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A note on endogenous restricted partecipation on financial markets: an existence result

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A note on endogenous restricted participation on financial markets: an existence result

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Abstract

This paper presents a GEI model with restricted participation where the access to the assets markets depends upon some endogenous variables. The proposed framework generalizes the ones described by Cass Siconolfi and Villanacci (1991) and by Siconolfi (1989) where the restriction on the assets market only depends on the assets demand. The existence result is obtained by means of a non standard degree argument.

1 Introduction

The idea under the so called restricted participation is the following one: consumers are not equal in front of assets markets. When someone asks for a loan to a bank or buys shares or derivatives at the stock exchange, must face some restriction depending basically on his wealth. In real life, there are many cases where the participation constraints on financial markets varies from a class of consumers to another. For example, we can think of collateral securities in American real estate market, of a credit line which is secured by financial assets and/or by a proportion of the consumer's wealth. So, in a two-period general equilibrium model with uncertainty and restricted participation, consumers have to face not only their budget constraints but also their financial constraints describing their different access to the assets market. Even if markets are complete, i.e. it is possible in general to move wealth from any state to any other one, personal constraints do not permit this possibility. In that sense, restricted participation can be seen as a generalization of the incompleteness of markets: in fact the different access to the financial markets makes them incomplete from the consumers' point of view.

In the recent literature several restricted participation models have been presented (see Siconolfi (1988), Balasko, Cass and Siconolfi (1990), Cass, Siconolfi and Villanacci (1992), Polemarchakis and Siconolfi (1997) and Carosi (2000)).

While Balasko Cass and Siconolfi (1990), Cass, Siconolfi and Villanacci (1992), and Carosi (2000) consider a model with numeraire assets and where the restricted participation depends basically on the demand of assets, this paper aims to introduce a financial constraint function which depends not only upon the assets' demand but also upon some other endogenous variables.

The set up of the model is described in section two while the third is devoted to the existence result which is obtained by a non standard degree argument; more precisely we present an original construction of the homotopy, which "links" the function F representing the equilibrium points, and the well chosen function g for which $\#g^{-1}(0)$ is known.

2 Set up of the model

We describe a competitive two-period exchange economy with uncertainty where there are S, $S > 1$, possible states of the world in the second period. Spot commodity markets open in the first and second period, and there are C, $C > 1$, commodities in each spot, labelled by $c = 1, 2, ..., C$. We label each spot by $s = 0, ..., S$, where $s = 0$ corresponds to the first period. There are H households, $H > 1$, labelled by $h = 1, 2, ...H$ and I assets, labelled by $i = 1, 2, ..., I$

The time structure of the model is as follows: in the first period, commodities and assets are exchanged and first period consumption takes place. Then uncertainty is resolved, assets pay their returns and finally households consume second-period commodities. $x_h^{s,c}$ is the consumption of commodity c in state s by household h; similar notation is used for the endowments, $e_h^{s,c}$. Both consumption and endowments are elements of $\mathbb{R}^G_{++}$ for each household, where $G = (S+1)C$.

Household h's preferences are represented by utility function $u_h : \mathbb{R}^G_{++} \to \mathbb{R}$.

As in most of the literature on smooth economies we will assume throughout that

Assumption 1 1. u_h is C^2,

2. differentiably strictly increasing, (i.e., $Du_h(x_h) \gg 0$),

3. differentiably strictly quasi-concave, (i.e., $\Delta x \neq 0$ and $Du_h(x_h)\Delta x = 0 \Rightarrow \Delta x^T D^2 u_h(x_h)\Delta x < 0$),

4. and it has indifference surfaces with closure (in the standard topology of $\mathbb{R}^G$) in $\mathbb{R}^G_{++}$(i.e., for any $\underline{u} \in \mathbb{R}$, $Cl\left\{x \in \mathbb{R}^C_{++} : u_h(x) = \underline{u}\right\} \subseteq \mathbb{R}^C_{++}$).

The following standard notation is also used:
$x_h^s \equiv (x_h^{s,c})_{c=1}^C$, $x_h \equiv (x_h^s)_{s=0}^S$, $x \equiv (x_h)_{h=1}^H$, with the obvious meaning,
$p^{s,c}$, the price of commodity c in spot s, $p \equiv (p^s)_{s=0}^S$, the commodity price vector, with the price p^{sC} of the numeraire commodity C. q^i , the price of the i-th asset, $q \equiv (q^i)_{i=1}^I$,
$y^{s,i}$ the yield of the i-th asset in state s in *units of commodity* C ,
b_h^i is the demand of asset i by household h.

Let Y be the $S \times I$ return matrix given by

$$Y \equiv \left[y^{si}\right]_{si}.$$

It greatly simplifies our analysis to assume that

Assumption 2 (No redundancy) We assume $S > I$ and $RankY = I$.

