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Abstract

The aim of this paper is to provide some first results regarding to the extension to the vector valued case of the b-invex functions. Some definitions will be given for both the nonsmooth case and the differentiable case; the inclusion relatioships among the introduced families of functions will be studied and some results regarding to vector valued optimization will be stated.

Keywords Generalized Convexity, Generalized Invexity, Vector Optimization, Multiobjective Programming.
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1 Introduction

Several classes of functions have been studied in these last years in order to generalize the concept of convex functions, the very well known classes of *generalized convex functions* deeply studied in the scalar case [1] and with many recent results also in the vector case [9, 10, 12, 13, 14, 15, 17, 18], such as the classes of *generalized invex functions* having many results in the scalar case and with some interesting recent results also in the vector case

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[7, 22, 23, 25, 27, 28, 36, 37, 38, 39, 43, 47, 49, 45, 50, 52, 53, 64], the classes of *generalized b-vex and b-invex functions* so far studied only in the scalar case [2, 3, 4, 5, 6, 57].

The aim of this paper is to present some results regarding to the extension to the vector case of the generalized b-invex concepts, providing also some results related to the use of these functions in multiobjective programming.

As it is known, the invexity concept is related, in the differentiable case, to scalar functions verifying for any couple of distinct feasible points x and y the following inequality:

$$f(y) \geq f(x) + \nabla f(x)^T \eta(x,y)$$

where $\eta(x,y)$ is a function generalizing the difference $y - x$ which appears in the differentiable convex functions. These invex functions trivially verify also the two following implications:

$$\begin{array}{lcl} f(y) \leq f(x) & \Rightarrow & \nabla f(x)^T \eta(x,y) \leq 0 \\ f(y) < f(x) & \Rightarrow & \nabla f(x)^T \eta(x,y) < 0 \end{array}$$

thus giving the chance to define the concepts of *quasinvexity* and *pseudoinvexity* (with the former and latter implication, respectively).

In the early literature, by means of the above conditions, a function has been said to be invex or generalized invex *if there exists a function* $\eta(x,y)$ *such that the corresponding property is verified.* Two main problems arise with this approach; the first one is that the behaviour of the functions greatly changes using different parameter functions $\eta(x,y)$, so that it may seem unproper to group all of them in the same class; the second one is that with this approach all the real functions are quasinvex ([1]), thus making useless such a definition, and the class of invex functions coincides with the one of pseudoinvex functions ([2]), making unuseful the definition of one of these two classes.

For this reason, in the recent literature, the generalized invex behaviour of a function has been linked with the particular $\eta(x,y)$ function used to verify the definition, so that the functions are classified to be η-invex, η-quasinvex and η-pseudoinvex, specifying in the name of the class the particular function $\eta(x,y)$ used.

[1] Just using the following $\eta(x,y)$ function, where v is any vector:

$$\eta(x,y) = \begin{cases} v & \text{if } \nabla f(x) = 0 \\ \frac{[f(y)-f(x)]\nabla f(x)}{\nabla f(x)^T \nabla f(x)} & \text{if } \nabla f(x) \neq 0 \end{cases}$$

we can easily see that any differentiable real function verify the quasinvexity property.

[2] Both the invex and pseudoinvex functions can be characterized, with this described approach, as the ones such that every critical point is also a global minimum.

In this paper we will use this last described approach, so that from now on the behaviour of a function will be studied with respect of the particular function $\eta(x, y)$ specified in the definition.

The same approach will be used also with respect to the b-vexity property, that is to say that such a property will be verified with respect to the particular function b specified in the definition.

In order to avoid misunderstandings, let us now remind the definitions of scalar generalized b-invex functions in both the nonsmooth case and the differentiable case. With this aim, the following preliminary definition of η-invex set is needed.

Definition 1.1 A set $A \subseteq \Re^n$ is said to be η-invex, with $\eta : (A \times A) \to \Re^n$, if the following implication holds:

$$x, y \in A,\ x \neq y \implies (x + \lambda\eta(x, y)) \in A \ \ \forall \lambda \in (0, 1)$$

The set A is said to be η_λ-invex, with $\eta_\lambda : (A \times A \times [0, 1]) \to \Re^n$, if the following implication holds:

$$x, y \in A,\ x \neq y \implies (x + \lambda\eta_\lambda(x, y, \lambda)) \in A \ \ \forall \lambda \in (0, 1).$$

Remark 1.1 The concept of η-invex set is the one used in the literature in order to define the nonsmooth generalized η-preinvex functions; such a concept analyze the behaviour of the function on the segment $[x, x+\eta(x, y)]$ which is necessarily required to be feasible. This concept can be generalized allowing the possibility to study the function not only on a straight segment but also on a curve; with this aim we have introduced the concept of η_λ-invex set which guarantees the feasibility of a curve with extremum points in x and $x + \eta_\lambda(x, y, 1)$. Note that an η-invex set is an η_λ-invex set where η_λ is independent to λ, that is to say that $\eta_\lambda(x, y, \lambda) = \eta(x, y)\ \forall \lambda \in [0, 1]$.

Note that if $\eta_\lambda(x, y, \lambda) = \eta(x, y) = y - x$ the previous definitions coincide with the convexity of A.

By means of the concept of η-invex set, the following classes of scalar generalized invex functions and generalized b-invex functions have been defined in the literature (see for example [2, 3, 4, 5, 6, 57]). Note that, by means of the chosen approach, we will specify in the definitions the particular functions η and b which will be used to verify the corresponding properties. This approach is again fundamental in order to avoid trivial cases, as it has been studied in [26] ([3]).

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[3]In [26] it has been proved that any scalar η-pre-quasinvex function is also $(\bar{b}, \eta)$-preinvex with

$$\bar{b}(x, y) = \begin{cases} 0 & \text{if } f(y) \leq f(x) \\ 1 & \text{if } f(y) > f(x) \end{cases}$$

and hence the classes of η-pre-quasinvex and (b_λ, η)-preinvex functions coincide if (b_λ, η)-

In the nonsmooth case, a scalar function $f : A \to \Re$, with $A \subset \Re^n$ η-invex set where $\eta : (A \times A) \to \Re^n$, is said to be:

i) [strictly] (b_λ, η)-preinvex if the following condition holds $\forall x, y \in A$, $x \neq y$, $\forall \lambda \in (0,1)$:

$$f(x + \lambda\eta(x,y)) - f(x) \leq \lambda b_\lambda(x,y,\lambda)(f(y) - f(x)) \quad [<]$$

ii) (b_λ, η)-pre-quasinvex if the following condition holds $\forall x, y \in A$, $x \neq y$, $\forall \lambda \in (0,1)$:

$$f(y) \leq f(x) \quad \Rightarrow \quad b_\lambda(x,y,\lambda)(f(x + \lambda\eta(x,y)) - f(x)) \leq 0$$

iii) (b_λ, η)-pre-pseudoinvex if the following condition holds $\forall x, y \in A$, $x \neq y$:

$$f(y) < f(x) \quad \Rightarrow \quad \begin{array}{c} \exists \xi_{x,y} < 0 \text{ such that } \forall \lambda \in (0,1) \\ b_\lambda(x,y,\lambda)(f(x + \lambda\eta(x,y)) - f(x)) \leq \lambda(1-\lambda)\xi_{x,y} \end{array}$$

where $b_\lambda : (A \times A \times (0,1)) \to \Re_+$ is a nonnegative scalar function not identically equal to 0, so that $b_\lambda(x,y,\lambda) \geq 0$ $\forall x, y \in A$, $x \neq y$, $\forall \lambda \in (0,1)$.

In the differentiable case, a scalar function $f : A \to \Re$, with $A \subset \Re^n$ η-invex set where $\eta : (A \times A) \to \Re^n$, is said to be:

iv) [strictly] (b, η)-invex if the following condition holds $\forall x, y \in A$, $x \neq y$:

$$\nabla f(x)^T \eta(x,y) \leq b(x,y)(f(y) - f(x)) \quad [<]$$

v) [strictly] (b, η)-pseudoinvex if the following condition holds $\forall x, y \in A$, $x \neq y$:

$$b(x,y)(f(y) - f(x)) < 0 \quad [\leq] \quad \Rightarrow \quad \nabla f(x)^T \eta(x,y) < 0$$

vi) (b, η)-quasinvex if the following condition holds $\forall x, y \in A$, $x \neq y$:

$$f(y) \leq f(x) \quad \Rightarrow \quad b(x,y)\nabla f(x)^T \eta(x,y) \leq 0$$

where $b : (A \times A) \to \Re_+$, is a nonnegative scalar function not identically equal to 0.

As it can be easily seen observing the above definitions, the basic idea of scalar b-invexity is to generalize the convexity concept by means of the use of a function $\eta(x,y)$ instead of $(y - x)$ (invexity approach) and adding a multiplier function $b_\lambda(x,y,\lambda)$ or $b(x,y)$ somewhere in the definition (b-vexity

preinvexity is defined requiring that *there exists a function $b_\lambda(x,y,\lambda)$ such that the corresponding property is verified.* Note that the property that any η-pre-quasinvex function is also $(\bar{b}, \eta)$-preinvex is based on the total ordering given in $\Re$ by the "$\leq$" binary relation; the same property will not hold in $\Re^n$ when a partial ordering induced by a closed convex cone $C \subset \Re^m$ will be used.

approach). There is no need to point out that when $b_\lambda(x,y,\lambda) = b(x,y) = 1$ the previous functions coincide with the scalar generalized invex functions, that when $\eta(x,y) = y - x$ the previous functions coincide with the scalar generalized b-vex functions, and that when $b_\lambda(x,y,\lambda) = b(x,y) = 1$ and $\eta(x,y) = y - x$ we obtain the very well known generalized convex functions.

The fact that the generalized b-invex scalar functions extend the concept of invexity and convexity is completed noticing that every generalized invex or generalized convex function is also generalized b-invex while the converse is not true; the classical way to find counter-examples is to use functions b_λ or b such that $b_\lambda(x,y,\lambda) = 0$ or $b(x,y) = 0$ for some x, y and λ (see for example [3]).

2 Nondifferentiable case

In this section we are going to extend to the vector case the classes of nonsmooth scalar generalized b-invex functions.

