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A finite Algorithm for a Particular d.c. Quadratic Programming Problem

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A finite algorithm for a particular d.c. quadratic programming problem

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Abstract

In this paper a particular quadratic problem, represented by a d.c. optimization problem is studied. Some theoretical properties of the problem will be stated, as well as some optimality conditions and a finite solving algorithm.

Keywords: Quadratic Programming, D.C. Functions, D.C. Optimization.

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1. Introduction

In this paper we consider the following problem:

$$(1.1) \qquad \min f(x) = 1/2\, x^TQx + c^Tx - (d^Tx)^2$$
$$x \in X = \{x \in R^n : Ax \geq b\}$$

where Q is a symmetric positive definite nxn matrix, c, $d \in R^n$, A is a mxn matrix, $b \in R^m$. Since the objective function f(x) is the difference between two convex quadratic functions it follows that problem (1.1) represent a particular d.c. optimization problem, that is to say that its quadratic objective function is a d.c. function (difference of convex functions). Let us note that in the recent literature many papers appeared regarding to d.c. optimization, because of its many applications in operations research, economics, engineering design and other fields (see [9, 10, 11, 13, 14]). Let us note that any linear multiplicative maximization problem of the kind:

$$\max f(x) = (u^Tx+u_0)(v^Tx+v_0),\ x \in X = \{x \in R^n : Ax \geq b\},\ (u^Tx)^2+(v^Tx)^2>0\ \forall x \neq 0$$

can be transformed to problem (1.1) with $Q=uu^T+vv^T$, $c=2uu_0+2vv_0$, $d=\frac{\sqrt{2}}{2}(u+v)$, since

$$(u^Tx+u_0)(v^Tx+v_0)=-1/2\ ((u^Tx+u_0)^2+(v^Tx+v_0)^2)+1/2\ (u^Tx+u_0+v^Tx+v_0)^2.$$

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In section 2 we will provide some theoretical properties of the problem, in section 3 we will state some optimality conditions which will allow us to determine, in section 4, a finite solving algorithm for problem (1.1), which will work also when the feasible region X is unbounded, finally in section 5 we will provide some further computational remarks.

2. Properties of the problem

For the sake of simplicity, from now on we will use the notation $B=Q-2dd^T$, so that the objective function can be rewritten as $f(x)=1/2\ x^TBx+c^Tx$.

Let us now point out some properties of the problem. The following result shows that problem (1.1) may be a convex or indefinite problem, but never a concave one.

Theorem 2.1

Consider problem (1.1). If $n\geq2$ then function f(x) is not concave.

Proof. Let $x\in R^n$, $x\neq0$, such that $d^Tx=0$ (note that such a vector x exists being $n\geq2$). We then have $x^TBx=x^TQx-2(d^Tx)^2=x^TQx>0$ so that B is not negative semidefinite and function f(x) is not concave. ♦

Note that theorem 2.1 shows that the function f(x) is not concave on R^n, obviously f(x) may still have a concave behaviour restricted on the feasible region X.

A sufficient condition in order to have a convex problem is the following.

Theorem 2.2

Consider problem (1.1). If the following condition holds:

$$\min_{\|y\|=1} y^TQy\geq2\ \|d\|^2 \quad [>]$$

that is to say that $\lambda\geq2\ \|d\|^2$ [>] where λ is the smaller eigenvalue of Q, then function f(x) is [strictly] convex.

Proof. Let $x\in R^n$, $x\neq0$; being $d^Tx\leq\|d\|\ \|x\|$ it is $(d^Tx)^2\leq\|d\|^2\ \|x\|^2$, by means of the hypothesis we also have that for any $x\in R^n$ it is $x^TQx=\|x\|^2\ y^TQy\geq2\ \|d\|^2\ \|x\|^2$ [$>2\ \|d\|^2\ \|x\|^2$] where $y=x/\|x\|$. By means of this condition we have $x^TBx=x^TQx-2(d^Tx)^2\geq0$ [>0] so that B is positive semidefinite [definite] and function f(x) is [strictly] convex. The whole thesis then follows being $\min_{\|y\|=1} y^TQy$ equal to the smaller eigenvalue of Q. ♦

Note that if f(x) is indefinite, then any minimum point (if it exists) belongs to the boundary of the feasible region X (being B an indefinite matrix no critical points in the interior of X may be minima).

Being the feasible region X a polyhedron, it can be decomposed as $X=K_p+C_p$, where K_p is a polyhedral compact set and C_p is a polyhedral cone which coincides with the so called *recession cone* of the feasible region X, defined in general as follows:

$$\begin{aligned} rec(X) &= \{y: \exists\{x_n\}\subset X, \exists\{t_n\}\subset R, t_n\to+\infty, (x_n/t_n)\to y\} \\ &= \{0\}\cup\{y: \exists\{x_n\}\subset X, \|x_n\|\to+\infty, (x_n/\|x_n\|)\to v, y=\lambda v, \lambda\geq 0\}. \end{aligned}$$

Remind that if a set X is closed and convex then its recession cone is closed and convex too and can be rewritten as follows:

$$\begin{aligned} rec(X) &= \{y: \exists x\in X \text{ such that } x+\lambda y\in X \ \forall\lambda>0\} \\ &= \{y: x+\lambda y\in X \ \forall x\in X \ \forall\lambda>0\}. \end{aligned}$$

Also the concept of copositivity of a matrix will be useful in the rest of the paper [7]; remind that a symmetric matrix B is said to be *[strictly] copositive with respect to a cone V* if and only if $v^TBv\geq 0$ [>0] $\forall v\in V$, $v\neq 0$.