Define

$$a_h : \mathbb{R}^I \times \mathbb{R}_{++}^{G-(S+1)} \times \mathbb{R}^I \times \mathbb{R}^G \to \mathbb{R}^{\#J_h}, \quad a_h : (b_h, p, q, e_h) \mapsto a_h(b_h, p, q, e_h),$$

where J_h is a finite set such that $\#J_h < I$. Then household h's maximization problem is the following one. For given $p \in \mathbb{R}_{++}^G$, $q \in \mathbb{R}^I$, $e \in \mathbb{R}_{++}^G$,

$$\begin{array}{lll} \max_{(x_h,b_h)} u_h(x_h) & s.t. & \\ p^0 x_h^0 + q b_h & \leq p^0 e_h^0 & \\ p^s x_h^s & \leq p^s e_h^s + \sum_{i=1}^I p^{sC} y^{si} b_h^i & (s = 1, .., S) \\ a_h(b_h, p, q, e_h) & \geq 0 & \end{array} \tag{1}$$

Assumption 3 a_h is spot by spot homogenous of degree zero with respect to the prices of goods and assets, i.e., for every (b_h, p, q, e_h) and every $\gamma \in \mathbb{R}_{++}^{S+1}$,

$$a_h(b_h, (\gamma^s p^s)_{s=0}^S, \gamma^0 q, e_h) = a_h(b_h, p, q, e_h).$$

Given the above assumption, we can normalize prices using the price of the good C in each spot. Define $p^{\setminus} = (p^{\setminus s})_{s=0}^S = ((p^{sc})_{c \neq C})_{s=0}^S$
With innocuous abuse of notation, we still denote by p and q normalized prices. Define

$$\Phi = \begin{bmatrix} p^0 & & & \\ & p^1 & & \\ & & \ddots & \\ & & & p^S \end{bmatrix}$$

and $R(q) = \begin{bmatrix} -q \\ Y \end{bmatrix}$, we can rewrite (1) as

$$\begin{array}{llll} max_{x_h,b_h} & u_h(x_h) & -\Phi(x_h - e_h) + Rb_h \geq 0 & \lambda_h = (\lambda_h^s)_{s=0,1,...,S} \\ & & a_h(b_h, p, q, e_h) \geq 0 & \mu_h = (\mu_h^j)_{j \in J_h}. \end{array} \tag{2}$$

where λ_h and μ_h are the Kuhn Tucker multipliers associated with the corresponding constraints.

Define J'_h as a subset of J_h, and

$$a^{J'_h} : \mathbb{R}^I \to \mathbb{R}^{\#J'_h}, a^{J'_h} : b_h \mapsto \left(a^j_h\left(b_h, p, q, e_h\right)\right)_{j \in J'_h}.$$

We assume that the restriction function a_h satisfies the following properties.

Assumption 4 i) a_h is $\mathcal{C}^2$.

ii) a_h is componentwise concave in the variable b_h, i.e., for any $j \in J_h$, for any b_h, p, q, e_h and for any $\Delta b \in \mathbb{R}^I$,

$$\Delta b^T D^2_{b_h b_h} a^j_h\left(b_h, p, q, e_h\right) \Delta b \leq 0.$$

iii) a_h permits no participation on the asset market, i.e., for any p, q, e_h it is the case that

$$a_h\left(0, p, q, e_h\right) \geq 0.$$

iv) For every $(b_h) \in \mathbb{R}^I$ such that $a^{J'_h}_h\left(b_h, p, q, e_h\right) = 0$,

$$rank\left(Da^{J'_h}_{b_h}\left(b_h, p, q, e_h\right)\right) = \#J'_h.$$

v) For every i there exists some $h(i)$ such that for every (b, p, q, e_h),

$$D_{b^i_{h(i)}} a_{h(i)}\left(b_{h(i)}, p, q, e_h\right) = 0.$$

Let us denote by $\mathcal{A}_h$ the set of functions a_h verifying Assumptions 4. Define also $\mathcal{A} = \underset{h \in H}{\times} \mathcal{A}_h$.

We present some easy examples of our kind of restricted participation.

Example 1 1. You cannot borrow more than a given proportion α_h of your expected real wealth:

$$qb_h \leq \alpha_h \min_s \left\{p^s e^s_h\right\}, \text{ or } \quad \alpha_h \min_s \left\{p^s e^s_h\right\} - qb_h \geq 0.$$

Observe that this function is concave and not $\mathcal{C}^2$ just on a zero measure set.

2. You have to buy a financial collateral, i.e., if you want to borrow, you have to partially cover your debt buying some "safe" bonds which could be use to repay your debt. Partitioning the set I in I^1 and I^2, where I^2 is the set of "safe" bonds, we must have

$$q^1 b^1_h \leq -\beta q^2 b^2_h, \quad \text{or} \quad -\beta q^2 b^2_h - q^1 b^1_h \geq 0.$$

Market clearing conditions are

$$\begin{array}{ll} (M1) & \sum_{h=1}^H \left(x_h - e_h\right) = 0, \\ (M2) & \sum_{h=1}^H b_h = 0. \end{array}$$

3 Existence of equilibria

In the proof of existence, fix (a, u). Consider the system of first order conditions to consumers' problems and market clearing conditions.