A very easy way to carry on such an extension, is to require for a vector valued function $f = (f_1, \ldots, f_m)$ the generalized b-invexity of every single component f_i; in this way, by means for example of the definition of (b_λ, η)-preinvexity, we obtain the following condition:

$$f_i(x + \lambda\eta(x,y)) - f_i(x) \leq \lambda b_\lambda(x,y,\lambda)(f_i(y) - f_i(x)) \quad \forall i = 1, \ldots, m$$

A further step is to allow the different components f_i to verify the b-vexity property with respect to different functions $b^{(i)}$, thus obtaining something like the next formula:

$$f_i(x + \lambda\eta(x,y)) - f_i(x) \leq \lambda b_\lambda^{(i)}(x,y,\lambda)(f_i(y) - f_i(x)) \quad \forall i = 1, \ldots, m$$

By means of the Paretian cone $C = \Re_+^m = \{v \in \Re^m : v \underline{\geq} 0\}$, the previous condition can be expressed in the following vector form:

$$f(x + \lambda\eta(x,y)) - f(x) - \lambda B_\lambda(x,y,\lambda)(f(y) - f(x)) \in -\Re_+^m$$

where $B_\lambda(x,y,\lambda)$ is a diagonal matrix function $B_\lambda : (A \times A \times (0,1)) \to \Re_+^{m\times m}$ such that $B_\lambda^{ij}(x,y,\lambda) = 0$ for $i \neq j$ and $B_\lambda^{ii}(x,y,\lambda) = b_\lambda^{(i)}(x,y,\lambda) \geq 0 \ \forall i$.

Using the same approach starting from the definition of (b_λ, η)-pre-quasinvexity, we obtain the following vector condition:

$$f(y) - f(x) \in -\Re_+^m \quad \Rightarrow \quad B_\lambda(x,y,\lambda)(f(x + \lambda\eta(x,y)) - f(x)) \in -\Re_+^m$$

which shows a sort of asimmetry with respect to the previously obtained one. In other words, using the preinvexity definition we obtain a condition where the diagonal matrix function B_λ multiplies $(f(y) - f(x))$, while using the pre-quasinvexity definition the matrix function B_λ multiplies $(f(x + \lambda\eta(x,y)) - f(x))$. This lack of simmetry is caused by the definition given in the literature for preinvexity and pre-quasinvexity; in the vector case we can solve the problem by means of the use of two diagonal matrix functions, say B_λ and D_λ, multiplying $(f(y) - f(x))$ and $(f(x + \lambda\eta(x,y)) - f(x))$ respectively. We can then obtain the two following properties:

$$D_\lambda(x,y,\lambda)(f(x+\lambda\eta(x,y)) - f(x)) - \lambda B_\lambda(x,y,\lambda)(f(y) - f(x)) \in -\Re^m_+$$
$$B_\lambda(x,y,\lambda)(f(y) - f(x)) \in -\Re^m_+ \;\Rightarrow\; D_\lambda(x,y,\lambda)(f(x+\lambda\eta(x,y)) - f(x)) \in -\Re^m_+$$

Some more general conditions can be obtained allowing the function η to be dependent to λ and using any closed convex pointed cone C with nonempty interior instead of just the Paretian cone $\Re^m_+$, as it has been already done for the vector valued generalized convex functions (see for example [12, 14]).

Using this approach, the total orderings in $\Re$ given by the binary relations "$\leq$" and "$<$" are "translated" in $\Re^n$ by means of the following partial orderings induced from a closed convex pointed cone $C \subset \Re^m$ having nonempty interior:

$$\begin{array}{lcl} a \leq b & \longleftrightarrow & a \in b - C \\ a < b & \longleftrightarrow & a \in b - C^1 \end{array} \tag{2.1}$$

with C^1 convex cone such that $\text{Int}(C) \subseteq C^1 \subseteq C \setminus \{0\}$.

It can be easily seen that the "$a < b$" relationship can be translated in infinitely many ways, from the stronger one $a \in b - \text{Int}(C)$ to the weaker $a \in b - (C \setminus \{0\})$.

Being C a closed convex pointed cone with nonempty interior then the following useful property holds.

Lemma 2.1 *Let $C \subset \Re^m$ be a closed convex pointed cone with nonempty interior and let C^1 and C^2 be convex cones such that $C^1 = C$ or $\text{Int}(C) \subseteq C^1 \subseteq C \setminus \{0\}$ and $C^2 = C$ or $\text{Int}(C) \subseteq C^2 \subseteq C \setminus \{0\}$.*
If $C^1 \subseteq C^2$ or $C^2 \subseteq C^1$ then:

$$a \in -C^1,\; b \in -C^2 \quad \Rightarrow \quad a + b \in -C^3 \;\; \textit{with}\;\; C^3 = C^1 \cap C^2$$

By means of the described approach, we are now able to give the following definitions of nonsmooth vector valued generalized b-invex functions.

Definition 2.1 Let $A \subset \Re^n$ be an η_λ-invex set, $\eta_\lambda : (A \times A \times [0,1]) \to \Re^n$, let $C \subset \Re^m$ be a closed convex pointed cone with nonempty interior and let $f : A \to \Re^m$. Let us consider also two diagonal matrix functions $B_\lambda, D_\lambda : (A \times A \times (0,1)) \to \Re^{m \times m}_+$, that is to say that $\forall x, y \in A$, $\forall \lambda \in (0,1)$, $\forall i,j \in \{1,\ldots,m\}$ it is $B^{ii}_\lambda(x,y,\lambda) \geq 0$, $D^{ii}_\lambda(x,y,\lambda) \geq 0$ and $B^{ij}_\lambda(x,y,\lambda) = D^{ij}_\lambda(x,y,\lambda) = 0$ for $i \neq j$. Then function f is said to be:

i) $(C^1, B_\lambda, D_\lambda, \eta_\lambda)$*-preinvex*, with C^1 convex cone such that $C^1 = C$ or $\text{Int}(C) \subseteq C^1 \subseteq C \setminus \{0\}$, if $\forall x, y \in A$, $x \neq y$, $\forall \lambda \in (0,1)$ it is:

$$D_\lambda(x,y,\lambda)[f(x+\lambda\eta_\lambda(x,y,\lambda))-f(x)]-\lambda B_\lambda(x,y,\lambda)[f(y)-f(x)]\in -C^1$$

ii) $(C^1,C^2,B_\lambda,D_\lambda,\eta_\lambda)$-*pre-quasinvex*, with C^1 and C^2 convex cones such that $C^1=C$ or $\text{Int}(C)\subseteq C^1\subseteq C\backslash\{0\}$ and $C^2=C$ or $\text{Int}(C)\subseteq C^2\subseteq C\backslash\{0\}$, if $\forall x,y\in A$, $x\neq y$, $\forall\lambda\in(0,1)$ the following implication holds:

$$B_\lambda(x,y,\lambda)[f(y)-f(x)]\in -C^1 \;\Rightarrow\; D_\lambda(x,y,\lambda)[f(x+\lambda\eta_\lambda(x,y,\lambda))-f(x)]\in -C^2$$

iii) $(C^1,C^2,B_\lambda,D_\lambda,\eta_\lambda)$-*pre-pseudoinvex*, with C^1 and C^2 convex cones such that $C^1=C$ or $\text{Int}(C)\subseteq C^1\subseteq C\backslash\{0\}$ and $C^2\subseteq C^1$, $\text{Int}(C)\subseteq C^2\subseteq C\backslash\{0\}$, if $\forall x,y\in A$, $x\neq y$, $\forall\lambda\in(0,1)$ the following implication holds:

$$B_\lambda(x,y,\lambda)[f(y)-f(x)]\in -C^1 \implies$$
$$\exists\xi_{x,y}\in -C^2 \text{ such that}$$
$$D_\lambda(x,y,\lambda)[f(x+\lambda\eta_\lambda(x,y,\lambda))-f(x)]\in\lambda(1-\lambda)\xi_{x,y}-C$$

Remark 2.1 Note that the previous definitions deal with three families of functions, since the cones C^1 and C^2 may be the cone C or any convex cone ([4]) contained in $C\setminus\{0\}$ and containing $\text{Int}(C)$. In this way we cover, with just one notation and simply specifying the cones C^1 and C^2, both the strict definitions and the non strict ones that we have in the scalar case (just remember the correspondences (2.1)).

It is now worth noticing the following particular subclasses of the defined families of functions:
- when $\eta_\lambda(x,y,\lambda)=y-x$ and $B_\lambda=D_\lambda=I_m$, where I_m is the identity matrix $m\times m$, we obtain the vector valued generalized convex functions studied in [12, 14] and named C^1-convex, (C^1,C^2)-quasiconvex and (C^1,C^2)-strictly-pseudoconvex;
- when $B_\lambda=b_\lambda(x,y,\lambda)I_m$ and $D_\lambda=d_\lambda(x,y,\lambda)I_m$, we require all the components f_i of f to verify the b-vexity properties with respect of the same functions b and d;
- when $\eta_\lambda(x,y,\lambda)=y-x$ the previous functions represent the extensions of the generalized b-vex nonsmooth scalar functions studied in [3];
- when $B_\lambda=D_\lambda=I_m$ whe have the extensions of the generalized invex nonsmooth scalar functions.

It is clear that many particular subclasses can be obtained assuming $\eta_\lambda(x,y,\lambda)=y-x$, $B_\lambda=I_m$ or $D_\lambda=I_m$; in order to use an uniform notation, thus avoiding misunderstandings, and since the choosen approach explicitly denote in the name of the classes the functions η_λ, B_λ and D_λ

[4]Note that, in order to have general results, we will consider also open cones and cones without the origin, that is to say sets K such that:

$$k\in K \implies \lambda k\in K\ \forall\lambda>0$$

used to verify the corresponding properties, we choose in these particular cases to delete from the names the corresponding symbol η_λ, B_λ or D_λ.

Definition 2.2 Let us consider the families of functions defined in Definition 2.1. If $\eta_\lambda(x,y,\lambda) = y - x$, $B_\lambda = I_m$ or $D_\lambda = I_m$ then we will not use in the name of the class of functions the symbol η_λ, B_λ or D_λ, respectively.

