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Abstract

We study the portfolio selection problem and asset pricing when returns are partially exchangeable random variables. We characterize the mean variance frontier of the portfolios and we show a K funds separation result. Exact arbitrage pricing holds. Some diversification results are proved. We also study the problem of ordering sets of exchangeable risks with the Stop-Loss criterion. Relationships between the Stop-Loss criterion and the Frechèt bounds are considered.

1 Introduction

In this paper we analyze portfolio selection and asset pricing when returns are partially exchangeable random variables. We also consider the problem of ordering sets of exchangeable risks with the Stop-Loss (SL) criterion.

Consider N real random variables $X_1, X_2, \ldots, X_N$ and a concave function u. The interpretation of our setting as a portfolio selection problem gives us X_n as stock return and u as the agent's utility function.

Let us assume that the agent's preference relation satisfies the Axioms for its Expected Utility representation and that the agent has one unit of wealth ($W_0 = 1$). Given $\gamma_1, \gamma_2, \ldots, \gamma_N$ real numbers summing up to 1, we call it a portfolio. The wealth associated with such a portfolio is

$$W = \sum_{n=1}^{N} \gamma_n X_n$$

The portfolio problem faced by the agent is as follows:

$$(1) \qquad \max_{\gamma_1,\gamma_2,\ldots,\gamma_N \in \Re} \int_{\Re^N} u(\sum_{n=1}^{N} \gamma_n x_n) dF(x_1, \ldots, x_N)$$

subject to $\sum_{n=1}^{N} \gamma_n = 1$. $F(x_1, \ldots, x_N)$ is the distribution function of the random vector $(X_1, X_2, \ldots, X_N)$.

Samuelson in a seminal paper, [Samuelson, 1956], addresses the problem assuming the N random variables to be independent and identically distributed or (in a weaker form) the distribution of the random variables to be symmetric. The solution of problem (1) is $\gamma_n = \frac{1}{N}$, $n = 1, \ldots, N$. Given a set of random variables independent and identically distributed or exchangeable, the optimal portfolio is characterized by full diversification: invest the same amount of money in every security. In this setting, a one fund separation result holds and the optimal portfolio is also the mean variance portfolio frontier.

In this paper we generalize the setting to the case of a set of random variables that are partially exchangeable, that is the N random variables can be divided in K groups in such a way that the random variables belonging to each group are exchangeable, that is the joint distribution of the random variables of each group is symmetric. Note that the case of independent and identically distributed random variables is a particular case of exchangeability. Our goal is to analyze the portfolio selection problem in this setting and to work out asset pricing implications. As a first step, we characterize the mean variance portfolio frontier, see Section 2. A portfolio belonging to the mean variance frontier foresees the same investment in each asset belonging to a given group of exchangeable assets, the amount of money invested depends on the expected return and on the number of assets of the group. In Section 3 we show that in this framework a K funds separation result holds: the optimal portfolio for an agent with a concave and monotone utility function is a linear combination of K mutual funds. Each mutual fund is made up only of the assets belonging to a group of exchangeable assets, the weights of the fund are one over the number of the assets belonging to the group, that is there is full diversification inside each group. Furthermore, we show that exact arbitrage pricing holds with respect to the returns of the mutual funds.

A mutual funds separation result through the funds obtained by full diversification inside each group represents a diversification result which is the most natural extension of the result obtained in [Samuelson, 1956]. Some other diversification results are obtained by exploiting results already established in the literature.

The second problem addressed in this paper (ordering sets of risks) considers an insurer who acts according to the premium principle based on a fixed loading of the expected value of the risk and who orders with the stop loss (SL) criterion two sets of exchangeable (or two sets of partially exchangeable) risks with the same expected gain. The SL order is not confirmed under addition of non independent risks. Limiting our attention to digital risks, we provide sufficient condition for such order to be maintained under addition. Moreover, we investigate the dependence structures among a set of risks with the safest and the riskiest distribution of the aggregate claim with respect to the SL criterion and we check the relationships between these dependence structures and the Frechèt bounds.

