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Pseudoconvexity of a class of quadratic fractional functions

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Abstract

We aim to characterize the pseudoconvexity of a quadratic fractional function f whose quadratic form is given by the product of two affine functions. Starting from a recent result given by [2], we specify necessary and sufficient conditions that can be easily checked, thus very useful from an algorithmic point of view.

Keywords Generalized Convexity, Fractional Programming, Quadratic Programming
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1 Introduction and statement of the problem

Many problems of Management Science such as maximization of productivity, maximization of return of investment, minimization of return to risk, can be seen as applications of fractional programming. In general, whenever we have a problem that describe some kind of an efficiency measure of a system, we can formulate it as a fractional program. In this context, many results about fractional programming has been developed in the last decades and both theoretical and algorithmic point of views have been handled (see for instance [1] and [9]). Several of these studies deal with the generalized convexity of the objective function, because of the key role played by this kind of property in minimization problems. (See for example [2],[3],[4], [8],[5], [6]).

*The paper has been discussed jointly by the authors. In particular, sections 2 and 3 has been developed by Laura Carosi while sections 1 and 4 by Riccardo Cambini.

In this paper, we aim to determine necessary and sufficient conditions guaranteeing the pseudoconvexity of a quadratic fractional function f whose quadratic form is determined by the product of two affine ones. More precisely we consider the following class of functions

$$f(x) = \frac{n(x)}{b^T x + b_0} = \frac{(a^T x + a_0)(c^T x + c_0) + (d^T x + d_0)}{b^T x + b_0} \tag{1}$$

on the set $X = \{x \in \Re^n : b^T x + b_0 > 0\}$, where $a, b, c, d, x \in \Re^n$, $a, b, c \neq 0$, and $a_0, b_0, c_0, d_0 \in \Re$. Observe that when $a = 0$ or $c = 0$, f becomes a linear fractional function which is known to be pseudolinear (see for example [4], [6]), while when $b = 0$ f is a quadratic function whose properties are very well known.

From now on, the number of the negative eingenvalues of a symmetric matrix Q is denoted by $\nu_-(Q)$ and similarly $\nu_+(Q)$ represents the number of the positive eingenvalues while $\nu_0(Q)$ is the algebraic multiplicity of the 0 eigenvalue.

A key tool which is going to be used throughout the paper is the following result given by [3].

Theorem 1 *Let us consider the following quadratic fractional function*

$$f(x) = \frac{n(x)}{b^T x + b_0} = \frac{\frac{1}{2} x^T Q x - q^T x + q_0}{b^T x + b_0} \tag{2}$$

on the set $X = \{x \in \Re^n : b^T x + b_0 > 0\}$, where Q is a $n \times n$ symmetric matrix, $q, x, b \in \Re^n$, $b \neq 0$, and $q_0, b_0 \in \Re$. Then the function f is pseudoconvex on X if and only if one of the following conditions hold:

i) $\nu_-(Q) = 0$ *(i.e. Q is positive semidefinite);*

ii) $\nu_-(Q) = 1$, $\exists \overline{x}, \overline{y} \in \Re^n$ *such that* $Q\overline{x} = q$ *and* $Q\overline{y} = b$, $b^T \overline{y} = 0$, $b^T \overline{x} + b_0 = 0$ *and* $n(\overline{x}) \geq 0$*;*

iii) $\nu_-(Q) = 1$, $\exists \overline{x}, \overline{y} \in \Re^n$ *such that* $Q\overline{x} = q$ *and* $Q\overline{y} = b$, $b^T \overline{y} < 0$ *and* $(b^T \overline{x} + b_0)^2 + 2n(\overline{x}) b^T \overline{y} \leq 0$.

Referring to this characterization, we derive new necessary and sufficient conditions which guarantee the pseudoconvexity of the function f given in (1); these new conditions are very easy to be checked and that suggests their potential use from an algorithmic point of view.

We proceed as follows; in section two we consider the function f defined in (1) where a and c are linearly dependent; first we deep on the case where the quadratic form can be derived by the square of an affine function, and then we deal with the general case with just the linear dependence between a and c. Section three characterizes the pseudoconvexity of the function f whenever a and c are linear independent. Finally, in section four, as a Corollary of our results, we recover conditions given by [3] for a function f which is the sum of an affine and a linear fractional ones.