For example, by means of the above definition:
- a (C^1, D_λ)-preinvex function has $B_\lambda = I_m$ and $\eta_\lambda(x,y,\lambda) = y - x$,
- a $(C^1, C^2, B_\lambda, \eta_\lambda)$-pre-quasinvex function is characterized to have $D_\lambda = I_m$,
- a (C^1, C^2, η_λ)-pre-pseudoinvex function has $B_\lambda = D_\lambda = I_m$.

Some examples to prove that the defined families of functions are not trivial can be found, for the case $B_\lambda = D_\lambda = I_m$ and $\eta_\lambda(x,y,\lambda) = y - x$, in [12, 14].

As it is well known, the generalization to the vector case of convexity concepts based on a partial ordering given by a convex cone is more general, even in the Paretian case (that is when we consider the cone $C = \Re^m_+ = \{y \in \Re^m : y \geq 0\}$), than the componentwise generalized convexity of the single component scalar functions (see for example [12, 14]). As it is shown in the following example, even in the Paretian case there are no relations between the $(C, C, B_\lambda, D_\lambda, \eta_\lambda)$-quasinvexity of the vector valued function and the componentwise generalized b-invexity of the function itself.

Example 2.1 Consider the cone $C = \Re^3_+$, assume $\eta_\lambda(x,y,\lambda) = y - x$ and $B_\lambda = D_\lambda = I_3$, and let $A = \Re$. The function $f : A \to \Re^3$ defined as $f(x) = (-x^2, x^2 + x, -x)$ results to be (C, C^2)-pre-pseudoinvex and also (C, C^2)-pre-quasinvex, since there are no $x, y \in A$, $x \neq y$, such that $f(y) \in f(x) - C$; on the other side, the first component of f is a scalar strictly concave function and hence it is not quasiconvex, that is b-quasinvex with $b = 1$ and $\eta = y - x$.

In the Paretian case there exists anyway a strict relation between the $(C^1, B_\lambda, \eta_\lambda)$-preinvexity (note that we assume $D_\lambda = I_m$) and the componentwise b-invexity of the function, as is pointed out in the following theorem which comes out directly from the definitions. Let us note that this property is already known for generalized convex and generalized invex vector valued functions.

Theorem 2.1 *Let $A \subset \Re^n$ be an η_λ-invex set, $\eta_\lambda : (A \times A \times [0,1]) \to \Re^n$, let $f : A \to \Re^m$, $f(x) = (f_1(x), \ldots, f_m(x))$, and let $C = \Re^m_+$ be the Paretian cone. Then i), ii) and iii) hold:*

i) f is $(C, B_\lambda, \eta_\lambda)$-preinvex if and only if the scalar functions f_i are $(B^{ii}_\lambda, \eta_\lambda)$-preinvex $\forall i \in \{1, \ldots, m\}$;

ii) if the scalar functions f_i are $(B_\lambda^{ii}, \eta_\lambda)$-preinvex $\forall i \in \{1, \ldots, m\}$ and at least one of them is strictly $(B_\lambda^{ii}, \eta_\lambda)$-preinvex, then $\exists C^1$ such that $\mathrm{Int}(C) \subseteq C^1 \subseteq C \setminus \{0\}$, such that f is $(C^1, B_\lambda, \eta_\lambda)$-preinvex;

iii) f is $(\mathrm{Int}(C), B_\lambda, \eta_\lambda)$-preinvex if and only if the scalar functions f_i are strictly $(B_\lambda^{ii}, \eta_\lambda)$-preinvex $\forall i \in \{1, \ldots, m\}$.

Let us now turn our study towards the inclusion relationships among the defined families of functions. Inside of the families, the inclusion relationships are given trivially by means of the inclusions of the used cones; for example, if $C^2 \subset C^3$ then:
- a $(C^2, B_\lambda, D_\lambda, \eta_\lambda)$-preinvex function is also $(C^3, B_\lambda, D_\lambda, \eta_\lambda)$-preinvex,
- a $(C^1, C^2, B_\lambda, D_\lambda, \eta_\lambda)$-pre-quasinvex function is also $(C^1, C^3, B_\lambda, D_\lambda, \eta_\lambda)$-pre-quasinvex,
- a $(C^3, C^1, B_\lambda, D_\lambda, \eta_\lambda)$-pre-quasinvex function is also $(C^2, C^1, B_\lambda, D_\lambda, \eta_\lambda)$-pre-quasinvex,
- a $(C^1, C^2, B_\lambda, D_\lambda, \eta_\lambda)$-pre-pseudoinvex function is also $(C^1, C^3, B_\lambda, D_\lambda, \eta_\lambda)$-pre-pseudoinvex,
- a $(C^3, C^1, B_\lambda, D_\lambda, \eta_\lambda)$-pre-pseudoinvex function is also $(C^2, C^1, B_\lambda, D_\lambda, \eta_\lambda)$-pre-pseudoinvex.

Some inclusion relationships among different families of functions are given in the next theorem. Note that, just as it happens in the generalized convex case, we will prove that a preinvex function is also pseudoinvex and that a pseudoinvex function is also quasinvex.

Theorem 2.2 *Let $A \subset \Re^n$ be an η_λ-invex set, $\eta_\lambda : (A \times A \times [0, 1]) \to \Re^n$, let $C \subset \Re^m$ be a closed convex pointed cone with nonempty interior and let $f : A \to \Re^m$.*

i) If f is $(C, B_\lambda, D_\lambda, \eta_\lambda)$-preinvex then it is also $(C^1, C^1, B_\lambda, D_\lambda, \eta_\lambda)$-pre-pseudoinvex, with $\mathrm{Int}(C) \subseteq C^1 \subseteq C \setminus \{0\}$;

ii) If f is $(\mathrm{Int}(C), B_\lambda, D_\lambda, \eta_\lambda)$-preinvex then it is also $(C, \mathrm{Int}(C), B_\lambda, D_\lambda, \eta_\lambda)$-pre-pseudoinvex;

iii) If f is $(C^1, B_\lambda, D_\lambda, \eta_\lambda)$-preinvex, with $C^1 = C$ or $\mathrm{Int}(C) \subseteq C^1 \subseteq C \setminus \{0\}$, then it is also $(C^2, C^3, B_\lambda, D_\lambda, \eta_\lambda)$-pre-quasinvex with $C^2 = C$ or $\mathrm{Int}(C) \subseteq C^2 \subseteq C \setminus \{0\}$, $C^1 \subseteq C^2$ or $C^2 \subseteq C^1$, and $C^3 = C^1 \cap C^2$;

iv) If f is $(C^1, C^2, B_\lambda, D_\lambda, \eta_\lambda)$-pre-pseudoinvex, with $C^1 = C$ or $\mathrm{Int}(C) \subseteq C^1 \subseteq C \setminus \{0\}$ and with $C^2 \subseteq C^1$, $\mathrm{Int}(C) \subseteq C^2 \subseteq C \setminus \{0\}$, then it is also $(C^1, C^2, B_\lambda, D_\lambda, \eta_\lambda)$-pre-quasinvex.

Proof *i)* We prove the result by contradiction. Suppose that f is not $(C^1, C^1, B_\lambda, D_\lambda, \eta_\lambda)$-pre-pseudoinvex, so that $\exists x, y \in A$, $x \neq y$, $\exists \lambda \in (0, 1)$

such that $B_\lambda(x, y, \lambda)[f(y) - f(x)] \in -C^1$ and $\forall \xi_{x,y} \in -C^1$ it is

$$D_\lambda(x, y, \lambda)[f(x + \lambda\eta_\lambda(x, y, \lambda)) - f(x)] \notin \lambda(1 - \lambda)\xi_{x,y} - C$$

Assuming $\xi_{x,y} = \frac{1}{1-\lambda}B_\lambda(x, y, \lambda)[f(y) - f(x)] \in -C^1$ we then have, being C a convex pointed cone,

$$D_\lambda(x, y, \lambda)[f(x + \lambda\eta_\lambda(x, y, \lambda)) - f(x)] \notin \lambda B_\lambda(x, y, \lambda)[f(y) - f(x)] - C$$

so that f is not $(C, B_\lambda, D_\lambda, \eta_\lambda)$-preinvex which is a contradiction.

ii) We prove the result by contradiction. Suppose that function f is not $(C, \text{Int}(C), B_\lambda, D_\lambda, \eta_\lambda)$-pre-pseudoinvex, so that $\exists x, y \in A$, $x \neq y$, $\exists \lambda \in (0, 1)$ such that $B_\lambda(x, y, \lambda)[f(y) - f(x)] \in -C$ and $\forall \xi_{x,y} \in -\text{Int}(C)$ it is

$$D_\lambda(x, y, \lambda)[f(x + \lambda\eta_\lambda(x, y, \lambda)) - f(x)] \notin \lambda(1 - \lambda)\xi_{x,y} - C$$

Assuming $k \in \text{Int}(C)$ and $\xi_{x,y} = \frac{1}{1-\lambda}B_\lambda(x, y, \lambda)[f(y) - f(x)] - \frac{1}{\lambda(1-\lambda)}\frac{1}{n}k \in -\text{Int}(C)$, we then have, being C a convex pointed cone,

$$D_\lambda(x, y, \lambda)[f(x + \lambda\eta_\lambda(x, y, \lambda)) - f(x)] \notin \lambda B_\lambda(x, y, \lambda)[f(y) - f(x)] - \frac{1}{n}k - C$$

Approaching n to $+\infty$ we then have:

$$D_\lambda(x, y, \lambda)[f(x + \lambda\eta_\lambda(x, y, \lambda)) - f(x)] \notin \lambda B_\lambda(x, y, \lambda)[f(y) - f(x)] - \text{Int}(C)$$

so that f is not $(\text{Int}(C), B_\lambda, D_\lambda, \eta_\lambda)$-preinvex which is a contradiction.

iii),iv) The thesis follow directly from the definitions and Lemma 2.1, being C a convex pointed cone and being $\lambda(1 - \lambda)\xi_{x,y} \in C^2$. □

Remark 2.2 Note that specifying the results of the previous theorem in the scalar case with $b = d = 1$ (that is without the b-vexity properties) and eventually with $\eta_\lambda = y - x$ (that is without the invexity property), we have that result *i)* shows that every preinvex [convex] function is also pre-pseudoinvex [pseudoconvex], result *ii)* shows that every strictly preinvex [strictly convex] function is also strictly pre-pseudoinvex [strictly pseudoconvex], and result *iii)* shows that every pre-pseudoinvex [strictly pre-pseudoinvex, pseudoconvex, strictly pseudoconvex] is also pre-quasinvex [strictly pre-quasinvex, quasiconvex, strictly quasiconvex, respectively].