$$\begin{array}{lll} (h.1) & D_{x_h} u_h(x_h) - \lambda_h \Phi & = 0 \\ (h.2) & -\Phi(x_h - e_h) + R b_h & = 0 \\ (h.3) & \lambda_h R + \mu_h D_{b_h} a_h(b_h, p, q, e_h) & = 0 \\ (h.4) & \min\left\{\mu_h^j, a_h^j(b_h, p, q, e_h)\right\} & = 0 \\ (M.1) & \sum_{h=1}^H \left(x_h^{\setminus} - e_h^{\setminus}\right) & = 0 \\ (M.2) & \sum_{h=1}^H b_h & = 0 \end{array} \qquad (3)$$

Define

$$\Xi \equiv \mathbb{R}_{++}^{GH} \times \mathbb{R}^{H\widehat{S}} \times \mathbb{R}^{HI} \times \mathbb{R}^{\sum \# J_h} \times \mathbb{R}_{++}^{G-(S-1)} \times \mathbb{R}^{I}, \quad \xi \equiv \left((x_h, \lambda_h, b_h, \mu_h)_{h=1}^{H}, p^{\setminus}, q\right) \in \Xi$$

$$F : \Xi \times \mathbb{R}_{++}^{GH} \to \mathbb{R}^{\dim \Xi}, \quad (\xi, e) \mapsto \text{left hand side of (3)}$$

To show existence, i.e. $F^{-1}(0) \neq \emptyset$, we are going to use a degree argument.

Definition 1 $C(M, N, 0)$ denotes the class of functions which satisfy the following Assumption i) to iv).

i) M, N are smooth boundaryless manifolds such that $\dim M = \dim N$;
ii) $f : M \to N$ is a C^0 function;
iii) $0 \in N$;
iv) $f^{-1}(0)$ is compact.

Theorem 5 . *Assume that*

1. $f, g \in C(M, N, 0)$;
2. 0 is a regular value for g and $g^{-1}(0)$ is odd;
3. there exists a C^0 homotopy $H : M \times [0,1] \to N$, $(x, \tau) \mapsto H(x, \tau)$ from f to g i.e. $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$, such that $H^{-1}(0)$ is compact.
Then $f^{-1}(0) \neq \emptyset$.

We are going first to construct the needed function g such that $g^{-1}(0)$ is odd, and then the needed homotopy . The basic idea is the following one. We need to be careful about how to "homotopize" the restricted participation function.

Roughly speaking, using a Pareto Optimal allocation x^* and an appropriate restricted participation function a^* we define g, i.e. H_1, as $g : \xi \mapsto$ (left hand side of (3)) associated with $e = x^*$, u and $a = a^*$. In order to have the differentiability of g we have to require $a_h^{*j}(0, \ldots) > 0$. On the other hand, we need the compactness of $H^{-1}(0)$ that is the sequentially compactness. Then along the sequence of τ^v, we need to be able to get the convergence of ξ^v and therefore of μ_h^v. To insure that we need either to have $a_h^{*j}(b_h(\tau^v),...) > 0$, so that μ_h^{jv} converges to *zero*, or $a_h^j(b_h(\tau^v),...) = 0$, *where there is no* $*$, *i.e., we need to get the true restricted participation function,* in order to be able to use Assumption 4.iv, and through the rank condition, get the convergence of μ_h.

To get the differentiability of g we can consider the following homotopy about a_h

$$a_h^*(.) \equiv a_h(b_h,...) + \tau\mathbf{1}.$$

For $(b_h,..)$ such that $a_h(b_h,...) \geq 0$, our choice guarantees $a_h^*(b_h,...) > 0$ as $\tau = 1$ and from Assumption 4.iii, we get $a_h^*(0,..) > 0$. Unfortunately $a_h^*(.)$ does not in general verifies Assumption 4.iv and so it is difficult to get the covergence of μ_h^v. To deal with this issue we take $\overline{\tau} \in (0,1)$ and consider the following continuous functions.

$$\Upsilon : \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} e^{-\frac{1}{(x-\overline{\tau})^2(1-x)^2}} & if \quad \overline{\tau} < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\Gamma : \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} e^{-\frac{1}{x^2(\overline{\tau}-x)^2}} & if \quad 0 < x < \overline{\tau} \\ 0 & \text{otherwise} \end{cases}$$

Then we construct the two following smooth bump functions
$\gamma : [0,1] \to [0,1], \tau \mapsto \frac{\int_{-\infty}^{\tau} \Upsilon(x)dx}{\int_{-\infty}^{+\infty} \Upsilon(x)dx}$ i.e.

$$\gamma : \tau \mapsto \begin{cases} 0 & if \quad \tau \in [0,\overline{\tau}) \\ \frac{\int_{-\infty}^{\tau} \Upsilon(x)dx}{\int_{-\infty}^{+\infty} \Upsilon(x)dx} & if \quad \tau \in (\overline{\tau},1) \\ 1 & if \quad \tau = 1 \end{cases}$$

$\psi : [0,1] \to [0,1], \tau \mapsto 1 - \frac{\int_{-\infty}^{\tau} \Gamma(x)dx}{\int_{-\infty}^{+\infty} \Gamma(x)dx}$ i.e.

$$\psi : \tau \mapsto \begin{cases} 1 & if \quad \tau = 0 \\ 1 - \frac{\int_{-\infty}^{\tau} \Gamma(x)dx}{\int_{-\infty}^{+\infty} \Gamma(x)dx} & if \quad \tau \in (0,\overline{\tau}) \\ 0 & if \quad \tau \in [\overline{\tau},1] \end{cases}$$

For any j and h, define

$$a_h^{j*}(b_h,p,q,e_h,\tau) \equiv a_h^j(\psi(\tau)b_h,p,q,(1-\tau)e_h + \tau x_h^*) + \gamma(\tau),$$

and $\beta_h \equiv \psi(\tau)b_h$.