2 a and c linearly dependent

In this section, before dealing with the general case, we first deep on the pseudoconvexity of function f, given in (1), when the quadratic form is determined by the square of a linear function (i.e. $c = ka$, $c_0 = ka_0$ with $k \neq 0$), that is

$$f(x) = \frac{n(x)}{b^T x + b_0} = \frac{k(a^T x + a_0)^2 + (d^T x + d_0)}{b^T x + b_0} \tag{3}$$

Note that when $k = 0$ f becomes a linear fractional function. It is useful observing that $n(x)$ can be written in the form $n(x) = \frac{1}{2}x^T Q x - q^T x + q_0$ where:

$$Q = 2kaa^T \qquad q = -2ka_0 a - d \qquad q_0 = ka_0^2 + d_0$$

Q is a $n \times n$ symmetric matrix and when $k < 0$ $n(x)$ belongs to the class of the D.C. functions, (i.e. $n(x)$ can be written as a difference of convex functions), which is widely studied in Global Optimization.

Lemma 2 *The real number $2k\|a\|^2 \neq 0$ is an eigenvalue of Q and a is one of its corresponding eigenvectors. Therefore:*

i) $k > 0$ *if and only if* Q *is positive semidefinite with* $\nu_-(Q) = 0$, $\nu_0(Q) = n - 1$ *and* $\nu_+(Q) = 1$;

ii) $k < 0$ *if and only if* Q *is negative semidefinite with* $\nu_-(Q) = 1$, $\nu_0(Q) = n - 1$ *and* $\nu_+(Q) = 0$;

Proof. Firstly note that $Qa = 2kaa^T a = (2k\|a\|^2)a$ so that a is an eigenvector of Q corresponding to the eigenvalue $2k\|a\|^2 \neq 0$.
Being $Q = 2kaa^T$ we have that

$$\{v \in \Re^n : a^T v = 0\} \subseteq \ker(Q) = \{v \in \Re^n : Qv = 0\}$$

so that $\dim(\ker(Q)) \geq n - 1$. Since $\forall v \in \ker(Q)$ $Qv = 0v$, so that v is an eigenvector of Q corresponding to the eigenvalue 0, we get $\nu_0(Q) \geq n - 1$. The thesis then follows being $2k\|a\|^2 \neq 0$ an eigenvalue of Q. ■

Lemma 3 *It results $Q(\Re^n) = \{\mu a, \mu \in \Re\}$, so that:*

i) $q \in Q(\Re^n)$ *if and only if* $\exists \delta \in \Re$ *such that* $d = \delta a$;

ii) $b \in Q(\Re^n)$ *if and only if* $\exists \beta \in \Re$ *such that* $b = \beta a$;

iii) $Q\overline{x} = q$ *for* $\overline{x} = \frac{-2ka_0-\delta}{2k\|a\|^2}a$;

iv) $Q\overline{y} = b$ *for* $\overline{y} = \frac{\beta}{2k\|a\|^2}a$.

Proof. Being $Q = 2kaa^T$, $a \in Q(\Re^n)$, and from Lemma 2 $\dim(\ker(Q)) \geq n-1$; then it is straightforward that $\dim(\mathrm{Im}(Q)) = 1$ and $Q(\Re^n) = \{\mu a, \mu \in \Re\}$.

i),ii) The previous result implies that $q \in Q(\Re^n)$ if and only if $\exists \mu \in \Re$ such that $q = -2ka_0 a - d = \mu a$, that is to say $d = (-2ka_0 - \mu)a = \delta a$; in the same way we prove *ii)*.

iii),iv) Being $Qa = (2k\|a\|^2)a$ the thesis follows directly from *i)* and *ii)*. ■

The following theorem characterizes the pseudoconvexity of f in (3).

Theorem 4 *Function f in (3) is pseudoconvex on X if and only if one of the following conditions hold:*

i) $k > 0$ *(i.e.* $\nu_-(Q) = 0$ *and hence Q is positive semidefinite);*

ii) $k < 0$, $\exists \delta, \beta \in \Re$ *such that* $d = \delta a$, $b = \beta a$, *and*

$$k(\beta a_0 - b_0)^2 \geq \beta(b_0\delta - d_0\beta).$$

Proof. We will prove the result by means of Theorem 1. From Lemma 2 we have that $\nu_-(Q) = 0$ if and only if $k > 0$ (case *i)* of Theorem 1), while $\nu_-(Q) = 1$ if and only if $k < 0$ (cases *ii)* and *iii)* of Theorem 1). From Lemma 3 condition

$$\exists \overline{x}, \overline{y} \in \Re^n \text{ such that } Q\overline{x} = q \text{ and } Q\overline{y} = b$$

is equivalent to the following one:

$$\exists \delta, \beta \in \Re \text{ such that } d = \delta a \text{ and } b = \beta a \tag{4}$$