Lemma 2.1

Let $B\in R^{nxn}$ be any matrix, $c\in R^n$ be any vector and X any closed subset of R^n; consider also the function $f(x)=x^TBx+c^Tx$. If B is strictly copositive with respect to the cone rec(X), then f(x) attains a minimum over X.

Proof. Let $\{x_n\}\subset X$ be the sequence such that $f(x_n)\to \inf\{f(x)\}$ and let $w_n=(x_n/\|x_n\|)\to y$. Let us now prove that $\lim_{n\to+\infty}\|x_n\|<+\infty$; suppose by contradiction that $\lim_{n\to+\infty}\|x_n\|=+\infty$, then $y\in rec(X)$ so that, being B strictly copositive with respect to the cone rec(X), we have:

$$\begin{aligned} \lim_{n\to+\infty} f(x_n) &= \lim_{n\to+\infty}(x_n{}^TBx_n+c^Tx_n) = \\ &= \lim_{n\to+\infty}\|x_n\|^2 \lim_{n\to+\infty}(w_n{}^TBw_n+(1/\|x_n\|)\,c^Tw_n) = +\infty \end{aligned}$$

which is impossible. Being $\lim_{n\to+\infty}\|x_n\|<+\infty$ then, by means of the closure of X, $x_n\to x^*\in X$ so that for the continuity of f(x) the infimum is reached as a minimum and x* is a minimum point. ♦

The following sufficient optimality conditions for problem (1.1) follow directly by means of the previous lemma.

Corollary 2.1

Let us consider problem (1.1). If at least one of the following conditions hold:

i) $v^TQv>2\,\|d^Tv\|^2 \;\; \forall v\in \text{rec}(X),\; v\neq 0$ (that is to say that B of problem (1.1) is strictly copositive with respect to the cone rec(X))

ii) $$\min_{\|y\|=1,\, y\in \text{rec}(X)} y^TQy>2\,\|d\|^2$$

then the minimum exists for problem (1.1).

The following further lemma will be helpful in stating a necessary and sufficient condition for the existence of a minimum for problem (1.1) [5].

Lemma 2.2

Let $B\in R^{nxn}$ be any matrix, $c\in R^n$ be any vector and X any closed subset of R^n; consider also the function $f(x)=x^TBx+c^Tx$. Suppose B to be copositive with respect to the cone rec(X) and define the following auxiliary function: $g_n(x)=f(x)+ (1/n)\, x^Tx$ $\quad n=1,2,3,...$

Then the following properties hold:

i) $\forall n$ the function $g_n(x)$ attains a minimum over X, say $x_n\in \text{argmin}\{g_n(x)\}$

ii) the sequence $\{f(x_n)\}$ is decreasing and $f(x_n)\rightarrow \inf\{f(x) \text{ over } X\}$

The next result follows from the definition of the feasible set X [1, 2, 3].

Lemma 2.3

Let us consider the feasible region X of problem (1.1); since $X=K_p+C_p$, where K_p is a polyhedral compact set and C_p is a polyhedral cone, then for every sequence $\{x_n\}\subset X$ such that $\|x_n\|\rightarrow +\infty$ and $(x_n/\|x_n\|)\rightarrow d\in \text{rec}(X)$ the following property holds: $\forall \rho>0$ $x_n-\rho d\in X$ for n sufficiently large.

The following theorem provides a necessary and sufficient condition for the existence of the minimum for problem (1.1).

Theorem 2.3

The minimum exists for problem (1.1) if and only if both the two following conditions hold:

i) $v^TQv \geq 2\,\|d^Tv\|^2$ $\forall v \in rec(X)$, that is to say that B is copositive with respect to the cone rec(X);

ii) $\forall v \in rec(X)$ such that $v^TQv = 2\,\|d^Tv\|^2$, it results $v^T(Q-2\,dd^T)x + v^Tc \geq 0$ $\forall x \in X$, that is to say that the function f(x) along the direction v is linear and nondecreasing.

Proof.$\Rightarrow$*) i)* Let us suppose by contradiction that $\exists v \in rec(X)$ such that $v^TBv<0$; being $v \neq 0$ we have also $y^TBy<0$ where $y=v/\|v\|$. Being $y \in rec(X)$ there exists a sequence $\{x_n\} \subset X$, $\|x_n\| \to +\infty$, such that $(x_n/\|x_n\|) \to y$; denoting with $w_n=(x_n/\|x_n\|)$ it then results:

$$\lim_{n\to+\infty} f(x_n) = \lim_{n\to+\infty} (1/2\; x_n^TBx_n + c^Tx_n) =$$

$$= \lim_{n\to+\infty} \|x_n\|^2 \lim_{n\to+\infty} (1/2\; w_n^TBw_n + (1/\|x_n\|)\, c^Tw_n) = -\infty$$

which is a contradiction.

ii) Suppose on the contrary that $\exists v \in rec(X)$, $v^TBv=0$, $\exists x \in X$ such that $v^TBx+v^Tc<0$ and consider the function f(x) restricted to the halfline $x+\lambda v \in X$ $\forall \lambda>0$. It results $f(x+\lambda v) = 1/2\; x^TBx + \lambda\, v^TBx + 1/2\,\lambda^2\, v^TBv + c^Tx + \lambda\, v^Tc$ so that

$f(x+\lambda v) = \lambda\,(v^TBx + v^Tc) + 1/2\; x^TBx + c^Tx$. We then have, being $v^TBx+v^Tc<0$, that for $\lambda \to +\infty$, $f(x+\lambda v) \to -\infty$ which is a contradiction.