The following theorem provides some more inclusion relationships which point out that under some particular conditions, the general classes of vector valued generalized b-invex functions coincide with some of their subclasses.

Theorem 2.3 *Let $A \subset \Re^n$ be an η_λ-invex set, $\eta_\lambda : (A \times A \times [0, 1]) \to \Re^n$, let $C \subset \Re^m$ be a closed convex pointed cone with nonempty interior and let*

$f : A \to \Re^m$. *Then the following results hold:*

i) if f is a $(C^1, C^2, B_\lambda, D_\lambda, \eta_\lambda)$-pre-quasinvex function [$(C^1, C^2, B_\lambda, \eta_\lambda)$-pre-quasinvex, $(C^1, C^2, B_\lambda, D_\lambda, \eta_\lambda)$-pre-pseudoinvex, $(C^1, C^2, B_\lambda, \eta_\lambda)$-pre-pseudoinvex] such that the following condition holds:

$$f(y) - f(x) \in -C^1 \quad \Rightarrow \quad B_\lambda(x, y, \lambda)[f(y) - f(x)] \in -C^1 \tag{2.2}$$

then it is also a $(C^1, C^2, D_\lambda, \eta_\lambda)$-pre-quasinvex function [(C^1, C^2, η_λ)-pre-quasinvex, $(C^1, C^2, D_\lambda, \eta_\lambda)$-pre-pseudoinvex, (C^1, C^2, η_λ)-pre-pseudoinvex, respectively];

ii) if f is a $(C^1, C^2, B_\lambda, D_\lambda, \eta_\lambda)$-pre-quasinvex function [$(C^1, C^2, D_\lambda, \eta_\lambda)$-pre-quasinvex] such that the following condition holds:

$$\begin{aligned} D_\lambda(x, y, \lambda)[f(x + \lambda\eta_\lambda(x, y, \lambda)) - f(x)] \in -C^2 \quad \Longrightarrow \\ f(x + \lambda\eta_\lambda(x, y, \lambda)) - f(x) \in -C^2 \end{aligned} \tag{2.3}$$

then it is also a $(C^1, C^2, B_\lambda, \eta_\lambda)$-pre-quasinvex function [(C^1, C^2, η_λ)-pre-quasinvex];

iii) if f is a $(C^1, C^2, B_\lambda, D_\lambda, \eta_\lambda)$-pre-pseudoinvex function [$(C^1, C^2, D_\lambda, \eta_\lambda)$-pre-pseudoinvex] such that the following condition holds:

$$\begin{aligned} D_\lambda(x, y, \lambda)[f(x + \lambda\eta_\lambda(x, y, \lambda)) - f(x)] \in \lambda(1-\lambda)\xi(x, y) - C \quad \Longrightarrow \\ f(x + \lambda\eta_\lambda(x, y, \lambda)) - f(x) \in \lambda(1-\lambda)\xi(x, y) - C \end{aligned} \tag{2.4}$$

then it is also a $(C^1, C^2, B_\lambda, \eta_\lambda)$-pre-pseudoinvex function [(C^1, C^2, η_λ)-pre-pseudoinvex].

Proof The thesis follows directly applying sequentially condition (2.2) (if the case), the definitions of the generalized b-invex vector valued functions, and then (if the case) condition (2.3) or condition (2.4). □

Remark 2.3 Note that conditions (2.2), (2.3) and (2.4) are trivially verified if, for example, $B_\lambda(x, y, \lambda) = b_\lambda(x, y, \lambda)I_m$ and $D_\lambda(x, y, \lambda) = d_\lambda(x, y, \lambda)I_m$, with $b_\lambda(x, y, \lambda)$ and $d_\lambda(x, y, \lambda)$ positive real valued scalar functions.

Note also that these conditions may not be verified if $B_\lambda^{ii}(x, y, \lambda)$ and $D_\lambda^{ii}(x, y, \lambda)$ are positive real valued scalar functions $\forall i \in \{1, \ldots, m\}$ but there exists i and j such that $B_\lambda^{ii}(x, y, \lambda) \neq B_\lambda^{jj}(x, y, \lambda)$ or $D_\lambda^{ii}(x, y, \lambda) \neq D_\lambda^{jj}(x, y, \lambda)$. Note finally that condition (2.2) does not hold if $\text{Int}(C) \subseteq C^1 \subseteq C \setminus \{0\}$ and $\exists x, y, \lambda$ such that $B_\lambda(x, y, \lambda) = 0$ or if $C^1 = \text{Int}(C)$ and $\exists x, y, \lambda, i$ such that $B_\lambda^{ii}(x, y, \lambda) = 0$.

As it is well known, the finite sum of scalar convex functions is a convex function too, and if at least one of the added functions is strictly convex then the sum is strictly convex too. The following theorem shows that this property holds also in the vector case, in other words we will prove that the sum of $(C, B_\lambda, D_\lambda, \eta_\lambda)$-preinvex functions is $(C, B_\lambda, D_\lambda, \eta_\lambda)$-preinvex too.

Theorem 2.4 *Let $A \subset \Re^n$ be an η_λ-invex set, $\eta_\lambda : (A \times A \times (0,1)) \to \Re^n$, let $C \subset \Re^m$ be a closed convex pointed cone with nonempty interior and let $f_i : A \to \Re^m$, $i = 1, \ldots, q$, be q vector valued functions. Consider also q nonnegative real values $\alpha_i \geq 0$ and the function $g(x) = \sum_{i=1}^{q} \alpha_i f_i(x)$. If all the functions $f_i(x)$, $i = 1, \ldots, q$, are $(C, B_\lambda, D_\lambda, \eta_\lambda)$-preinvex then:*

i) $g(x)$ is a $(C, B_\lambda, D_\lambda, \eta_\lambda)$-preinvex function,

ii) if $\exists j \in \{1, \ldots, q\}$ such that $\alpha_j > 0$ and $f_j(x)$ is $(C^1, B_\lambda, D_\lambda, \eta_\lambda)$-preinvex, with $\mathrm{Int}(C) \subseteq C^1 \subseteq C \setminus \{0\}$, then $g(x)$ is a $(C^1, B_\lambda, D_\lambda, \eta_\lambda)$-preinvex function.

Proof The thesis follows directly from the hypothesis, being C a convex pointed cone, since it results:

$$D_\lambda(x,y,\lambda)[g(x + \lambda\eta_\lambda(x,y,\lambda)) - g(x)] - \lambda B_\lambda(x,y,\lambda)[g(y) - g(x)] =$$

$$= \sum_{i=1}^{q} \alpha_i \left(D_\lambda(x,y,\lambda)[f_i(x + \lambda\eta_\lambda(x,y,\lambda)) - f_i(x)] - \lambda B_\lambda(x,y,\lambda)[f_i(y) - f_i(x)]\right)$$

□

3 Differentiable case

By means of the same approach described in the previous section, we are now able to define some families of differentiable vector valued generalized b-invex functions. Note that in the names we will use the symbols B, D and η instead of B_λ, D_λ and η_λ since in the differentiable case these functions are required to be independent to λ.

Definition 3.1 Let $A \subset \Re^n$ be an open η-invex set, $\eta : (A \times A) \to \Re^n$, let $C \subset \Re^m$ be a closed convex pointed cone with nonempty interior and let $f : A \to \Re^m$ be a differentiable function. Let us consider also two diagonal matrix functions $B, D : (A \times A) \to \Re_+^{m \times m}$, that is to say that $\forall x, y \in A$, $\forall i, j \in \{1, \ldots, m\}$ it is $B_{ii}(x,y) \geq 0$, $D_{ii}(x,y) \geq 0$ and $B_{ij}(x,y) = D_{ij}(x,y) = 0$ for $i \neq j$. Then function f is said to be:

i) (C^1, B, D, η)-*invex*, with C^1 convex cone such that $C^1 = C$ or $\mathrm{Int}(C) \subseteq C^1 \subseteq C \setminus \{0\}$, if $\forall x, y \in A$, $x \neq y$, it is:

$$D(x,y)J_f(x)\eta(x,y) - B(x,y)[f(y) - f(x)] \in -C^1$$

ii) (C^1, B, D, η)*-quasinvex,* with C^1 convex cone such that $C^1 = C$ or $\text{Int}(C) \subseteq C^1 \subseteq C \setminus \{0\}$, if $\forall x, y \in A$, $x \neq y$, it is:

$$B(x,y)[f(y) - f(x)] \in -C^1 \;\Rightarrow\; D(x,y)J_f(x)\eta(x,y) \in -C$$

iii) (C^1, C^2, B, D, η)*-pseudoinvex,* with C^1 and C^2 convex cones such that $C^1 = C$ or $\text{Int}(C) \subseteq C^1 \subseteq C \setminus \{0\}$ and $C^2 \subseteq C^1$, $\text{Int}(C) \subseteq C^2 \subseteq C \setminus \{0\}$, if $\forall x, y \in A$, $x \neq y$, it is:

$$B(x,y)[f(y) - f(x)] \in -C^1 \;\Rightarrow\; D(x,y)J_f(x)\eta(x,y) \in -C^2$$

Remark 3.1 Note that, analogously to the nondifferentiable case, the previous definitions deal with three families of functions, since the cones C^1 and C^2 may be any convex cone contained in $C \setminus \{0\}$ and containing $\text{Int}(C)$ and C^1 may also coincide with C.

Like in the nonsmooth case, with the used notations we cover, simply specifying the cones C^1 and C^2, both the strict definitions and the non strict ones of the scalar case (just remember the correspondences (2.1)).

Some particular subclasses of the defined families of differentiable functions are the followings:
- when $\eta(x,y) = y - x$ and $B = D = I_m$ we obtain the vector valued differentiable generalized convex functions studied in [12, 14] and named C^1-convex, (C^1, C)-weakly-quasiconvex and (C^1, C^2)-pseudoconvex;
- when $B = b(x,y)I_m$ and $D = d(x,y)I_m$, we require all the components f_i of f to verify the b-vexity properties with respect of the same functions b and d; these functions have been used in [28] in order to state some optimality conditions;
- when $\eta(x,y) = y - x$ the previous functions represent the extensions of the generalized b-vex differentiable scalar functions studied in [3];
- when $B = D = I_m$ whe have the extensions of the generalized invex differentiable scalar functions.