Whenever $k < 0$ and condition (4) holds we get

$$b^T\overline{y} = \frac{\beta}{2k\|a\|^2}b^T a = \frac{\beta^2}{2k\|a\|^2}a^T a = \frac{\beta^2}{2k} < 0$$

and hence case *ii)* of Theorem 1 never occurs. By means of simple calculations we have $b^T\overline{x} + b_0 = b_0 - a_0\beta - \frac{\beta\delta}{2k}$, $n(\overline{x}) = d_0 - a_0\delta - \frac{\delta^2}{4k}$ so that:

$$(b^T\overline{x} + b_0)^2 + 2n(\overline{x})b^T\overline{y} = (b_0 - a_0\beta)^2 - \frac{\beta}{k}(b_0\delta - d_0\beta)$$

and hence $(b^T\overline{x} + b_0)^2 + 2n(\overline{x})b^T\overline{y} \leq 0$ if and only if

$$k(b_0 - a_0\beta)^2 \geq \beta(b_0\delta - d_0\beta)$$

The whole thesis is then proved by means of Theorem 1. ■

The following example underlines that pseudoconvexity of f can be easily checked by means of Theorem 4.

Example 5 Consider $f(x) = \frac{-(2x_1+3x_2+1)^2+x_1+3/2x_2+5}{4x_1+6x_2+1}$ and $X = \{(x_1, x_2) \in \Re^2 : 4x_1 + 6x_2 + 1 > 0\}$. Since the gradient of f never vanishes, f is pseudoconvex. We obtain the same result by means of Theorem 4 . Observe that $k = -1 < 0$, $a = (2,3)$, $b = (4,6)$ $d = (1,3/2)$ hence $b = \beta a$ with $\beta = 2$ and $d = \delta a$ with $\delta = 1/2$. By simple calculation $b^T\overline{y} = -2 < 0$ and $k(b_0 - a_0\beta)^2 = -1(1-2)^2 = -1$, $\beta(b_0\delta - d_0\beta) = 2(1/2 - 10)$ and consequentely conditions *ii)* of Theorem 4 are verified.

Using the previous results, we can now handle the general case with a and c linearly dependent, that is

$$f(x) = \frac{n(x)}{b^Tx + b_0} = \frac{(a^Tx + a_0)(c^Tx + c_0) + (d^Tx + d_0)}{b^Tx + b_0} \quad \text{with} \quad c = ka, \ k \neq 0 \tag{5}$$

Firstly note that $n(x)$ can be specified in the form

$$n(x) = k(a^Tx + a_0)^2 + (\overline{d}^Tx + \overline{d}_0)$$

where:

$$\overline{d} = (c_0 - a_0k)a + d \qquad \overline{d}_0 = (c_0 - a_0k)a_0 + d_0$$

In this case, pseudoconvexity can be characterized as a corollary of Theorem 4.

Corollary 6 *Function f in (5) is pseudoconvex on X if and only if one of the following conditions hold:*

i) $k > 0$ (i.e. $\nu_-(Q) = 0$ and hence Q is positive semidefinite);

ii) $k < 0$, $\exists \delta, \beta \in \Re$ *such that* $d = \delta a$, $b = \beta a$, *and*

$$k(\beta a_0 - b_0)^2 \geq \beta(b_0\delta - d_0\beta) + \beta(c_0 - a_0 k)(b_0 - a_0\beta). \qquad (6)$$

Proof. Recalling that $n(x) = k(a^T x + a_0)^2 + (\overline{d}^T x + \overline{d}_0)$ condition ii) in Theorem 4 can be specified as follows

$$k < 0, \exists \overline{\delta}, \beta \in \Re \text{ such that } \overline{d} = \overline{\delta} a, b = \beta a,$$

and

$$k(\beta a_0 - b_0)^2 \geq \beta(b_0\overline{\delta} + \overline{d}_0\beta)$$

Note that $\overline{d} = \overline{\delta} a$ implies $(c_0 - a_0 k)a + d = \overline{\delta} a$ and hence $d = (\overline{\delta} - c_0 + a_0 k)a = \delta a$, where $\delta = \overline{\delta} - c_0 + a_0 k$. In other words, $\exists \delta \in \Re$ such that $d = \delta a$ and it results $\overline{\delta} = \delta + (c_0 - a_0 k)$. The thesis then follows from Theorem 4 observing that:

$$(b_0\overline{\delta} + \overline{d}_0\beta) = (b_0\delta - d_0\beta) + (c_0 - a_0 k)(b_0 - a_0\beta)$$