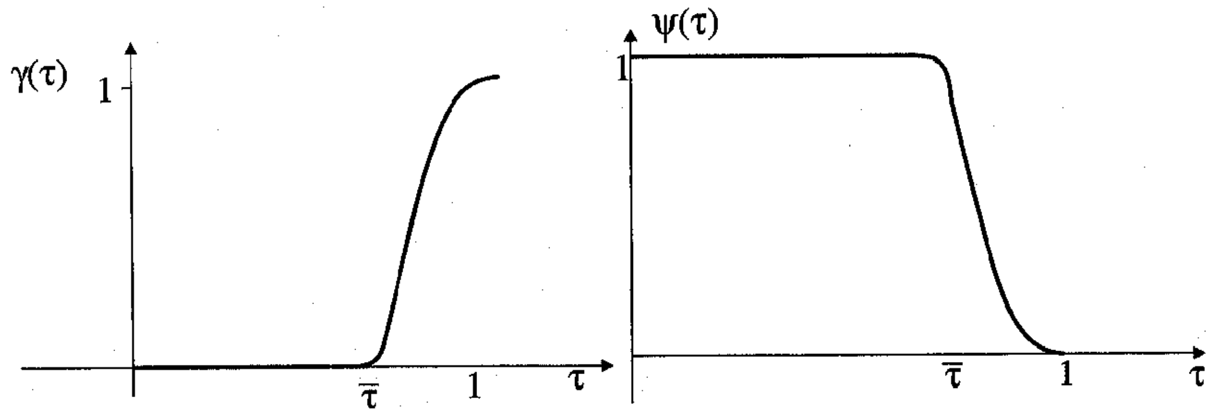

Figure 1:

Observe that the above function takes the values indicated in the following table with respect to τ:

τ		$a_h^{j*}(b_h, p, q, e_h, \tau)$
0		$a_h^j(b_h, p, q, e_h)$
$(0, \overline{\tau})$		$a_h^j(\psi(\tau) b_h, p, q, (1-\tau) e_h + \tau x_h^*)$
$\overline{\tau}$		$a_h^j(0, p, q, (1-\overline{\tau}) e_h + \overline{\tau} x_h^*)$
$(\overline{\tau}, 1)$		$a_h^j(0, p, q, (1-\tau) e_h + \tau x_h^*) + \gamma(\tau)$
1		$a_h^j(0, p, q, x_h^*) + 1$

Now we are ready to construct the desired homotopy function.

Consider a Pareto Optimal allocation $x^* \in \mathbb{R}_{++}^{GH}$. It is known that (Balasko 1988) there exists $(\theta_{\setminus 1}^*, \chi^*) \in \mathbb{R}_{++}^{H-1} \times \mathbb{R}_{++}^{G}$ such that $(x^*, \theta_{\setminus 1}^*, \chi^*)$ is the unique solution to the following system

$$\begin{array}{lll} (1) & D_{x_1} u_1(x_1^*) - \chi^* & = 0 \\ (2) & \theta_h^* D_{x_h} u_h(x_h) - \chi & = 0 \\ (3) & (u_h(x_h) - u_h(x_h^*))_{h \neq 1} & = 0 \\ (4) & -\sum_{h=1}^{H} x_h + \sum_{h=1}^{H} x_h^* & = 0 \end{array} \qquad (4)$$

Let $\theta_1^* \equiv 1$.

Consider now the following system we use to define the needed homotopy.

$$
\begin{array}{lll}
(h.1) & D_{x_h} u_h(x_h) - \lambda_h \Phi & = 0 \\
(h.2) & -\Phi\left(x_h - \left((1-\tau) e_h + \tau x_h^*\right)\right) + R b_h & = 0 \\
(h.3) & \lambda_h R + \mu_h D_{\beta_h} a_h^j\left(\psi(\tau) b_h, p, q, (1-\tau) e_h + \tau x_h^*\right) & = 0 \\
(h.4) & \min\left\{\mu_h^j, a_h^j\left(\psi(\tau) b_h, p, q, (1-\tau) e_h + \tau x_h^*\right) + \gamma(\tau)\right\} & = 0 \\
(M.1) & \sum_{h=1}^{H}\left(x_h^{\backslash} - \left((1-\tau) e_h^{\backslash} + \tau x_h^{\backslash *}\right)\right) & = 0 \\
(M.2) & \sum_{h=1}^{H} b_h & = 0
\end{array}
\qquad (5)
$$

For given e and x^*, define

$$
\begin{aligned}
H &: \Xi \times [0,1] \longrightarrow \mathbb{R}^{\dim \Xi}, \\
H &: (\xi, \tau) \mapsto (\text{left hand side of system (5)})
\end{aligned}
$$

Remark 1 ξ is a solution to $H(\xi, 0) = 0$ iff ξ is an equilibrium at e, i.e., $F(\xi, e) = 0$.