Just like in the nonsmooth case, many particular subclasses can be obtained assuming $\eta(x,y) = y - x$, $B = I_m$ or $D = I_m$; maintaining the same kind of notation we have used so far, we will delete from the names, in these particular cases, the corresponding symbol η, B or D.

Definition 3.2 Let us consider the families of functions defined in Definition 3.1. If $\eta(x,y) = y - x$, $B = I_m$ or $D = I_m$ then we will not use in the name of the class of functions the symbol η, B or D, respectively.

For example, by means of the above definition:
- a (C^1, D, η)-invex function is characterized to have $B = I_m$,

- a (C^1, B)-quasinvex function has $D = I_m$ and $\eta(x, y) = y - x$,
- a (C^1, C^2, η)-pseudoinvex function has $B = D = I_m$.

Some examples to prove that the defined families of functions are not trivial can be found, for the case $B = D = I_m$ and $\eta(x, y) = y - x$, in [12, 14].

Let us now study the inclusion relationships among the defined families of functions. Inside of the families, the inclusion relationships are given by means of the inclusions of the used cones; for example, if $C^2 \subset C^3$ then:
- a (C^2, B, D, η)-invex function is also (C^3, B, D, η)-invex,
- a (C^3, B, D, η)-quasinvex function is also (C^2, B, D, η)-quasinvex,
- a (C^3, C^1, B, D, η)-pseudoinvex function is also (C^2, C^1, B, D, η)-pseudoinvex,
- a (C^1, C^2, B, D, η)-pseudoinvex function is also (C^1, C^3, B, D, η)-pseudoinvex.

Some inclusion relationships among different families of functions are given in the next theorem, which follows directly from the definitions being C a pointed convex cone. Note that, just as it happens in the generalized convex case, we will prove that an invex function is also pseudoinvex and that a pseudoinvex function is also quasinvex.

Theorem 3.1 *Let $A \subset \Re^n$ be an open η-invex set, $\eta : (A \times A) \to \Re^n$, let $C \subset \Re^m$ be a closed convex pointed cone with nonempty interior and let $f : A \to \Re^m$ be a differentiable function.*

i) If f is (C, B, D, η)-invex then it is also (C^1, C^1, B, D, η)-pseudoinvex, with $\mathrm{Int}(C) \subseteq C^1 \subseteq C \setminus \{0\}$;

ii) If f is (C^1, B, D, η)-invex, with $\mathrm{Int}(C) \subseteq C^1 \subseteq C \setminus \{0\}$, *then it is also (C^2, C^3, B, D, η)-pseudoinvex, with $C^2 = C$ or* $\mathrm{Int}(C) \subseteq C^2 \subseteq C \setminus \{0\}$, $C^1 \subseteq C^2$ *or* $C^2 \subseteq C^1$, *and* $C^3 = C^1 \cap C^2$;

iii) If f is (C, B, D, η)-invex, then it is also (C^1, B, D, η)-quasinvex with $C^1 = C$ or $\mathrm{Int}(C) \subseteq C^1 \subseteq C \setminus \{0\}$;

iv) If f is (C^1, C^2, B, D, η)-pseudoinvex, with $C^1 = C$ or $\mathrm{Int}(C) \subseteq C^1 \subseteq C \setminus \{0\}$ *and with* $C^2 \subseteq C^1$, $\mathrm{Int}(C) \subseteq C^2 \subseteq C \setminus \{0\}$, *then it is also (C^1, B, D, η)-quasinvex.*

Remark 3.2 Note that specifying the results of the previous theorem in the scalar case with $b = d = 1$ (that is without the b-vexity properties) and eventually with $\eta(x, y) = y - x$ (that is without the invexity property), we have that result *i)* shows that every invex [convex] function is also pseudoinvex [pseudoconvex], result *ii)* shows that every strictly invex [strictly convex] function is also strictly pseudoinvex [strictly pseudoconvex], and result *iii)* shows that every pseudoinvex [strictly pseudoinvex, pseudoconvex, strictly pseudoconvex] is also quasinvex [strictly quasinvex, quasiconvex, strictly quasiconvex, respectively].

Let us now study the inclusion relationships among the families of differentiable and nondifferentiable functions defined so far. Note that, even in the case $B_\lambda = D_\lambda = B = D = I_m$ and $\eta_\lambda(x,y,\lambda) = \eta(x,y) = y - x$, all the defined families of functions are disjointed (see [12, 14]).

Theorem 3.2 *Let $A \subset \Re^n$ be an open η_λ-invex and η-invex set, $\eta_\lambda : (A \times A \times [0,1]) \to \Re^n$ and $\eta : (A \times A) \to \Re^n$ such that $\eta = \lim_{\lambda \to 0^+} \eta_\lambda$, let $C \subset \Re^m$ be a closed convex pointed cone with nonempty interior and let $f : A \to \Re^m$ be a differentiable function. Let us consider also the diagonal matrix functions $B_\lambda, D_\lambda : (A \times A \times (0,1)) \to \Re_+^{m\times m}$ and $B, D : (A \times A) \to \Re_+^{m \times m}$ such that $B = \lim_{\lambda \to 0^+} B_\lambda$ and $D = \lim_{\lambda\to 0^+} D_\lambda$.*

i) If f is $(C, B_\lambda, D_\lambda, \eta_\lambda)$-preinvex then it is also (C, B, D, η)-invex,

ii) If f is $(C^1, B_\lambda, D_\lambda, \eta_\lambda)$-preinvex, with $\mathrm{Int}(C) \subseteq C^1 \subseteq C \backslash \{0\}$, $D_\lambda^{ii}(x,y,\lambda) > 0$ $\forall x, y, \lambda, i$, and with $\eta_\lambda(x,y,\lambda) = \eta(x,y)$ $\forall \lambda \in (0,1)$ (that is η_λ is independent from λ) then $\forall \tilde{\lambda} \in (0,1)$ it is also $(C^1, \tilde{B}, \tilde{D}, \tilde{\eta})$-invex where:

$$\tilde{B}(x,y) = B(x, x + \tilde{\lambda}\eta(x,y)) D_\lambda^{-1}(x,y,\tilde{\lambda}) B_\lambda(x,y,\tilde{\lambda}),$$
$$\tilde{D}(x,y) = D(x, x+\tilde{\lambda}\eta(x,y)),\ \tilde{\eta}(x,y) = \frac{1}{\tilde{\lambda}}\eta(x, x+\tilde{\lambda}\eta(x,y))$$

iii) If f is $(\mathrm{Int}(C), C^2, B_\lambda, D_\lambda, \eta_\lambda)$-pre-quasinvex, with $C^2 = C$ or $\mathrm{Int}(C) \subseteq C^2 \subseteq C \setminus \{0\}$, then it is also $(\mathrm{Int}(C), B, D, \eta)$-quasinvex,

iv) If f is $(\mathrm{Int}(C), C^2, B_\lambda, D_\lambda, \eta_\lambda)$-pre-pseudoinvex, with $\mathrm{Int}(C) \subseteq C^2 \subseteq C \setminus \{0\}$, then it is also $(\mathrm{Int}(C), C^2, B, D, \eta)$-pseudoinvex.

Proof *i)* Since f is $(C, B_\lambda, D_\lambda, \eta_\lambda)$-preinvex we have that for any $\lambda \in (0,1)$:

$$D_\lambda(x,y,\lambda)[\tfrac{1}{\lambda}(f(x+\lambda\eta_\lambda(x,y,\lambda)) - f(x))] - B_\lambda(x,y,\lambda)[f(y) - f(x)] \in -C$$

so that approaching λ to 0^+ we have the thesis, being C a closed cone.

ii) Let now be $\mathrm{Int}(C) \subseteq C^1 \subseteq C\backslash\{0\}$; since f is $(C^1, B_\lambda, D_\lambda, \eta_\lambda)$-preinvex then it is also $(C, B_\lambda, D_\lambda, \eta_\lambda)$-preinvex so that for *i)* it is (C, B, D, η)-invex and hence, fixed $\tilde{\lambda} \in (0,1)$ and defined $z = x + \tilde{\lambda}\eta(x,y)$:

$$D(x,z)J_f(x)\eta(x,z) - B(x,z)[f(z) - f(x)] \in -C;$$

since f is $(C^1, B_\lambda, D_\lambda, \eta_\lambda)$-preinvex we have also that:

$$D_\lambda(x,y,\tilde{\lambda})[f(z) - f(x)] \in \tilde{\lambda} B_\lambda(x,y,\tilde{\lambda})[f(y) - f(x)] - C^1,$$

and hence, being $\det(D_\lambda) > 0$,

$$[f(z) - f(x)] \in \tilde{\lambda} D_\lambda^{-1}(x,y,\tilde{\lambda}) B_\lambda(x,y,\tilde{\lambda})[f(y) - f(x)] - C^1.$$

The following result then holds, being C a pointed convex cone:

$$D(x,z)J_f(x)\frac{1}{\tilde{\lambda}}\eta(x,z) - B(x,z)D_\lambda^{-1}(x,y,\tilde{\lambda})B_\lambda(x,y,\tilde{\lambda})[f(y)-f(x)] \in -C^1$$

so that the thesis holds, being $\tilde{B}(x,y) = B(x,z)D_\lambda^{-1}(x,y,\tilde{\lambda})B_\lambda(x,y,\tilde{\lambda})$, $\tilde{D}(x,y) = D(x,z)$ and $\tilde{\eta}(x,y) = \frac{1}{\tilde{\lambda}}\eta(x,z)$.

iii) Let $x,y \in A$, $x \neq y$, be such that $B(x,y)[f(y)-f(x)] \in -\text{Int}(C)$; then since $B = \lim_{\lambda\to 0^+} B$ we have, by means of a well known limit theorem, that $\exists\epsilon \in (0,1)$ such that $B_\lambda(x,y,\lambda)[f(y)-f(x)] \in -\text{Int}(C)$ $\forall\lambda \in (0,\epsilon)$. By means of the $(\text{Int}(C), C^2, B_\lambda, D_\lambda, \eta_\lambda)$-pre-quasinvexity of f we then have that $D_\lambda(x,y,\lambda)[\frac{1}{\lambda}(f(x+\lambda\eta_\lambda(x,y,\lambda))-f(x))] \in -C^2$ so that the thesis holds approaching λ to 0^+.