$\Leftarrow$*)* Let $g_n(x)=f(x)+(1/n)\, x^Tx$, n=1,2,3,...; being B copositive with respect to the cone rec(X) then for Lemma 2 the function $g_n(x)$ attains a minimum over X $\forall n$, say $x_n \in \operatorname{argmin}\{g_n(x)\}$, and the sequence $\{f(x_n)\}$ is decreasing with $f(x_n) \to \inf\{f(x)$ over $X\}$. Let us now prove that $\lim_{n\to+\infty} \|x_n\| < +\infty$; suppose by contradiction that $\lim_{n\to+\infty} \|x_n\| = +\infty$ and let $v \in rec(X)$ such that $w_n=(x_n/\|x_n\|) \to v$. Note that:

$$+\infty > \lim_{n\to+\infty} f(x_n) = \lim_{n\to+\infty} (1/2\; x_n^TBx_n + c^Tx_n) =$$

$$= \lim_{n\to+\infty} \|x_n\|^2 \lim_{n\to+\infty} (1/2\; w_n^TBw_n + (1/\|x_n\|)\, c^Tw_n)$$

so that $v^TBv \leq 0$; being B copositive with respect to the cone rec(X) it then follows $v^TBv=0$. For condition ii), being $f(x_n+\lambda v)=\lambda(v^TBx_n+v^Tc)$, we have that f(x) is nondecreasing along the direction v; this along with Lemma 3 implies that $\forall\rho>0$ and for n sufficiently large $x_n-\rho v\in X$ and $f(x_n-\rho v)\leq f(x_n)$; note also that for n sufficiently large $\|x_n-\rho v\|^2<\|x_n\|^2$. Being $x_n\in \text{argmin}\{g_n(x)\}$ we have $g_n(x_n)\leq g_n(x_n-\rho v)$ so that:

$$f(x_n)+(1/n)\|x_n\|^2=g_n(x_n)\leq g_n(x_n-\rho v)=f(x_n-\rho v)+(1/n)\|x_n-\rho v\|^2<f(x_n)+(1/n)\|x_n\|^2$$

which is a contradiction. Being $\lim_{n\to+\infty}\|x_n\|<+\infty$ then, by means of the closure of X, $x_n\to x^*\in X$ so that for the continuity of f(x) the infimum is reached as a minimum and x^* is a minimum point. ♦

Corollary 2.2

The following properties hold:

i) if problem (1.1) has no minimum then $\inf\{f(x) \text{ over } X\}=-\infty$;

ii) if $\inf\{f(x) \text{ over } X\}>-\infty$ then problem (1.1) admits minimum points.

Proof. i) By means of the previous theorem if problem (1.1) has no minimum then $\exists d\in \text{rec}(X)$ such that $v^TQv<2\,\|d^Tv\|^2$ or $\exists d\in \text{rec}(X)$, $\exists x\in X$ such that $v^TQv=2\,\|d^Tv\|^2$ and $d^T(Q-2\,dd^T)x+d^Tc<0$

In both cases, being v a feasible direction for problem (1.1), we have that $f(x+\lambda v)\to -\infty$.

ii) Follows trivially from i). ♦

Remark 2.1

Note that the previous theorem gives us some useful stop criterions for the solving algorithm:

(2.1) if a feasible direction v is found such that $v^TQv<2\,\|d^Tv\|^2$ then there is no minimum for problem (1.1) and $\inf\{f(x) \text{ over } X\}=-\infty$;

(2.2) if a feasible direction v and a feasible point x are found such that $v^TQv=2\,\|d^Tv\|^2$ and $v^T(Q-2\,dd^T)x+v^Tc<0$ then there is no minimum for problem (1.1) and $\inf\{f(x) \text{ over } X\}=-\infty$.

The concept of copositivity allow us to state the following global optimality conditions for problem (1.1), which will be helpful in the solving algorithm.

Theorem 2.4

Let us consider problem (1.1), a feasible point $x_0 \in X$ and a convex cone $V \subseteq R^n$; let us also define the following subset of the feasible region $Y = X \cap (x_0 + V) \subseteq X$. Suppose finally that $v^T Q v \geq 2 \|d^T v\|^2 \ \forall v \in V$, that is to say that matrix B of problem (1.1) is copositive with respect to the cone V. Then the following properties hold:

i) if $v^T \nabla f(x_0) = v^T B x_0 + v^T c \geq 0 \ \forall v \in V$ then x_0 is a global minimum point over Y;

ii) if x_0 is a local minimum point over Y then it is also a global minimum point over Y.