Define

$$
g : \Xi \to \mathbb{R}^{\dim \Xi}, \;\; \xi \mapsto H(\xi, 1),
$$

and therefore $g(\xi) = 0$ is

$$
\begin{array}{lll}
(h.1) & D_{x_h} u_h(x_h) - \lambda_h \Phi & = 0 \\
(h.2) & -\Phi\left(x_h - x_h^*\right) + R b_h & = 0 \\
(h.3) & \lambda_h R + \mu_h D_{\beta_h} a_h^j\left(0, p, q, x_h^*\right) & = 0 \\
(h.4) & \min\left\{\mu_h^j, a_h^j\left(0, p, q, x_h^*\right) + 1\right\} & = 0 \\
(M.1) & \sum_{h=1}^{H}\left(x_h^{\backslash} - x_h^{\backslash *}\right) & = 0 \\
(M.2) & \sum_{h=1}^{H} b_h & = 0
\end{array}
$$

We are now ready to show that $g^{-1}(0)$ is odd and 0 is a regular value for g.

That result is obtained showing that $g^{-1}(0) = \{\xi^*\}$ in Lemmas 1 and 2, and that *rank* $D_\xi g(\xi^*) = \dim \Xi$ in Lemma 3.

Lemma 1 $\xi^{**} \in g^{-1}(0)$, *where*

$$
\xi^{**} = \left(\begin{array}{l}
\left(x_h^{**} = x_h^*, \; \lambda_h^{**} = \left(\frac{\chi^{*sC}}{\theta_h^*}\right)_{s=0}^{S}, \; b_h^{**} = 0, \mu^{**} = 0\right)_{h \in H}, \\
p^{**} = \left(\frac{\chi^{*s}}{\chi^{*sC}}\right)_{s=0}^{S}, q^{**} = \sum_{s=1}^{S}\left(\frac{\chi^{*sC}}{\chi^{*0C}}\right) y^s
\end{array} \right).
$$

Proof.

The result follows simply computing $g(\xi^{**})$. We want to underline that :

$$
a_h^j\left(0, p, q, x_h^*\right) + 1 > 0 \text{ and } \mu_h^j = 0.
$$

■

Remark 2 From the fact that for $s \geq 0$,we have that

$$D_{x^s} u_h\left(x_h^*\right) - \lambda^{s**} p^{s**} = 0,$$

and

$$D_{x^{sC}} u_h\left(x_h^*\right) = \lambda^{s**}.$$

Therefore

$$p^{s**} = \frac{D_{x^s} u_h\left(x_h^*\right)}{D_{x^{sC}} u_h\left(x_h^*\right)}$$

Lemma 2 $\{\xi^{**}\} = g^{-1}(0)$.

Proof.

The first 3 steps are the basically the same as in the case of numeraire assets.

Consider an arbitrary $\widehat{\xi} \equiv \left(\widehat{x}, \widehat{\lambda}, \widehat{b}, \widehat{\mu}, \widehat{p}, \widehat{q}\right)$. From the previous Lemma, it is enough to show that if $F\left(\widehat{\xi}, x^*\right) = 0$, then $\widehat{\xi} = \xi^{**}$.

1. $\widehat{x} = x^*$.

Suppose otherwise, i.e., $\widehat{x} \neq x^*$. Consider $\widetilde{x} \equiv \frac{1}{2}\left(\widehat{x} + x^*\right)$. Since $F\left(\widehat{\xi}, x^*\right) = 0$, $\sum_h \widehat{x}_h = \sum_h x_h^* =$ and

$$\sum_h \widetilde{x}_h = \frac{1}{2}\left(\sum_h \widehat{x}_h + \sum_h x_h^*\right) = \sum_h x_h^*. \tag{6}$$

From Assumption 1.iii (strict quasiconcavity of the utility functions), we then have

$$u\left(\widetilde{x}\right) \gg u\left(x^*\right) \tag{7}$$

But (6) and (7) contradict the Pareto Optimality of x^*. Therefore, we have that for $h = 1, ..., H$

$$\begin{array}{lll} (h.1) & D_{x_h} u_h\left(x_h^*\right) - \lambda_h \Phi\left(\widehat{p}\right) & = 0 \\ (h.2) & -\Phi\left(x_h^* - x_h^*\right) + R\widehat{b}_h & = 0 \\ (h.3) & \widehat{\lambda}_h R\left(\widehat{q}\right) + \widehat{\mu}_h D_{\beta_h} a_h^j\left(0, \widehat{p}, \widehat{q}, x_h^*\right) & = 0 \\ (h.4) & \min\left\{\widehat{\mu}_h^j, a_h^j\left(\widehat{b}_h, \widehat{p}, \widehat{q}, x_h^*\right) + 1\right\} & = 0 \end{array} \tag{8}$$

2. $\widehat{\lambda} = \lambda^{**}$.