iv) Let $x,y \in A$, $x \neq y$, be such that $B(x,y)[f(y)-f(x)] \in -\text{Int}(C)$; then since $B = \lim_{\lambda\to 0^+} B$ we have, by means of a well known limit theorem, that $\exists\epsilon \in (0,1)$ such that $B_\lambda(x,y,\lambda)[f(y)-f(x)] \in -\text{Int}(C)$ $\forall\lambda \in (0,\epsilon)$. By means of the $(\text{Int}(C), C^2, B_\lambda, D_\lambda, \eta_\lambda)$-pre-pseudoinvexity of f we then have that $D_\lambda(x,y,\lambda)[\frac{1}{\lambda}(f(x+\lambda\eta_\lambda(x,y,\lambda))-f(x))] \in (1-\lambda)\xi(x,y) - C$ so that approaching λ to 0^+ it results $D(x,y)J_f(x)\eta(x,y) \in \xi(x,y) - C$ with $\xi(x,y) \in -C^2$, so that the thesis holds. □

Some more inclusion relationships can be stated in the particular case of matrices $B_\lambda(x,y,\lambda)$ and $D_\lambda(x,y,\lambda)$ independent to λ.

Theorem 3.3 *Let $A \subset \Re^n$ be an open η_λ-invex and η-invex set, $\eta_\lambda : (A \times A \times [0,1]) \to \Re^n$ and $\eta : (A \times A) \to \Re^n$ such that $\eta = \lim_{\lambda\to 0^+} \eta_\lambda$, let $C \subset \Re^m$ be a closed convex pointed cone with nonempty interior and let $f : A \to \Re^m$ be a differentiable function. Let us suppose also that the diagonal matrix functions B_λ and D_λ are independent to λ, so that $B_\lambda(x,y,\lambda) = B(x,y)$ and $D_\lambda(x,y,\lambda) = D(x,y)$.*

i) If f is $(C^1, C^2, B_\lambda, D_\lambda, \eta_\lambda)$-pre-quasinvex, with $C^1 = C$ or $\text{Int}(C) \subseteq C^1 \subseteq C \setminus \{0\}$ and with $C^2 = C$ or $\text{Int}(C) \subseteq C^2 \subseteq C \setminus \{0\}$, then it is also (C^1, B, D, η)-quasinvex

ii) If f is $(C^1, C^2, B_\lambda, D_\lambda, \eta_\lambda)$-pre-pseudoinvex, with $C^1 = C$ or $\text{Int}(C) \subseteq C^1 \subseteq C \setminus \{0\}$ and with $C^2 \subseteq C^1$, $\text{Int}(C) \subseteq C^2 \subseteq C \setminus \{0\}$, then it is also (C^1, C^2, B, D, η)-pseudoinvex

Proof *i)* Let $x,y \in A$, $x \neq y$, be such that $B(x,y)[f(y)-f(x)] \in -C^1$; being $B_\lambda = B$, $D_\lambda = D$ and f $(C^1, C^2, B_\lambda, D_\lambda, \eta_\lambda)$-pre-quasinvex then $D(x,y)[\frac{1}{\lambda}(f(x+\lambda\eta_\lambda(x,y,\lambda))-f(x))] \in -C^2$ $\forall\lambda \in (0,1)$ so that the thesis follows approaching λ to 0^+.

ii) Let $x,y \in A$, $x \neq y$, be such that $B(x,y)[f(y)-f(x)] \in -C^1$; being $B_\lambda = B$, $D_\lambda = D$ and f $(C^1, C^2, B_\lambda, D_\lambda, \eta_\lambda)$-pre-pseudoinvex then $D(x,y)[\frac{1}{\lambda}(f(x+\lambda\eta_\lambda(x,y,\lambda))-f(x))] \in (1-\lambda)\xi(x,y) - C$ $\forall\lambda \in (0,1)$ so that the thesis follows approaching λ to 0^+ since $\xi(x,y) \in -C^2$. □

Remark 3.3 Note that specifying the results of the two previous theorems in the scalar case with $b = d = 1$ and with $\eta(x,y) = y - x$, we have that results *i)* and *ii)* of Theorem 3.2 correspond to one implication of the first order characterizations of convex and strictly convex functions, while results *iii)* and *iv)* of Theorem 3.2 and *i)* and *ii)* of Theorem 3.3 correspond to one implication of the first order characterizations of quasiconvex, pseudoconvex and strictly pseudoconvex functions.

Note that it has not been possible to state some first order characterizations for the previously defined nonsmooth families of functions; this behaviour is not strange and is already known, for example, regarding to the vector valued generalized convex functions (see [12, 14]). The following example point out that the previous implications are proper and are not first order characterizations; in Theorem 3.2 we have stated that if f is $(C, B_\lambda, D_\lambda, \eta_\lambda)$-preinvex then it is also (C, B, D, η)-invex, where $B = \lim_{\lambda\to0^+} B$, $D = \lim_{\lambda\to0^+} D$ and $\eta = \lim_{\lambda\to0^+} \eta$, in the following Example 3.1 we will show that the converse is not true in general.

Example 3.1 Assume $\eta_\lambda(x,y,\lambda) = \eta(x,y) = y - x$ and let $f(x) = (x^2, x)$, $D = D_\lambda = I_m$, $B_\lambda(x,y,\lambda) = \begin{bmatrix} 1-\lambda & 0 \\ 0 & 1 \end{bmatrix}$ so that $B = \lim_{\lambda\to0^+} B_\lambda = I_m$. $f(x)$ is (C, B, D, η)-invex since all the components of f are convex and $B = D = I_m$; $f(x)$ is not $(C, B_\lambda, D_\lambda, \eta_\lambda)$-preinvex since for $x = 0$, $y = 1$ and $\lambda = \frac{2}{3}$ we have:

$$f_1(x + \lambda(y-x)) - f_1(x) - \lambda B_{11}(x,y,\lambda)[f_1(y) - f_1(x)] =$$
$$(\tfrac{4}{9}) - 0 - (\tfrac{2}{9})[1-0] = \tfrac{2}{9} > 0.$$

The following theorem shows that under some particular hypothesis it is possible that a (C, B, D, η)-invex function is also $(C, B_\lambda, D_\lambda, \eta_\lambda)$-preinvex.

Theorem 3.4 *Let $A \subset \Re^n$ be an open η-invex set, $\eta : (A \times A) \to \Re^n$, let $C \subset \Re^m$ be a closed convex pointed cone with nonempty interior and let $f : A \to \Re^m$ be a differentiable function. If f is a (C^1, D, η)-invex function, with $C^1 = C$ or $\mathrm{Int}(C) \subseteq C^1 \subseteq C \setminus \{0\}$, verifying the following properties $\forall x, y \in A$, $x \neq y$, $\forall \lambda \in (0,1)$:*

$$D(x,y) = D(x) \tag{3.1}$$
$$D(z)J_f(z)[\lambda\eta(z,y) + (1-\lambda)\eta(z,x)] \in C, \quad z = x + \lambda\eta(x,y) \tag{3.2}$$

then f is also a (C^1, η_λ)-preinvex function with $\eta_\lambda = \eta$.

Proof Let $x, y \in A$, $x \neq y$, and $\lambda \in (0,1)$; being f (C^1, D, η)-invex and being $D(x,y) = D(x)$ we have:

$$D(z)J_f(z)\eta(z,y) - [f(y) - f(z)] \in -C^1$$
$$D(z)J_f(z)\eta(z,x) - [f(x) - f(z)] \in -C^1$$

where $z = x + \lambda\eta(x,y)$; multiplying the first vector inequality by $\lambda > 0$, multiplying the second inequality by $(1-\lambda) > 0$ and adding the two obtained inequalities it results, being C a convex pointed cone:

$$k - [\lambda(f(y) - f(z)) + (1-\lambda)(f(x) - f(z))] \in -C^1$$

where $k = D(z)J_f(z)[\lambda\eta(z,y) + (1-\lambda)\eta(z,x)] \in C$. It then results:

$$[f(x + \lambda\eta(x,y)) - f(x)] - \lambda[f(y) - f(x)] \in -k - C^1$$

and the theorem is proved, being C a convex pointed cone. □

Remark 3.4 Note that if we avoid in the previous theorem the invexity property, that is to say that we assume $\eta(x,y) = y - x$, then we have:

$$[\lambda\eta(z,y) + (1-\lambda)\eta(z,x)] = x - z + \lambda(y - z + z - x) = z - z = 0$$

so that condition (3.2) trivially holds since $0 \in C$; in the case $D = I_m$ we then have that the previous theorem states a first order characterization for vector valued convex functions. This points out the different behaviour of invexity towards convexity; in other words with respect to invex functions, we do not have a first order characterization even in the scalar case, unless some very particular hypothesis are assumed, while with respect to convex functions we have a first order characterization even in the vector case (see [12]).

Analogously to the nondifferentiable case, the following theorem provides some more inclusion relationships, showing conditions which force the defined classes of differentiable functions to coincide with some of their subclasses.