Proof. i) We will prove the result by contradiction. Suppose on the contrary that $\exists y \in Y$ such that $f(y) < f(x_0)$ and define $v = y - x_0$. Firstly note that Y is a convex set (being the intersection of two convex sets) and that, being $y \in Y$, $v = y - x_0 \in V$ is a feasible direction. It then results:

$$f(y) = f(x_0 + v) = [\ 1/2\, x_0^T B x_0 + c^T x_0\ ] + [\ v^T B x_0 + v^T c\ ] + 1/2\, v^T B v <$$
$$< [\ 1/2\, x_0^T B x_0 + c^T x_0\ ] = f(x_0)$$

so that it follows, being $\nabla f(x_0)^T v = v^T B x_0 + v^T c \geq 0 \ \forall v \in V$:

$$0 > [\ v^T B x_0 + v^T c\ ] + 1/2\, v^T B v \geq 1/2\, v^T B v$$

which is a contradiction since B is copositive with respect to the cone V.

ii) The thesis follows directly from property i) since if x_0 is a local minimum point over Y then $\nabla f(x_0)^T v \geq 0 \ \forall v \in V$. ♦

3. Some local optimality conditions

In this section we give some local optimality conditions for problem (1.1). If we add the constraint $d^T x = \xi$, $\xi \in R$, to problem (1.1), the following strictly convex quadratic problem is obtained:

$$P(\xi) \qquad \begin{array}{l} z(\xi) = -\xi^2 + \min\ (1/2\, x^T Q x + c^T x) \\ x \in X(\xi) \end{array}$$

where $X(\xi)=X\cap\{x\in R^n: d^Tx=\xi\}$. The parameter ξ is said to be a *feasible level* if the set $X(\xi)$ is nonempty. An optimal solution of problem $P(\xi)$ is called an *optimal level solution* [4, 8, 12].

Clearly problem (1.1) is equivalent to problem $P(\xi)$, when ξ is the level corresponding to an optimal solution of problem (1.1).

In this section we give some optimality conditions which allow us to detect if an optimal level solution is a local minimum of problem (1.1).

Let x' be the optimal solution of problem $P(\xi')$ and let $Nx=k$ be the equations of the constraints binding at x'. We can always choose a subset of these constraints, making a submatrix M of N and correspondingly a subvector h of k, such that the rows of M and the vector d are linearly independent. Being problem $P(\xi)$ convex, then x' is an optimal solution if and only if the Kuhn-Tucker conditions are verified.

Since Q is positive definite and the rows of M and d are linearly independent, the matrix of the following Kuhn-Tucker linear system is non singular:

$$\begin{aligned} Qx - M^T\mu - d\lambda &= -c \\ Mx &= h \\ d^Tx &= \xi' \end{aligned} \qquad (3.1)$$

where μ is the vector of the Lagrange multipliers associated to the constraints $Mx=h$ and λ is the Lagrange multiplier of the parametric constraint $d^Tx=\xi'$. The solution x', μ', λ' of (3.1) is then unique, note also that being x' an optimal solution then $\mu'\geq 0$.

Let us consider the parametric program:

$$P(\xi'+\theta) \qquad \begin{aligned} &z(\xi'+\theta) = -(\xi'+\theta)^2 + \min\,(1/2\, x^TQx + c^Tx) \\ &x\in X(\xi'+\theta) \end{aligned}$$

where $X(\xi'+\theta)=X\cap\{x\in R^n: d^Tx=\xi'+\theta\}$. Let

$$\begin{aligned} x'(\theta) &= x' + \theta\,\alpha \\ \mu'(\theta) &= \mu' + \theta\,\gamma \\ \lambda'(\theta) &= \lambda' + \theta\,\beta \end{aligned} \qquad (3.2)$$

be the solutions of the Kuhn-Tucker system:

$$\begin{aligned} Qx - M^T\mu - d\lambda &= -c \\ Mx &= h \\ d^Tx &= \xi'+\theta. \end{aligned} \qquad (3.3)$$

Note that (α,γ,β) is the unique solution of the linear system

$$(3.4)\qquad \begin{aligned} Qx - M^T\mu - d\lambda &= 0 \\ Mx &= 0 \\ d^Tx &= 1 \end{aligned}$$

so that it results $Q\alpha = M^T\gamma + d\beta$, $M\alpha = 0$, $d^T\alpha = 1$ and $\beta = \alpha^TQ\alpha$. Note also that, being Q positive definite, it is $\beta > 0$ if and only if $\alpha \neq 0$.

Set $F(\theta)=\{\theta\colon x'(\theta)\in X\}$, $O(\theta)=\{\theta\colon \mu'(\theta)\geq 0\}$, $H(\theta)=F(\theta)\cap O(\theta)$. Clearly, $x'(\theta)$ is an optimal level solution for $\theta\in H(\theta)$. Set $z(\theta)=z(\xi'+\theta)$, $z'=1/2\ x'^TQx' + c^Tx'$. The following lemma gives an explicit form for the function $z(\theta)$, $\theta\in H(\theta)$.

Lemma 3.1

If $H(\theta)\neq\{0\}$, then $z(\theta) = (1/2\beta - 1)\,\theta^2 + (\lambda' - 2\xi')\,\theta + z' - \xi'^2$ where $\beta = \alpha^TQ\alpha$.