For any h and s,from (8.1), $D_{x^{sC}} u_h\left(x_h^*\right) - \widehat{\lambda}_h^s = 0$. Therefore, from Remark 2 $\widehat{\lambda}_h^s = \lambda_h^{s**}$.

3. $\widehat{p} = p^{**}$.

From (8.1), and from Remark 2,

$$\widehat{p}^s = \frac{D_{x^s} u_h\left(x_h^*\right)}{\widehat{\lambda}_h^s} = \frac{D_{x^s} u_h\left(x_h^*\right)}{D_{x^{sC}} u_h\left(x_h^*\right)} = p^{s**}.$$

4. $\widehat{b} = b^{**} = 0$.
From (3.h2), we have

$$\widehat{b}_h = \left[\widetilde{Y}\right]^{-1} \cdot 0 = 0,$$

where $\widetilde{Y}$ a $I \times I$ full rank submatrix of Y.
5. $\widehat{\mu} = \mu^{**}$.
From Assumption 4.iii), we have that $a_h\left(0, p^{**}, \hat{q}, x_h^*\right) \geq 0$ and therefore $a_h\left(0, p^{**}, \widehat{q}, x_h^*\right) + \mathbf{1} > 0$. Then, from (8.4),

$$0 = \min\left\{\widehat{\mu}_h^j, a_h^j\left(\widehat{b}_h, \widehat{p}, \widehat{q}, x_h^*\right) + 1\right\} = \widehat{\mu}_h^j.$$

6. $\widehat{q} = q^{**}$.
From (8.3) and from Remark 2,

$$\widehat{q} = \sum_{s=1}^{S} \frac{\widehat{\lambda}_h^s}{\widehat{\lambda}_h^0} y^s = \sum_{s=1}^{S} \frac{\lambda_h^{s**}}{\lambda_h^{0**}} r^s = q^{**}.$$

■

Lemma 3 *rank* $D_\xi g\left(\xi^*\right) = \dim \Xi$.

Proof.

	x_h	λ_h	b_h	μ_h	$p^\backslash$	q
$D_{x_h} u_h\left(x_h\right) - \lambda_h \Phi$	$D^2 u_h\left(x_h^*\right)$	$-\Phi^T$			Λ_h^{**}	
$-\Phi\left(x_h - x_h^*\right) + R b_h$	$-\Phi\left(p^{**}\right)$		$R\left(q^{**}\right)$			
$\lambda_h R + \mu_h D_{\beta_h} a_h\left(0, p, q, x_h^*\right)$		$R^T\left(q^{**}\right)$		$\left[D_{b_h} a_h\left(0, p^{**}, q^{**}, x_h^*\right)\right]^T$		$-\lambda_h^{0*} I_I$
$\min\left\{\mu_h^j, a_h^j\left(0, p, q, x_h^*\right) + \mathbf{1}\right\}$				I		
$\sum_{h=1}^H \left(x_h^\backslash - x_h^{\backslash *}\right)$	$I0$					
$\sum_{h=1}^H b_h$			I			

where

$$\Lambda_h^{**} \equiv \begin{bmatrix} \lambda_h^{0**} I_{C-1} & & & \\ 0 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \lambda_h^{S**} I_{C-1} \\ & & & 0 \end{bmatrix}.$$

We can use the unit matrix I in superrow of min $\left\{\mu_h^j, a_h^j\left(0, p, q, x_h^*\right) + \mathbf{1}\right\}$ to clean up the column of μ_h. We can then erase those superrow and supercolumn. Then we get the same matrix of the case of numeraire asset economy. ■

Lemma 4 $H^{-1}(0)$ *is compact.*

Proof. To show this result we are going to prove that $H^{-1}(0)$ is sequentially compact, i.e. every sequence admits a converging subsequence. Take a sequence $\{(\xi^n, \tau^n)\}$ such that $(\xi^n, \tau^n) \in H^{-1}(0)$. First of all observe that since $\{\tau^n\} \subseteq [0,1]$ it admits a converging subsequence $\{\tau^n\} \to \tilde{\tau}$. Then we are left to show that every component of the sequence $\{\xi^n\} = \{(x_h^n, \lambda_h^n, b_h^n, \mu_h^n)_{h\in H}, p^{\setminus n}, q^n\}$ such that $(\xi^n, \tau^n) \in H^{-1}(0)$ admits a converging subsequence in Ξ since in this case $\xi^n \to \widetilde{\xi} = \left(\left(\widetilde{x}_h, \widetilde{\lambda}_h, \widetilde{b}_h, \widetilde{\mu}_h\right)_{h\in H}, \widetilde{p}^{\setminus}, \widetilde{q}\right)$ and from the continuity of H we get $\widetilde{\xi} \in H^{-1}(0)$ and then we are done.

The following steps 1-6 show that every component of the sequence $\{\xi^n\}$ admits a converging subsequence in Ξ.