Theorem 3.5 *Let $A \subset \Re^n$ be an open η-invex set, $\eta : (A \times A) \to \Re^n$, let $C \subset \Re^m$ be a closed convex pointed cone with nonempty interior and let $f : A \to \Re^m$ be a differentiable function. Then the following results hold:*

i) if function f is a (C^1, B, D, η)-quasinvex function [(C^1, B, η)-quasinvex, (C^1, C^2, B, D, η)-pseudoinvex, (C^1, C^2, B, η)-pseudoinvex] such that the following condition holds:

$$f(y) - f(x) \in -C^1 \quad \Rightarrow \quad B(x,y)[f(y) - f(x)] \in -C^1 \qquad (3.3)$$

then it is also (C^1, D, η)*-quasinvex [*(C^1, η)*-quasinvex,* (C^1, C^2, D, η)*-pseudoinvex,* (C^1, C^2, η)*-pseudoinvex, respectively];*

ii) if f is a (C^1, B, D, η)*-quasinvex function [*(C^1, D, η)*-quasinvex] such that the following condition holds:*

$$D(x, y)J_f(x)\eta(x, y) \in -C \quad \Rightarrow \quad J_f(x)\eta(x, y) \in -C \tag{3.4}$$

then it is also a (C^1, B, η)*-quasinvex function [*(C^1, η)*-quasinvex];*

iii) if f is a (C^1, C^2, B, D, η)*-pseudoinvex function [*(C^1, C^2, D, η)*-pseudoinvex] such that the following condition holds:*

$$D(x, y)J_f(x)\eta(x, y) \in -C^2 \quad \Rightarrow \quad J_f(x)\eta(x, y) \in -C^2 \tag{3.5}$$

then it is also a (C^1, C^2, B, η)*-pseudoinvex function [*(C^1, C^2, η)*-pseudoinvex].*

The following theorem shows that even in the differentiable case, the finite sum of (C, B, D, η)-invex functions is (C, B, D, η)-invex too, thus generalizing the property of the scalar convex functions.

Theorem 3.6 *Let* $A \subset \Re^n$ *be an open* η*-invex set,* $\eta : (A \times A) \to \Re^n$*, let* $C \subset \Re^m$ *be a closed convex pointed cone with nonempty interior and let* $f_i : A \to \Re^m$*,* $i = 1, \ldots, q$*, be q differentiable functions. Consider also q nonnegative real values* $\alpha_i \geq 0$ *and the function* $g(x) = \sum_{i=1}^{q} \alpha_i f_i(x)$*. If all the functions* $f_i(x)$*,* $i = 1, \ldots, q$*, are* (C, B, D, η)*-invex then:*

i) $g(x)$ *is a* (C, B, D, η)*-invex function,*

ii) if $\exists j \in \{1, \ldots, q\}$ *such that* $\alpha_j > 0$ *and* $f_j(x)$ *is* (C^1, B, D, η)*-invex, with* $\mathrm{Int}(C) \subseteq C^1 \subseteq C \setminus \{0\}$*, then* $g(x)$ *is a* (C^1, B, D, η)*-invex function.*

4 Optimality Conditions

The aim of this section is to show how the defined families of functions may be used in order to extend to the vector case the very well known properties of generalized convex scalar functions regarding to optimality conditions.

The optimality conditions we are going to study will be related to the following multiobjective problems:

$$P_U = \begin{cases} \text{C-min } f(x) \\ x \in A \end{cases} , \quad P_C = \begin{cases} \text{C-min } f(x) \\ g(x) \in -V \\ x \in A \end{cases}$$

where $A \subset \Re^n$, $C \subset \Re^m$ and $V \subset \Re^s$ are closed convex pointed cones with nonempty interior and $f : A \to \Re^m$ and $g : A \to \Re^s$ are vector valued functions. Problem P_U is an unconstrained vector optimization problem and, as usual, a point $x_0 \in A$ will be said to be a C^1*-efficient point* for problem P_U, with $C^1 = C$ or $\text{Int}(C) \subseteq C^1 \subseteq C \setminus \{0\}$, if:

$$\nexists y \in A,\ y \neq x_0, \text{ such that } f(y) \in f(x_0) - C^1,$$

while it will be said to be a *local* C^1*-efficient point* for problem P_U, with $C^1 = C$ or $\text{Int}(C) \subseteq C^1 \subseteq C \setminus \{0\}$, if there exists a suitable neighbourhood I_{x_0} of x_0 such that:

$$\nexists y \in A \cap I_{x_0},\ y \neq x_0, \text{ such that } f(y) \in f(x_0) - C^1.$$

Problem P_C is a constrained vector optimization problem and a feasible point $x_0 \in A$ (that is $g(x_0) \in -V$) will be said to be a C^1*-efficient point* for problem P_C, with $C^1 = C$ or $\text{Int}(C) \subseteq C^1 \subseteq C \setminus \{0\}$, if:

$$\nexists y \in A,\ y \neq x_0, \text{ such that } f(y) \in f(x_0) - C^1 \text{ and } g(y) \in -V.$$

In the following subsections, we will analyze the properties of vector valued b-invex functions regarding to the global optimality of local optima, critical points, and points verifying the Kuhn-Tucker conditions (see also the various papers listed in the References for results using generalized invexity or generalized convexity). Note that generalized invex functions and generalized b-vex functions have been also used in Duality (for such results refer to [4, 5, 6, 23, 28, 63]).

4.1 Global efficiency of local optima

Let us consider Problem P_U and let us assume the set $A \subset \Re^n$ to be η_λ-invex, $\eta_\lambda : (A \times A \times [0,1]) \to \Re^n$.

The scalar quasiconvex functions are known to have the property that every local minima are also global optima; the following theorem generalizes this property to the vector case.

Theorem 4.1 *Let us consider problem P_U with $A \subset \Re^n$ η_λ-invex set, $\eta_\lambda : (A \times A \times [0,1]) \to \Re^n$. If f is a (C^1, C^2, η_λ)-pre-quasinvex function verifying the following property:*

$$\lim_{\lambda \to 0^+} \lambda \eta_\lambda(x, y, \lambda) = 0 \tag{4.1}$$

then every local C^2-efficient point is also a global C^1-efficient one.

Proof We prove the result by contradiction. Suppose that $x_0 \in A$ is a local C^2-efficient point but not a global C^1-efficient one, that is to say that $\exists y \in A$ such that $f(y) \in f(x_0) - C^1$. This along with the (C^1, C^2, η_λ)-pre-quasinvexity of f yelds that $f(x_0 + \lambda\eta(x_0, y, \lambda)) \in f(x_0) - C^2\ \forall \lambda \in (0,1)$ which in turn implies, being $\lim_{\lambda \to 0^+} \lambda\eta(x_0, y, \lambda) = 0$, that x_0 is not a local C^2-efficient point which is a contradiction. □

4.2 Global efficiency of critical points

Let us consider Problem P_U and let us assume $A \subset \Re^n$ to be an η-invex set, $\eta : (A \times A) \to \Re^n$, and f to be a differentiable function.

The following theorem generalizes the very well known property of pseudoconvex functions, which guarantees the global optimality of critical points. With this aim we will denote with $C^+ = \{\alpha \in \Re^m : \alpha^T c \geq 0 \ \forall c \in C\}$ the positive polar cone of C and with $C^{++} = \{\alpha \in \Re^m : \alpha^T c > 0 \ \forall c \in C, \ c \neq 0\}$ the strictly positive polar cone of C (remind that if $\alpha \in C^+$, $\alpha \neq 0$, and $c \in \text{Int}(C)$ then $\alpha^T c > 0$).

Remind that in the vector valued case $x_0 \in A$ is said to be a *critical point* if one of the two following conditions hold:

$$\exists \alpha \in C^{++} \text{ such that } \alpha^T J_f(x_0) = 0$$
$$\exists \alpha \in C^+ \setminus \{0\} \text{ such that } \alpha^T J_f(x_0) = 0$$

Theorem 4.2 *Let us consider problem P_U with f differentiable and $A \subset \Re^n$ open η-invex set, $\eta : (A \times A) \to \Re^n$. Let f be (C^1, C^2, D, η)-pseudoinvex, with $\text{Int}(C) \subseteq C^2 \subseteq C \setminus \{0\}$, such that $D_{ii}(x, y) > 0 \ \forall i = 1, \ldots, m$.*

If $\exists \alpha \in C^{++}$ such that $\alpha^T J_f(x_0) = 0$ and the following condition holds:

$$\alpha \in C^{++} \implies \alpha^T (D(x, y))^{-1} \in C^{++} \tag{4.2}$$

then x_0 is a global C^1-efficient point.

Proof We prove the results by contradiction. Suppose that $x_0 \in A$ is not a global C^1-efficient point, that is to say that $\exists y \in A$ such that $f(y) \in f(x_0) - C^1$. This along with the definition of (C^1, C^2, D, η)-pseudoinvexity yields that $D(x_0, y) J_f(x_0) \eta(x_0, y) \in -C^2$.

Being $C^2 \subseteq C \setminus \{0\}$ and $\alpha \in C^{++}$ and taking into account condition (4.2) we then have that:

$$0 > (\alpha^T (D(x_0, y))^{-1})(D(x_0, y) J_f(x_0) \eta(x_0, y)) = \alpha^T J_f(x_0) \eta(x_0, y)$$

which contradicts $\alpha^T J_f(x_0) = 0$. □

In the same way it is possible to prove the following analogous result.

Theorem 4.3 *Let us consider problem P_U with f differentiable and $A \subset \Re^n$ open η-invex set, $\eta : (A \times A) \to \Re^n$. Let f be $(C^1, \text{Int}(C), D, \eta)$-pseudoinvex, such that $D_{ii}(x, y) > 0 \ \forall i = 1, \ldots, m$.*

If $\exists \alpha \in C^+ \setminus \{0\}$ such that $\alpha^T J_f(x_0) = 0$ and the following condition holds:

$$\alpha \in C^+ \setminus \{0\} \implies \alpha^T (D(x, y))^{-1} \in C^+ \setminus \{0\} \tag{4.3}$$

then x_0 is a global C^1-efficient point.

Remark 4.1 Note, for example, that particular diagonal matrix functions $D(x,y)$ such that $D_{ii}(x,y) > 0 \ \forall i = 1,\ldots,m$ and verifying conditions (4.2) and (4.3) are the ones of the kind $D(x,y) = b(x,y)I_m$, where I_m is the $m \times m$ identity matrix and $b(x,y)$ is a positive scalar function; just note that in this case it results $(D(x,y))^{-1} = (b(x,y)I_m)^{-1} = \frac{1}{b(x,y)}I_m$.

Remark 4.2 Note that the same result of Theorem 4.2 and Theorem 4.3 holds with no need of conditions (4.2) and (4.3) assuming the function f to be (C^1, C^2, η)-pseudoinvex and $(C^1, \text{Int}(C), \eta)$-pseudoinvex, respectively.

4.3 Sufficiency of Kuhn-Tucker conditions

Let us now consider Problem P_C and let us assume $A \subset \Re^n$ to be an η-invex set, $\eta : (A \times A) \to \Re^n$, and f and g to be differentiable functions.