Proof. We have $z(\theta) = -(\xi'+\theta)^2 + 1/2\,(x' + \theta\,\alpha)^TQ(x' + \theta\,\alpha) + c^T(x' + \theta\,\alpha) = -\xi'^2 - \theta^2 - 2\xi'\theta + 1/2\,x'^TQx' + \alpha^TQx'\,\theta + 1/2\,\alpha^TQ\alpha\,\theta^2 + \theta\,c^T\alpha + c^Tx'$; note also that from (3.4) it results $\alpha^TQx' = \lambda' - \alpha^Tc$. From direct substitution we obtain $z(\theta) = (1/2\beta - 1)\,\theta^2 + (\lambda' - 2\xi')\,\theta + z' - \xi'^2$. ♦

Now, the following lemma can be derived.

Lemma 3.2

If $\lambda' > 2\xi'$ ($\lambda' < 2\xi'$), then $z(\theta)$ is increasing (decreasing) at $\theta=0$.

Proof. We have $z'(\theta) = (\beta - 2)\,\theta + (\lambda' - 2\xi')$. Hence $z'(0) = \lambda' - 2\xi'$. ♦

Set

$U(\theta) = H(\theta)\cap[0, +\infty)$, if $\lambda' > 2\xi'$;

$U(\theta) = H(\theta)\cap(-\infty, 0]$, if $\lambda' < 2\xi'$;

$$\theta' = \frac{2\xi' - \lambda'}{\beta - 2},\ \text{if } \beta > 2.$$

The following theorem holds:

Theorem 3.1

a) If $\lambda' = 2\xi'$ and $\beta \geq 2$, then x' is a local minimum for problem (1.1).

b) If $\theta' \in U(\theta)$, then $x'(\theta')$ is a local minimum for problem (1.1).

Proof. a) $\lambda' = 2\xi'$ and $\beta \geq 2$ imply $z'(0) = 0$ and $z''(0) = \beta - 2 \geq 0$; hence $x'(0)=x'$ is a local minimum. b) We have $z'(\theta') = 0$ and $z''(\theta') = \beta - 2 \geq 0$; this implies that $x'(\theta')$ is a local minimum for problem (1.1). ♦

Let x' be a vertex of X; in x' at least n constraints of X are binding as well as the parametric constraint and thus x' is a degenerate basic solution. Clearly, the different bases containing the parametric constraint are n if x' is a non degenerate vertex of X; more than n if x' is a degenerate vertex of X. A basis B is said to be feasible if $\mu'_B \geq 0$. To point out the dependence of $z(\theta)$, $H(\theta)$, etc. on the basis B, we write $z_B(\theta)$, $H_B(\theta)$, etc..

Theorem 3.2

a) If there are two different feasible bases B_1 and B_2 such that either $\lambda'_{B_1} > 2\xi'$, $\sup H_{B_1}(\theta) > 0$, $\lambda'_{B_2} < 2\xi'$, $\inf H_{B_2}(\theta) < 0$ or $\lambda'_{B_1} < 2\xi'$, $\inf H_{B_1}(\theta) < 0$, $\lambda'_{B_2} > 2\xi'$, $\sup H_{B_2}(\theta) > 0$, then x' is a local minimum for problem (1.1).

b) If we have $U_B(\theta)=\{0\}$ for any feasible basis B, then x' is a local minimum for problem (1.1).

Proof. a) In wiew of Lemma 3.2 condition $\lambda'_{B_1} > 2\xi'$, $\lambda'_{B_2} < 2\xi'$ ($\lambda'_{B_1} < 2\xi'$, $\lambda'_{B_2} > 2\xi'$) implies $z(\theta) \geq z(0)$ in a neighborhood of 0. Hence x' is a local minimum for problem (1.1). b) This follows directly from the definition of $U_B(\theta)$. ♦

4. A finite algorithm for problem (1.1)

Since problem (1.1) is nonconvex, in general, it is necessary to solve problem $P(\xi)$ for all feasible levels in order to find a global minimum, assuming one exists. In this section we will show that this can be done by means of a finite number of iterations, using the results of the previous section.

Let ξ' be a feasible level and suppose that x^* is the incumbent global minimum for $\xi \leq \xi'$, i.e. is the best optimal level solution for $\xi \leq \xi'$. Clearly $UB=f(x^*)$ is an upper bound for the value of $z(\xi)$ for $\xi > \xi'$.

Let $\xi_{max} = \sup \{d^T x, x \in X\}$ (of course ξ_{max} may be equal to $+\infty$).

Let us consider the parametric problem $P(\xi'+\theta)$ for $\theta \geq 0$ and determine $x'(\theta)$, $\mu'(\theta)$, $\lambda'(\theta)$, $z(\theta)$, $\theta' = \frac{2\xi'-\lambda'}{\beta-2}$, if $\beta > 2$, $F(\theta)$, $O(\theta)$, $H(\theta)$ as well sup $F(\theta)$, sup $O(\theta)$, sup $H(\theta)$. For each $\theta \in O(\theta)$, $z(\theta)$ is a lower bound for $P(\xi'+\theta)$; in fact if $\theta \in F(\theta)$, then $x'(\theta)$ is an optimal level solution; otherwise, if $\theta \notin F(\theta)$, $x'(\theta)$ is unfeasible for $P(\xi'+\theta)$ but is an optimal solution of a problem with the same objective function of $P(\xi'+\theta)$ and a feasible region containing $X(\xi'+\theta)$.