■

It is worthy remarking that if $c_0 = a_0 k$ function f defined in (5) is equal to the function in (3) and condition *ii)* of Corollary 6 coincides with the *ii)* of Theorem 4.

3 a and c linearly independent

In this section we study the pseudoconvexity of function (1) when a and c are linearly independent that is:

$$f(x) = \frac{n(x)}{b^T x + b_0} = \frac{(a^T x + a_0)(c^T x + c_0) + (d^T x + d_0)}{b^T x + b_0} \quad \text{with } a, c \text{ l.i.} \qquad (7)$$

Firstly note that $n(x)$ can be written in the form $n(x) = \frac{1}{2}x^T Q x - q^T x + q_0$ where:

$$Q = ac^T + ca^T \qquad q = -a_0 c - c_0 a - d \qquad q_0 = a_0 c_0 + d_0$$

with Q a $n \times n$ symmetric matrix.

Lemma 7 *Matrix* Q *is indefinite with* $\nu_-(Q) = 1 = \nu_+(Q)$ *and* $\nu_0(Q) = n - 2$.

Proof. Being $Q = ac^T + ca^T$ we have that

$$\{w \in \Re^n : a^T w = 0, c^T w = 0\} \subseteq \ker(Q) = \{w \in \Re^n : Qw = 0\}$$

so that $\dim(\ker(Q)) \geq n - 2$ Since $\forall w \in \ker(Q)$ $Qw = 0w$ means that w is an eigenvector of Q corresponding to the eigenvalue 0, $\nu_0(Q) \geq n-2$. Being a and c linearly independent it is possible to determine a vector $u \in \Re^n$ such that $c^T u = 0$ and $a^T u \neq 0$. Then we have $Qu = (a^T u)c \neq 0$ and $u^T Q u = 0$ and this implies, for a known property of semidefinite matrices, that Q is indefinite. Consequently Q has at least one positive and one negative eigenvalue. ■

Lemma 8 *The following statements hold:*

i) there exist $u, v \in \Re^n$ such that $c^T u = 0$, $a^T u = 1$ and $a^T v = 0$, $c^T v = 1$; therefore $Qu = c$, $Qv = a$;

ii) $Q(\Re^n) = \{\mu_1 a + \mu_2 c, \mu_1, \mu_2 \in \Re\}$;

iii) $q \in Q(\Re^n)$ if and only if $\exists \delta_1, \delta_2 \in \Re$ such that $d = \delta_1 a + \delta_2 c$;

iv) $b \in Q(\Re^n)$ if and only if $\exists \beta_1, \beta_2 \in \Re$ such that $b = \beta_1 a + \beta_2 c$;

v) $Q\overline{x} = q$ for $\overline{x} = -(a_0 + \delta_2)u - (c_0 + \delta_1)v$;

vi) $Q\overline{y} = b$ for $\overline{y} = \beta_2 u + \beta_1 v$.

Proof. *i)* Follows directly observing a and c linearly independent.

ii) From *i)* $a, c \in Q(\Re^n)$ and since from Lemma 7 $\dim(\ker(Q)) \geq n - 2$, it is straightforward that $\dim(\mathrm{Im}(Q)) = 2$ and hence $Q(\Re^n) = \{\mu_1 a + \mu_2 c, \mu_1, \mu_2 \in \Re\}$.

iii),vi) The previous result implies that $q \in Q(\Re^n)$ if and only if $\exists \mu_1, \mu_2 \in \Re$ such that $q = -a_0 c - c_0 a - d = \mu_1 a + \mu_2 c$, that is to say $d = -(c_0 + \mu_1)a - (a_0 + \mu_2)c = \delta_1 a + \delta_2 c$; in the same way we prove also *ii)*.