Step 1. $\{x_h^n\}$ has a converging subsequence in $\mathbb{R}_{++}^{GH}$.
By definition we have $x_h^n \gg 0$, for every h and therefore $\{x_h^n\}$ is bounded below. Since $\sum_h \left(x_h^{\setminus n} - e_h^{\setminus}(\tau^n)\right) = 0$, we get $\sum_h x_h^{\setminus n} \leq \sum_h e_h^{\setminus}(\tau^n)$ and so $(x_h^{sc})^n \leq \sum_h e_h^{sc}(\tau^n)$. Observing that $e_h^{\setminus}(\tau^n)$ is continuous and $\{\tau^n\}$ is a converging sequence, we obtain $\{x_h^n\}$ is bounded above and hence it admits a converging subsequence $\{x_h^n\} \to \widetilde{x}_h$. We are left to show that $\widetilde{x}_h \in \mathbb{R}_{++}^{GH}$. Since $\{e_h^n\} \to \widetilde{e}_h$ there exists a compact set $I_{\widetilde{e}_h}$such that $\widetilde{e}_h \in I_{\widetilde{e}_h}$ and for any n which is sufficiently big, we have $e_h^n \in I_{\widetilde{e}_h}$. Let be $\widetilde{u} = \min\limits_{e_h \in I_{\widetilde{e}_h}} u_h(e_h)$. For a well chosen n', since $(\xi^n, \tau^n) \in H^{-1}(0)$, $u_h(x_h^n) \geq \widetilde{u}$ for every $n > n'$. Hence $x_h^n \in L_{\widetilde{u}} = \{x_h \in \mathbb{R}_{++}^G : u_h(x_h^n) \geq \widetilde{u}\}$. From Assumption on the utility function $clL_{\widetilde{u}} \subseteq \mathbb{R}_{++}^G$, hence $\widetilde{x}_h \gg 0$.

- **Step 2. $\{\lambda_h^{sn}\}$ has a converging subsequence in $\mathbb{R}_{++}$** for every $s = 0, ..., S$.
From the First Order Conditions, recalling $p^{sC} = 1$ for every s we have $D_{x_h^{sC}} u_h(x_h^n) - \lambda_h^{sn} = 0$ and $\lambda_h^{sn} = D_{x_h^{sC}} u_h(x_h^n)$. Hence

$$\lambda_h^{sn} \to \tilde{\lambda}_h^s = D_{x_h^{sC}} u_h(\tilde{x}_h) > 0.$$

- Step 3. $\{p^{\backslash n}\}$ has a converging subsequence in $\mathbb{R}^{G-\hat{S}}_{++}$.

From the First Order Conditions we have $p^{s\backslash n} = \frac{D_{x_h^s C} u_h(x_h^n)}{\lambda_h^{sn}}$ for every $s = 0, ...S$ and $h \in H$.

Hence

$$p^{s\backslash n} \rightarrow \tilde{p}^{s\backslash} = \frac{D_{x_h^{s\backslash}} u_h(\widetilde{x}_h)}{\widetilde{\lambda}_h^s} > 0.$$

Step 4. $\{b^n\}$ has a converging subsequence in $\mathbb{R}^I$

From $-\Phi(x_h - e_h) + Rb_h = 0$ we get $Rb_h = -\Phi(x_h - e_h)$ Consider the S equations referring to states $s = 1, ..., S$. Recall that $rank[Y] = I$, the vector b_h is a continuous function of (x_h, e_h, p). Then since $\{(x_h^n, e_h^n, p^{\backslash n})\}$ admits a converging subsequence we have $b_h^n \rightarrow \widetilde{b}_h$.

Step 5. $\{q^n\}$ has a converging subsequence in $\mathbb{R}^I$.

Due to the Assumption 4.v, for every asset i, there exists a consumer h' such that equation (5.h.5) becomes $\lambda_{h'}R = 0$ and so

$$q^{in} \rightarrow \frac{\sum_{s=1}^{S} \widetilde{\lambda}_{h'}^s y^{si}}{\widetilde{\lambda}_h^0} = \widetilde{q}^i.$$

Step 6. $\{\mu^n\}$ has a converging subsequence in $\mathbb{R}^{\#J_h} \times \mathbb{R}_+$.

In this last step we have to distinguish some cases according to the value of $\widetilde{\tau} \equiv \lim_{n \rightarrow +\infty} \tau^n$.

Case 1. $\tau^n \rightarrow 0$.

Let $\{J_h^A, J_h^B\}$ be a partition of the set of index J_h such that

$$J_h^A = \left\{ j \in J_h : \ a_h^{*j}\left(\tilde{b}_h, \tilde{p}, \tilde{q}, \tilde{e}_h, \widetilde{\tau}\right) = 0 \right\} \quad \text{and} \quad J_h^B = \left\{ j \in J_h : \ a_h^{*j}\left(\tilde{b}_h, \tilde{p}, \tilde{q}, \tilde{e}_h, \widetilde{\tau}\right) > 0 \right\}.$$

If $j \in J_h^B$, there exists a n^* such that $a_h^{j*}(\beta_h^n, p^n, q^n, e_h^n, \tau^n) > 0$ for every $n > n^*$. Hence for every $n > n^*$ we have $\mu_h^{jn} = 0$, i.e., $\mu_h^{jn} \rightarrow 0$ for every $j \in J_h^B$.