The following four cases can occur:

A1) $\beta > 2$, $\lambda' < 2\xi'$ ($z(\theta)$ convex and decreasing at $\theta=0$): three subcases need to be considered:

A1a) $\theta' \in F(\theta)$, sup $O(\theta) = +\infty$, problem (1.1) is solved; in fact if $UB < z(\theta')$, then x^* is a global minimum; otherwise $x'(\theta')$ is a global minimum;

A1b) $\theta' \in F(\theta)$, sup $O(\theta) = \theta'' \geq \theta'$, if $\xi''=\xi'+\theta'' \geq \xi_{max}$ problem (1.1) is solved; in fact if $UB < z(\theta')$, then x^* is a global minimum; otherwise $x'(\theta')$ is a global minimum; if $\xi'' < \xi_{max}$ then we consider the new feasible level ξ'' and the corresponding parametric problem $P(\xi''+\theta)$ with $x^*=x'(\theta')$, $UB=z(\theta')$ if $UB > z(\theta')$;

A1c) sup $H(\theta) = \theta'' < \theta'$, if $\xi''=\xi'+\theta''=\xi_{max}$, then problem (1.1) is solved; in fact if $UB < z(\theta'')$, then x^* is a global minimum; otherwise $x'(\theta'')$ is a global minimum; if $\xi'' < \xi_{max}$, then we consider the new feasible level ξ'' and the corresponding parametric problem $P(\xi''+\theta)$ with $x^*=x'(\theta'')$, $UB=z(\theta'')$ if $UB > z(\theta'')$;

A2) $\beta \geq 2$, $\lambda' \geq 2\xi'$ ($z(\theta)$ convex (or linear) and nondecreasing at $\theta=0$): two subcases need to be considered:

A2a) sup $O(\theta) = +\infty$, problem (1.1) is solved; x^* is a global minimum;

A2b) sup $O(\theta) = \theta'' < +\infty$, if $\xi''=\xi'+\theta'' \geq \xi_{max}$, then problem (1.1) is solved and x^* is a global minimum; if $\xi'' < \xi_{max}$, then we consider the new feasible level ξ'' and the corresponding parametric problem $P(\xi''+\theta)$;

A3) $\beta \leq 2$, $\lambda' < 2\xi'$ ($z(\theta)$ concave (or linear) and decreasing at $\theta=0$): two subcases need to be considered:

A3a) sup $H(\theta) = +\infty$, problem (1.1) is unbounded, i.e. inf $f(x)=-\infty$ (note that this is the stop criterion (2.1) or (2.2) if $z(\theta)$ is linear, that is $\beta=2$);

A3b) sup $H(\theta) = \theta'' < +\infty$, if $\xi''=\xi'+\theta''=\xi_{max}$, then problem (1.1) is solved; in fact if $UB < z(\theta'')$, then x^* is a global minimum; otherwise $x'(\theta'')$ is a global minimum; if $\xi'' < \xi_{max}$ then we consider the new feasible level ξ'' and the corresponding parametric problem $P(\xi''+\theta)$ with $x^*=x'(\theta'')$, $UB=z(\theta'')$ if $UB > z(\theta'')$;

A4) $\beta < 2$, $\lambda' \geq 2\xi'$ ($z(\theta)$ concave and nondecreasing at $\theta=0$): let $\sup F(\theta) = \theta^1$, $\sup O(\theta) = \theta^2$, θ_r be the positive root of the equation $z(\theta)=UB$ and $\theta^* = \min\{\theta^2, \theta_r\}$; four subcases need to be considered:

A4a) $\theta^1 < \theta^* \leq \theta^2$, if $\xi'' = \xi' + \theta^* \geq \xi_{max}$, then problem (1.1) is solved and x^* is a global minimum; if $\xi'' < \xi_{max}$ then we consider the new feasible level ξ'' and the corresponding parametric problem $P(\xi''+\theta)$;

A4b) $\theta^* \leq \theta^1 \leq \theta^2$, if $\xi'' = \xi' + \theta^1 = \xi_{max}$, then problem (1.1) is solved and $x'(\theta^1)$ is a global minimum; if $\xi'' < \xi_{max}$ then we consider the new feasible level ξ'' and the corresponding parametric problem $P(\xi''+\theta)$ with $x^* = x'(\theta^1)$, $UB = z(\theta^1)$;

A4c) $\theta^2 \leq \theta^1$, if $\xi'' = \xi' + \theta^2 = \xi_{max}$, then problem (1.1) is solved; in fact if $UB < z(\theta^2)$, then x^* is a global minimum; otherwise $x'(\theta^2)$ is a global minimum; if $\xi'' < \xi_{max}$, then we consider the new feasible level ξ'' and the corresponding parametric problem $P(\xi''+\theta)$ with $x^* = x'(\theta^2)$, $UB = z(\theta'')$ if $UB > z(\theta'')$;

A4d) $\sup H(\theta) = +\infty$, problem (1.1) is unbounded, i.e. $\inf f(x) = -\infty$ (note that this is the stop criterion (2.1)).

