bounded $\exists \overline{\lambda} > 0$ such that $x_0 + \lambda \overline{y} \in (D + D_\infty) \setminus D$, $\forall \lambda > \overline{\lambda} > 0$. The set of this minimum points is unbounded, which contradicts condition *i)*. If *ii)* holds then, being $D + D_\infty$ strictly convex, $x_0 + \lambda \overline{y} \in \mathrm{Int}(D + D_\infty)$, $\forall \lambda > 0$, and hence $x_0 + \lambda \overline{y} \in \mathrm{Int}(D + D_\infty) \setminus D$, $\forall \lambda > \overline{\lambda} > 0$, which is again a contradiction being this set of minimum points unbounded.

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References

1. H. Attouch, "Viscosity solutions of minimization problems", *SIAM J. Optim.*, Vol.6, no.3, pp.769-806, 1996.
2. A. Auslender, "Noncoercive Optimization Problems", *Math. Oper. Res.*, Vol.21, no.4, pp.769-782, 1996.
3. A. Auslender, "How to deal with the unbounded in optimization: Theory and Algorithms", *Math. Program.*, Vol.79, pp.3-18, 1997.
4. A. Auslender, "Existence of optimal solutions and duality results under weak conditions", *Math. Program.*, Vol.88, pp.45-59, 2000.
5. M.Avriel, W.E. Diewert, S. Schaible, and I. Zang, *Generalized Concavity, Mathematical Concepts and Methods in Science and Engineering*, vol.36, Plenum Press, New York, 1988.
6. C. Baiocchi, G. Buttazzo, F. Gastaldi and F. Tomarelli, "General existence theorems for unilateral problems in continuum mechanics", *Arch. Ration. Mech. Anal.*, Vol.100, pp.149-189, 1988.
7. G. Buttazzo and F. Tomarelli, "Nonlinear Neuman problems", *Adv. Math.*, Vol.89, pp.126-142, 1991.
8. J. B. Hiriart-Hurruti, Conditions for global optimality, R. Horst and P.M. Pardalos (eds.), *Handbook of Global Optimization*. Kluwer Academic Publishers, 1995, 1-26.

9. A. Mas Colell. M. D. Whinston, and J.R. Green , *Microeconomic Theory*, Oxford University Press, New York, NY, 1995.
10. R.T. Rockafellar, *Convex Analisys*. Princeton University Press. Princeton NJ, 1970
11. C. Zălinescu, Stability for a class of nonlinear optimization, in *Proceedings of the summer school in nonsmooth optimization and related topics - Erice*, F.H. Clarke, V. F. Demyanov and F. Giannessi (eds.), Plenum Press, New York, 1989, 437-458.