Since $\tau^n \rightarrow 0$ there exists a $\hat{n}$ such that for any $n > \hat{n}$ $a_h^*(b_h^n, p^n, q^n, e_h^n, \tau^n) = a(b_h^n, p^n, q^n, e_h^n, \tau^n)$.

If $j \in J_h^A$, from Assumption 4.iv, $rank\left(D_{\beta_h} a_h^{J^A}\left(\tilde{b}_h, \tilde{p}, \tilde{q}, \tilde{e}_h, \widetilde{\tau}\right)\right) = \#J_h^A$.

Let $D_{\beta_h} \tilde{a}_h^{J^A}\left(\left(\tilde{b}_h, \tilde{p}, \tilde{q}, \tilde{e}_h, \widetilde{\tau}\right)\right)$ be the square submatrix of $D_{\beta_h} a_h^{J^A}$ (whose dimension is $\#J_h^A \times \#J_h^A$) such that $\left|det D_{b_h} \tilde{a}_h^{J^A}\left(\tilde{b}_h, \tilde{p}, \tilde{q}, \tilde{e}_h, \widetilde{\tau}\right)\right| > 0$ and $D_{b_h} \tilde{a}_h^{J^B}\left(\tilde{b}_h, \tilde{p}, \tilde{q}, \tilde{e}_h, \widetilde{\tau}\right)$ is the matrix of dimension $(\#J_h - \#J_h^A) \times (I - \#J_h^A)$ which is the complement of $D_{b_h} \tilde{a}_h^{J^A}$. Then, there exists $n' > \hat{n}$ such that $\left|det D_{\beta_h} a_h^{J^A}(b_h^n, p^n, q^n, e_h^n, \tau^n)\right| > 0$ for every $n > n'$. Let us take $n^{**} = max\{n^*, n'\}$. Making the needed permutations, from equation $(h.3)$ of (5) we get:

$$\begin{array}{c|ccc|c} & 1 & I & 1 & \\ \begin{array}{c}\#J_h^B \\ \#J_h^A\end{array} & \begin{bmatrix} \mu_h^{j_h^B n} \\ \mu_h^{j^A n} \end{bmatrix}^T & \begin{bmatrix} D_{b_h}\tilde{a}_h^{j^B}(b_h^n, p^n, q^n, e_h^n, \tau^n) \\ D_{b_h}\tilde{a}_h^{j^A}(b_h^n, p^n, q^n, e_h^n, \tau^n) \end{bmatrix} & \equiv [\eta^n] & I \end{array}$$

for every $n > n^{**}$ i.e

$$\begin{bmatrix} 0 \\ \mu_h^{jAn} D_{b_h}\tilde{a}_h^{j^A}(b_h^n, p^n, q^n, e_h^n, \tau^n) \end{bmatrix} = \begin{bmatrix} \eta^{Bn} \\ \eta^{An} \end{bmatrix}$$

Then $\mu_h^{J^A n} = \eta^{nA}\left[D_{b_h}\tilde{a}_h^{J^A}(b_h^n, p^n, q^n, e_h^n, \tau^n)\right]^{-1}$ follows and $\mu_h^{J^A n} \to \widetilde{\mu}_h^{J^A}$.
Case 2. $\tau^n \to \widetilde{\tau} \in (0, \overline{\tau})$.
Similarly to Case 1, let $\{J_h^A, J_h^B\}$ be a partition of the set of index J_h such that

$$J_h^A = \left\{j \in J_h : \ a_h^j\left(\psi(\widetilde{\tau})\tilde{b}_h, \tilde{p}, \tilde{q}, \tilde{e}_h\right) = 0\right\} \quad \text{and} \quad J_h^B = \left\{j \in J_h : \ a_h^{j*}\left(\psi(\widetilde{\tau})\tilde{b}_h, \tilde{p}, \tilde{q}, \tilde{e}_h\right) > 0\right\}.$$

Then everything goes as in Case 1. Observe that the derivative of a_h with respect to β_h in fact computed at $\psi(\widetilde{\tau})\tilde{b}_h$ - see equation (5.h3).
Case 3. $\tau^n \to \widetilde{\tau} = \overline{\tau}$.
Again, similarly to Case 1, let $\{J_h^A, J_h^B\}$ be a partition of the set of index J_h such that

$$J_h^A = \left\{j \in J_h : \ a_h^j(0, \tilde{p}, \tilde{q}, \tilde{e}_h) = 0\right\} \quad \text{and} \quad J_h^B = \left\{j \in J_h : \ a_h^{j*}(0, \tilde{p}, \tilde{q}, \tilde{e}_h) > 0\right\},$$

and again everything goes as in Case 1.
Case 4 and 5. $\tau^n \to \widetilde{\tau} \in (\overline{\tau}, 1]$.
In this case

$$a_h^j(.) = a_h^j(0, \tilde{p}, \tilde{q}, \tilde{e}_h) + \gamma(\widetilde{\tau}) > 0$$

since, from Assumption 4.iii, $a_h^j(0, \tilde{p}, \tilde{q}, \tilde{e}_h) \geq 0$ and, by construction, $\gamma(\widetilde{\tau}) > 0$.
When using the same argument presented in Case 1, we get that $\mu^n \to 0$. ■

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