Remark 4.1

Let us note that in A1a) and A2a), in order to verify the global optimality, we just used the concept of optimal level solution; actually we have also implicitly verified the copositivity of matrix B with respect to the cone V defined as the cone of feasible directions from x' with respect to the set Y intersection of X and the halfspace given by $d^T x \geq \xi'$. This can be proved noticing that along the direction $\alpha \in V$ the copositivity has been explicitly checked, along the directions belonging to the hyperplane $d^T x = \xi'$ it is given since $P(\xi')$ is a strictly convex problem and x' is its global minimum, and finally since the objective function has a lower bound γ on the halfline $x'+\theta\alpha$, made by optimal level solution, so that we have

$$\lim f(x'+\theta v) \geq \lim f(x'+\theta\alpha) \geq \gamma$$

which implies that $\lim f(x'+\theta v) > -\infty$ so that, necessarily, it is $v^T B v \geq 0$.

Starting from the solution x' corresponding to the level ξ', we arrive at one of the following situations:

i) x^* is an optimal solution;

ii) the problem is unbounded;

iii) a level greater than ξ' has been found together with the best incumbent solution.

In order to propose a finite algorithm to solve problem (1.1), it remains to consider an appropriate initialization and to show how it is possible to obtain the optimal level solution corresponding to the new level ξ'' in a finite number of iterations.

Let us solve one of the following linear programs:

(P_1) $\quad$ $\min\ d^T x,\ x \in X;$

(P_2) $\quad$ $\max\ d^T x,\ x \in X.$

If x' is the unique optimal solution of (P_1) and $\xi'=d^T x'$ is the corresponding level then $X(\xi')=\{x'\}$ and clearly x' is an optimal level solution; in this case $x^*=x'$ and only increasing value of ξ need to be considered. Analogously, if x' is the unique optimal solution of (P_2) and $\xi'=d^T x'$ is the corresponding level, then $x^*=x'$ and only decreasing value of ξ need to be considered. Otherwise we can start from the optimal level solution x' corresponding to a feasible level ξ'; also in this case $x^*=x'$; but it is necessary to consider either increasing or decreasing values of the parameter.

It remains to consider the problem of obtaining the optimal level solution x' corresponding to the new level $\xi''=\xi'+\tilde{\theta}$ in a finite number of iterations. If $\tilde{\theta}=\sup H(\theta)=\sup F(\theta)$, then $x'=x'(\tilde{\theta})$ and at least one new constraint is binding at x', while if $\tilde{\theta}=\sup H(\theta)=\sup O(\theta)$ at least one of the Lagrange multipliers $\mu'(\tilde{\theta})$ are zero and the corresponding constraint can be deleted. If $\tilde{\theta}>\sup H(\theta)$, then $x'(\tilde{\theta})$ is unfeasible and the optimal level solution x' must be determined. Starting from the level ξ' the level $\xi'+\sup H(\theta)$ is obtained together with the optimal level solution $x'(\sup H(\theta))$, then starting from the level $\xi'=\xi'+\sup H(\theta)$ the new level $\xi'+\sup H(\theta)$ is obtained and so on until a level $\xi'\geq\xi''$ is reached. The proposed procedure is finite since for each new level either at least one new constraint is added or at least one old constraint is deleted.

Let us consider the following numerical example:

$$\min f(x) = 1/2\,(x_1, x_2)\begin{bmatrix} 8 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - (x_1 - 2x_2)^2,$$

(1) $x_2 \geq 0$, (2) $x_1 - x_2 \geq -2$, (3) $x_1 - 3x_2 \geq -10$.

Starting from the optimal solution $x'=(2, 4)$ of the linear program

$\min\ \{x_1 - 2x_2\colon x_2 \geq 0,\ x_1 - x_2 \geq -2,\ x_1 - 3x_2 \geq -10\}$

we obtain the following steps:

- base {(2), P}, $\xi'=-6$, $x'=(2, 4)$, $x'(\theta)=(2-\theta, 4-\theta)$, $\mu'_2(\theta)=50-21\theta$, $\lambda'(\theta)=-30+12\theta$, $z'=40$, $x^*=(2, 4)$, UB=4, $z(\theta)=5\theta^2-18\theta+4$, sup $O(\theta)=50/21$, sup $F(\theta)=4$, sup $H(\theta)=50/21$, $\theta'=9/5$; case A1b) holds;
- $x'(9/5)=(1/5, 11/5)$, $z(9/5)=-61/5<UB=4$, $x^*=(1/5, 11/5)$, UB=-61/5=-12.2, base {P}, $\xi'=-6+50/21=-76/21$, $x'=x'(50/21)=(-8/21,34/21)$, $x'(\theta)=(-8/21+4/38\theta, 34/21-17/38\theta)$, $\lambda'(\theta)=-30/21+15/38\theta$, $z'=380/147$, $z(\theta)=-61/76\theta^2+122/21\theta-4636/441=-0.80260^2+5.80950-10.5124$, $z(\theta)=UB=-12.2$, $\theta_r=7.51803$, sup $O(\theta)=+\infty$, sup $F(\theta)=76/21$, sup $H(\theta)=76/21$, case A4a) holds;
- $\xi'=-76/21+76/21=0$, $x'=x'(76/21)=(0, 0)$, base {(1), P}, $x'(\theta)=(\theta, 0)$, $\mu'_1(\theta)=17\theta$, $\lambda'(\theta)=8\theta$, $z'=0$, $z(\theta)=3\theta^2$, case A2a) holds, sup $O(\theta)=+\infty$, sup $F(\theta)=+\infty$, $x^*=(1/5, 11/5)$ is the optimal solution.