It is very well known that, in a scalar minimization problem, if the objective function f is pseudoconvex and the constraints are quasiconvex then the Kuhn-Tucker conditions are sufficient global optimality conditions. By means of the following theorems, we will extend such a result to the vector case; note that, without loss of generality, the sufficiency of the Kuhn-Tucker conditions will be stated with respect to a feasible point x_0 binding all the constraints, that is to say that $g(x_0) = 0$.

Theorem 4.4 *Let us consider problem P_C with f and g differentiable functions and $A \subset \Re^n$ open η-invex set, $\eta : (A \times A) \to \Re^n$. Assume $x_0 \in A$ to be such that $g(x_0) = 0$ and let f be $(C^1, \text{Int}(C), D_f, \eta)$-pseudoinvex and g be (V, D_g, η)-quasinvex such that $D_f^{ii}(x,y) > 0$ and $D_g^{ii}(x,y) > 0 \ \forall i = 1,\ldots,m$. Suppose that the following condition holds:*

$$\exists \alpha_f \in C^+ \setminus \{0\}, \ \exists \alpha_g \in V^+, \ \textit{such that } \alpha_f^T J_f(x_0) + \alpha_g^T J_g(x_0) = 0$$

as well as the next properties:

$$\alpha_g \in V^+ \implies \alpha_g^T (D_g(x,y))^{-1} \in V^+ \tag{4.4}$$

$$\alpha_f \in C^+ \setminus \{0\} \implies \alpha_f^T (D_f(x,y))^{-1} \in C^+ \setminus \{0\} \tag{4.5}$$

then x_0 is a global C^1-efficient point.

Proof We prove the result by contradiction. Suppose that $\exists y \in A$ such that $f(y) - f(x_0) \in -C^1$ and $g(y) = g(y) - g(x_0) \in -V$; by means of the hypothesis we have that $D_f(x_0,y)J_f(x_0)\eta(x_0,y) \in -\text{Int}(C)$ and $D_g(x_0,y)J_g(x_0)\eta(x_0,y) \in -V$. Being $\alpha_f \in C^+ \setminus \{0\}$ and taking into account the hypothesis we then have that:

$$0 > (\alpha_f^T (D_f(x_0,y))^{-1})(D_f(x_0,y)J_f(x_0)\eta(x_0,y)) = \alpha_f^T J_f(x_0)\eta(x_0,y)$$

$$0 \geq (\alpha_g^T (D_g(x_0,y))^{-1})(D_g(x_0,y)J_g(x_0)\eta(x_0,y)) = \alpha_g^T J_g(x_0)\eta(x_0,y)$$

so that $0 > (\alpha_f^T J_f(x_0) + \alpha_g^T J_g(x_0))\eta(x_0, y)$ which contradicts $\alpha_f^T J_f(x_0) + \alpha_g^T J_g(x_0) = 0$. □

Note the the previous result slightly generalizes the one presented in [28] which is related to the particular case $D_f(x, y) = d_f(x, y)I_m$ and $D_g(x, y) = d_g(x, y)I_s$ with $d_f(x, y) > 0$ and $d_g(x, y) > 0$; note that under these assumptions it is $(D_f(x, y))^{-1} = \frac{1}{d_f(x,y)}I_m$ and $(D_g(x, y))^{-1} = \frac{1}{d_g(x,y)}I_s$ so that conditions (4.4) and (4.5) trivially holds.

Remark 4.3 Note that the same result can be obtained without the use of conditions (4.4) and (4.5) assuming f to be $(C^1, \text{Int}(C), \eta)$-pseudoinvex and g to be (V, η)-quasinvex.

Specializing the previous result for a scalar minimization problem, that is for a problem P_C with a scalar objective function f, we can easily verify that the previous theorem extends the known sufficiency of the Kuhn-Tucker conditions.

Corollary 4.1 *Let us consider problem P_C with f differentiable scalar function, g differentiable vector valued function and $A \subset \Re^n$ open η-invex set, $\eta : (A \times A) \to \Re^n$. Assume $x_0 \in A$ to be such that $g(x_0) = 0$ and let f be [strictly] η-pseudoinvex and g be (V, D_g, η)-quasinvex such that $D_g^{ii}(x, y) > 0$ $\forall i = 1, \ldots, m$. Suppose that the following condition holds:*

$$\exists \alpha_g \in V^+ \text{ such that } \nabla f(x_0) + \alpha_g^T J_g(x_0) = 0$$

as well as the next property:

$$\alpha_g \in V^+ \implies \alpha_g^T (D_g(x, y))^{-1} \in V^+$$

then x_0 is a global [strict] minimum point.

Proof Firstly note that in the scalar case $m = 1$ it is $C = \Re_+$ and $\text{Int}(C) = C \setminus \{0\} = \Re_{++}$, so that a $(C^1, \text{Int}(C), D_f, \eta)$-pseudoinvex function with $D_f^{11}(x, y) > 0$ results to be η-pseudoinvex if $C^1 \neq C$ while it is strictly η-pseudoinvex if $C^1 = C$. We have also that $C^+ = C$ so that $\alpha_f \in C^+ \setminus \{0\}$ simply means that α_f is a positive scalar number; being $D_f^{11}(x, y) > 0$ we then also have that condition (4.5) trivially holds. The thesis then follows directly from Theorem 4.4 assuming $\alpha_f = 1$. □

It is possible to state some different versions of Theorem 4.4 just changing the generalized b-invexity property of the objective function f and the constraints g; these versions are listed in the followings and their proofs are analogous to the one of Theorem 4.4.

Theorem 4.5 *Let us consider problem P_C with f and g differentiable functions and $A \subset \Re^n$ open η-invex set, $\eta : (A \times A) \to \Re^n$. Assume $x_0 \in A$ to be such that $g(x_0) = 0$ and let f be (C^1, C^2, D_f, η)-pseudoinvex, with $C^2 = C \setminus \{0\}$, and g be (V, D_g, η)-quasinvex such that $D_f^{ii}(x,y) > 0$ and $D_g^{ii}(x,y) > 0\ \forall i = 1, \ldots, m$. Suppose that the following condition holds:*

$$\exists \alpha_f \in C^{++},\ \exists \alpha_g \in V^+, \text{ such that } \alpha_f^T J_f(x_0) + \alpha_g^T J_g(x_0) = 0$$

as well as the next properties:

$$\begin{aligned} \alpha_g \in V^+ &\implies \alpha_g^T (D_g(x,y))^{-1} \in V^+ \\ \alpha_f \in C^{++} &\implies \alpha_f^T (D_f(x,y))^{-1} \in C^{++} \end{aligned}$$

then x_0 is a global C^1-efficient point.

Theorem 4.6 *Let us consider problem P_C with f and g differentiable functions and $A \subset \Re^n$ open η-invex set, $\eta : (A \times A) \to \Re^n$. Assume $x_0 \in A$ to be such that $g(x_0) = 0$ and let f be $(C^1, \mathrm{Int}(C), D_f, \eta)$-pseudoinvex and g be $(V, \mathrm{Int}(V), D_g, \eta)$-pseudoinvex such that $D_f^{ii}(x,y) > 0$ and $D_g^{ii}(x,y) > 0$ $\forall i = 1, \ldots, m$. Suppose that the following condition holds:*

$$\exists \alpha_f \in C^+,\ \exists \alpha_g \in V^+,\ (\alpha_f, \alpha_g) \neq 0, \text{ such that } \alpha_f^T J_f(x_0) + \alpha_g^T J_g(x_0) = 0$$

as well as the next property:

$$\begin{gathered} (\alpha_f, \alpha_g) \in (C^+ \times V^+) \setminus \{0\} \implies \\ (\alpha_f^T (D_f(x,y))^{-1}, \alpha_g^T (D_g(x,y))^{-1}) \in (C^+ \times V^+) \setminus \{0\} \end{gathered}$$

then x_0 is a global C^1-efficient point.

Theorem 4.7 *Let us consider problem P_C with f and g differentiable functions and $A \subset \Re^n$ open η-invex set, $\eta : (A \times A) \to \Re^n$. Assume $x_0 \in A$ to be such that $g(x_0) = 0$ and let f be (C^1, D_f, η)-quasinvex and g be $(V, \mathrm{Int}(V), D_g, \eta)$-pseudoinvex such that $D_f^{ii}(x,y) > 0$ and $D_g^{ii}(x,y) > 0$ $\forall i = 1, \ldots, m$. Suppose that the following condition holds:*

$$\exists \alpha_f \in C^+,\ \exists \alpha_g \in V^+ \setminus \{0\}, \text{ such that } \alpha_f^T J_f(x_0) + \alpha_g^T J_g(x_0) = 0$$

as well as the next properties:

$$\begin{aligned} \alpha_f \in C^+ &\implies \alpha_f^T (D_f(x,y))^{-1} \in C^+ \\ \alpha_g \in V^+ \setminus \{0\} &\implies \alpha_g^T (D_g(x,y))^{-1} \in V^+ \setminus \{0\} \end{aligned}$$

then x_0 is a global C^1-efficient point.

Theorem 4.8 *Let us consider problem P_C with f and g differentiable functions and $A \subset \Re^n$ open η-invex set, $\eta : (A \times A) \to \Re^n$. Assume $x_0 \in A$ to be such that $g(x_0) = 0$ and let f be (C^1, D_f, η)-quasinvex and g be (V, V^2, D_g, η)-pseudoinvex, with $V^2 = V \setminus \{0\}$, such that $D_f^{ii}(x, y) > 0$ and $D_g^{ii}(x, y) > 0 \; \forall i = 1, \ldots, m$. Suppose that the following condition holds:*

$$\exists \alpha_f \in C^+, \;\; \exists \alpha_g \in V^{++}, \;\; \textit{such that } \alpha_f^T J_f(x_0) + \alpha_g^T J_g(x_0) = 0$$

as well as the next properties:

$$\begin{aligned} \alpha_f \in C^+ \quad &\Longrightarrow \quad \alpha_f^T (D_f(x, y))^{-1} \in C^+ \\ \alpha_g \in V^{++} \quad &\Longrightarrow \quad \alpha_g^T (D_g(x, y))^{-1} \in V^{++} \end{aligned}$$

then x_0 is a global C^1-efficient point.

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