v),vi) Being $Qu = c$ and $Qv = a$ the thesis follows from *i)* and *ii)*. ■

Theorem 9 *Function f in (7) is pseudoconvex on X if and only if the following conditions hold:*

i) $\exists \delta_1, \delta_2 \in \Re$ such that $d = \delta_1 a + \delta_2 c$;

ii) $\exists \beta_1, \beta_2 \in \Re$ such that $b = \beta_1 a + \beta_2 c$;

iii) defining $\gamma_1 = a_0 + \delta_2$ *and* $\gamma_2 = c_0 + \delta_1$ *one of the following conditions holds:*
iii-a) $\beta_1\beta_2 = 0,\ b_0 = \gamma_1\beta_1 + \gamma_2\beta_2,\ a_0c_0 + d_0 \geq \gamma_1\gamma_2,$
iii-b) $\beta_1\beta_2 < 0,\ (b_0 - \gamma_1\beta_1 - \gamma_2\beta_2)^2 + 4\beta_1\beta_2(a_0c_0 + d_0 - \gamma_1\gamma_2) \leq 0$

Proof. We will prove the result by means of Theorem 1. Lemma 7 implies that *i)* of Theorem 1 never occurs; Lemma 8 implies that condition

$$\exists \overline{x}, \overline{y} \in \Re^n \text{ such that } Q\overline{x} = q \text{ and } Q\overline{y} = b$$

is equivalent to conditions *i)* and *ii)*. The thesis then follows directly from Theorem 1 noticing that:

$$\begin{aligned} b^T\overline{y} &= 2\beta_1\beta_2 \\ b^T\overline{x} + b_0 &= b_0 - \gamma_1\beta_1 - \gamma_2\beta_2 \\ n(\overline{x}) &= a_0c_0 + d_0 - \gamma_1\gamma_2 \end{aligned}$$

■

4 The sum of an affine function and a linear fractional one

Using the previous results we are able to recover as a corollary the result given by [3] related to the following function

$$f(x) = a^Tx + \frac{d^Tx + d_0}{b^Tx + b_0} \tag{8}$$

on the set $X = \{x \in \Re^n : b^Tx + b_0 > 0\}$, where $a, d, b \in \Re^n$, $b \neq 0$, $d_0, b_0 \in \Re$. f can be easily seen as a particular case of function (1) when $c = b$, $c_0 = b_0$ and $a_0 = 0$.

Corollary 10 *Function f in (8) is pseudoconvex on X if and only if one of the following conditions hold:*

i) $a = kb,\ \ k \geq 0;$

ii) $\exists \delta \in \Re$ *such that* $d = \delta b$ *and* $d_0 \geq \delta b_0$.

Proof. *i)* When $k = 0$ f becomes a linear fractional function and hence it is pseudolinear. Otherwise, being $c = b$, the result follows directly from *i)* of Corollary 6.

ii) Let us firstly consider the case $a = kb, \; k < 0$.

Being $c = b, c_0 = b_0$, and $a_0 = 0$,from *ii)* of Corollary 6 there exists a $\bar{\delta} \in \Re$ such that $d = \bar{\delta}a$ and condition (6) can be specified as follows :

$$\frac{1}{k}(-b_0)^2 \geq \frac{1}{k}(b_0\bar{\delta} - \frac{1}{k}d_0) + \frac{1}{k}(b_0)(b_0)$$

and so $(b_0\bar{\delta} - \frac{1}{k}d_0) \geq 0$, that is $d_0 \geq k\bar{\delta}b_0$. Since $a = kb$ we get $d = \bar{\delta}kb$ so that $\delta = \bar{\delta}k$.

To complete the proof we are left to deal with the case a and b linearly independent using Theorem 9. More precisely since $c = b$ in *i)* of Theorem 9 we have $d = \delta_1 a + \delta_2 b$, while in *ii)* we get $\beta_1 = 0$, $\beta_2 = 1$. This implies that condition $\beta_1\beta_2 < 0$ in *iii-b)* never occurs. We have also $\gamma_1 = \delta_2$ and $\gamma_2 = b_0 + \delta_1$; hence $b_0 = \gamma_1\beta_1 + \gamma_2\beta_2$ in *iii-a)* becomes $\delta_1 = 0$ hence $d = \delta_2 b$ and $\gamma_2 = b_0$. We conclude observing that condition $a_0c_0 + d_0 \geq \gamma_1\gamma_2$ is specified as $d_0 \geq \delta_2 b_0$. The proof is now complete. ∎

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