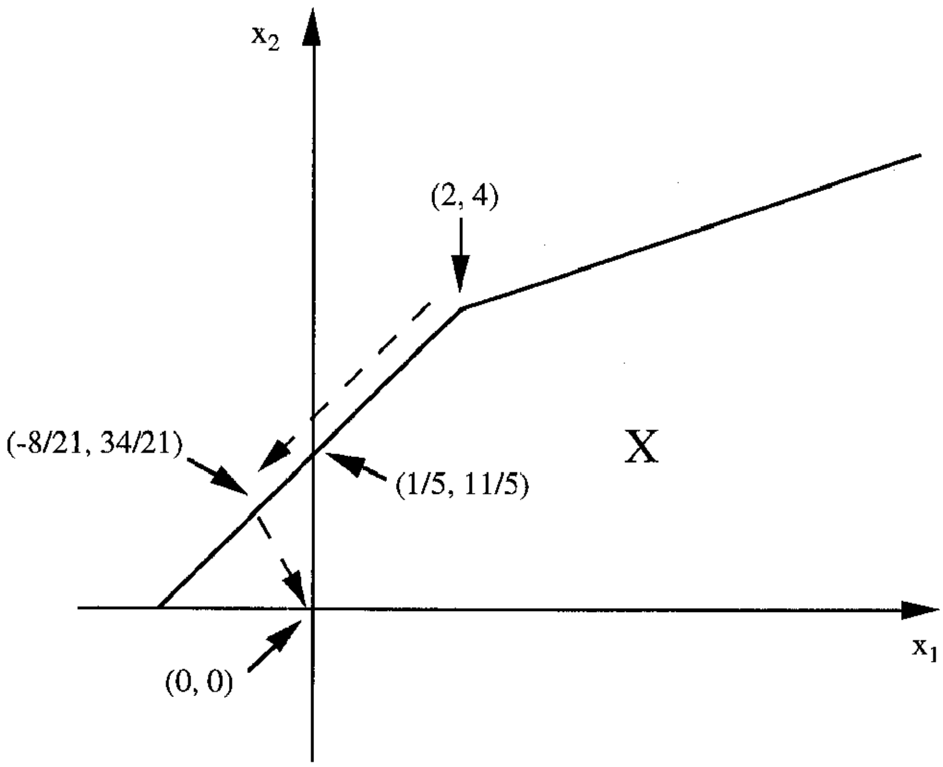

fig. 1

The path followed by the algorithm is depicted in fig. 1.

5. Further remarks

It is interesting to point out, even from a computational point of view, that it is possible to determine an explicit inverse of the constraint matrix related to the Kuhn-Tucker conditions used in Section 3. More precisely, let us define the following matrix:

$$D = \begin{bmatrix} Q & -M^T & -d \\ M & 0 & 0 \\ d^T & 0 & 0 \end{bmatrix}$$

so that the Kuhn-Tucker system (3.3) becomes:

$$D \begin{bmatrix} x \\ \mu \\ \lambda \end{bmatrix} = \begin{bmatrix} -c \\ h \\ \xi' \end{bmatrix} + \theta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

so that the unique solution of the system itself is:

$$\begin{bmatrix} x'(\theta) \\ \mu'(\theta) \\ \lambda'(\theta) \end{bmatrix} = \begin{bmatrix} x' \\ \mu' \\ \lambda' \end{bmatrix} + \theta \begin{bmatrix} \alpha \\ \gamma \\ \beta \end{bmatrix} = D^{-1} \begin{bmatrix} -c \\ h \\ \xi' \end{bmatrix} + \theta D^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Having an explicit form of D^{-1} allows us to directly calculate such a solution. The inverse can be stated with the following steps:

Preliminary Step:

Calculate Q^{-1}

$q:=Q^{-1}d$

$\delta:=1/(d^Tq)$

$H:=Q^{-1}-\delta qq^T$ (symmetric matrix)

First Step: Chosen the binding constraints given by matrix M do:

$v:=Mq$

$S:=(MHM^T)^{-1}$ (symmetric matrix)

$\gamma:=-\delta Sv$

$\beta:=\delta(1-v^T\gamma)$

$\alpha:=Q^{-1}(M^T\gamma+d\beta)$

$M_r:=HM^TS$

$B:=H-M_rMH$ (symmetric matrix)

Second Step: The inverse of D is then given by the following matrix:

$$D^{-1} = \begin{bmatrix} B & M_r & \alpha \\ -M_r^T & S & \gamma \\ -\alpha^T & \gamma^T & \beta \end{bmatrix}$$

By means of the following properties it is very easy to prove that the previous matrix is the correct inverse of D (that is verify that DD^{-1} is correctly equal to I):

$$MM_r=I \qquad d^TM_r=0 \qquad HQq=0 \qquad HQH=H$$

$$(1/\delta)=d^TQ^{-1}d=q^TQq \qquad \alpha=Q^{-1}(M^T\gamma+d\beta)=\delta(q-M_rMq)$$

These properties allow us to verify, by means of simple calculations, also that:

$$M\alpha = 0, \quad d^T\alpha = 1 \quad \text{and} \quad \beta = \alpha^TQ\alpha.$$

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