The notion of exchangeability, which goes back to [de Finetti, 1930], is a generalization of that of independent and identically distributed random variables. In fact, if $(X_1, \ldots, X_N)$ are independent and identically distributed then the joint distribution density is a product (N times) of the same function, therefore it is a symmetric distribution. On the contrary it is easy to produce examples of exchangeable random variables which are not independent. Nevertheless, for exchangeable random sequences it is possible to obtain convergence results analogous to those obtained for independent and identically distributed random variables.

This paper is organized as follows. In Section 2 we characterize the mean variance portfolio frontier. In Section 3 we present the K funds separation result and we analyze asset prices-risk premia in this framework. In Section 4 we present the problem of ordering two sets of exchangeable (or partially exchangeable) risks with the SL criterion. In Section 5 the relationships between SL criterion and the Frechèt bounds are considered. In Appendix A we present some preliminary results and some technical results.

2 Mean Variance Frontier

Let $(X_i)_{1\leq i\leq N}$ be a family of real random variables on a probability space $(\Omega, \mathcal{A}, P)$ and $\mathcal{G}$ a partition of the set of indices $I = \{1, \ldots, N\}$.

We will say that the family $(X_i)_{1\leq i\leq N}$ is *partially exchangeable* with respect to $\mathcal{G}$ if for any permutation σ of I which is compatible with $\mathcal{G}$ (that is $\sigma(G) = G$ for any element G of $\mathcal{G}$) we have that $(X_i)_{1\leq i\leq N}$ and $(X_{\sigma(i)})_{1\leq i\leq N}$ have the same distribution with respect to P. Obviously, the notion of partial exchangeability reduces to the notion of exchangeability when $\mathcal{G}$ is the trivial partition with I as the unique element. Partial exchangeability of the random variables implies some interesting properties, among them we recall that the random variables belonging to a group of exchangeable assets have the same marginal probability distribution, moreover the covariance is constant for every couple of random variables belonging to the same group and it is also constant for every couple of random variables belonging to two (given) different groups.

We state our result in a particular setting and then we show how it can be generalized to a general partially exchangeable setting. Let us assume that the N random variables $(X_i)_{1\leq i\leq N}$ are partially exchangeable with respect to the partition of the indices $\mathcal{G} = \{G_1, G_2\}$, where $G_1 = \{1, \ldots, N_1\}$ and $G_2 = \{N_1 + 1, \ldots, N_1 + N_2\}$ with $N_1 + N_2 = N$.

We reformulate the portfolio selection problem by denoting with μ the amount of money invested in the first group of assets as a whole and with $(1 - \mu)$ the amount of money invested in the second group of assets. μ and $1 - \mu$ are then invested in each asset through the following weights: $(\lambda_1, \ldots, \lambda_{N_1})$, $(\lambda_{N_1+1}, \ldots, \lambda_{N_1+N_2})$, where $\sum_{i\in G_1} \lambda_i = \sum_{i\in G_2} \lambda_i = 1$. The wealth associated with this portfolio becomes

$$W = \mu \sum_{i\in G_1} \lambda_i X_i + (1-\mu) \sum_{i\in G_2} \lambda_i X_i.$$

With this formulation, the expected return of the portfolio is exclusively determined by an appropriate choice of μ. The result we obtain is as follows.

Proposition 2.1 *Let $(X_i)_{1\leq i\leq N}$ be a family of real random variables partially exchangeable with respect to the partition $\mathcal{G} = \{G_1, G_2\}$, where $G_1 = \{1, \ldots, N_1\}$ and $G_2 = \{N_1 + 1, \ldots, N\}$. Fixed the expected return of the portfolio, the portfolio with the smallest variance is given by $\lambda_i = \frac{1}{N_1}$, $i \in G_1$, and $\lambda_i = \frac{1}{N_2}$, $i \in G_2$.*

Proof:

It is enough to prove that the portfolio obtained with the weights $\lambda_i = \frac{1}{N_1}$, $i \in G_1$, and $\lambda_i = \frac{1}{N_2}$, $i \in G_2$, is the one with minimum variance among the portfolios with given expected return. Therefore, fixed μ, it is enough to prove that

$$E[(\mu \sum_{i\in G_1} \frac{1}{N_1} X_i + (1-\mu) \sum_{i\in G_2} \frac{1}{N_2} X_i)^2] \leq E[(\mu \sum_{i\in G_1} \lambda_i X_i + (1-\mu) \sum_{i\in G_2} \lambda_i X_i)^2]$$

for any weights $(\lambda_1, \ldots, \lambda_{N_1})$, $(\lambda_{N_1+1}, \ldots, \lambda_{N_1+N_2})$, where $\sum_{i\in G_1} \lambda_i = 1$ and $\sum_{i\in G_2} \lambda_i = 1$. The above inequality is satisfied if the following holds

$$E[\mu^2 (\sum_{i\in G_1} \frac{1}{N_1} X_i)^2 + (1-\mu)^2 (\sum_{i\in G_2} \frac{1}{N_2} X_i)^2 + \mu(1-\mu) \sum_{i\in G_1} \frac{1}{N_1} X_i \sum_{i\in G_2} \frac{1}{N_2} X_i]$$

$$\leq E[\mu^2 (\sum_{i\in G_1} \lambda_i X_i)^2 + (1-\mu)^2 (\sum_{i\in G_2} \lambda_i X_i)^2 + \mu(1-\mu) \sum_{i\in G_1} \lambda_i X_i \sum_{i\in G_2} \lambda_i X_i].$$

Remark that

$$E[\mu^2 (\sum_{i\in G_1} \frac{1}{N_1} X_i)^2] \leq E[\mu^2 (\sum_{i\in G_1} \lambda_i X_i)^2]$$

with $\sum_{i\in G_1} \lambda_i = 1$, because $(X_1, \ldots, X_{N_1})$ is an exchangeable sequence. Analogously for the second group. Then it is enough to prove that

$$E[\sum_{i\in G_1} \frac{1}{N_1} X_i \sum_{i\in G_2} \frac{1}{N_2} X_i] - E[\sum_{i\in G_1} \lambda_i X_i \sum_{i\in G_2} \lambda_i X_i] \leq 0. \tag{2}$$

We will prove that equality holds in (2). This will follow from the following fact

$$(\lambda_j - \frac{1}{N_2}) E[X_j \sum_{i\in G_1} (\lambda_i X_i - \frac{1}{N_1} X_i)] = 0 \quad \forall j \in G_2. \tag{3}$$

Observe that (3) can be written as

$$(\lambda_j - \frac{1}{N_2}) E[X_j \sum_{i\in G_1} \lambda_i (X_i - M_{N_1})] = 0. \tag{4}$$

where M_{N_1} is defined as in (25). For any $i \in G_1$ and $j \in G_2$

$$E[X_j \lambda_i (X_i - M_{N_1})] = E[E[X_j \lambda_i (X_i - M_{N_1}) | \mathcal{F}_{N_2}]]$$

$$= E[X_j E[\lambda_i (X_i - M_{N_1}) | \mathcal{F}_{N_2}]] = 0$$

because $M_{N_1} = E[X_i | \mathcal{F}_{N_2}]$, $1 \leq i \leq N_1$, and X_j, $N_1 + 1 \leq j \leq N_1 + N_2$ is $\mathcal{F}_{N_2}$ measurable, where $\mathcal{F}_{N_2}$ is the $\sigma-$algebra which leaves fixed the last N_2 indices, see Proposition A.1. □

This Proposition can be generalized to the case of partial exchangeability with respect to a partition of the indices $\mathcal{G} = \{G_1, \ldots, G_K\}$ denoting by

$$W = \sum_{k=1}^{K} \mu_k \sum_{i \in G_k} \lambda_i X_i,$$

where $\sum_{i \in G_k} \lambda_i = 1,\ k = 1, \ldots, K$, and $\sum_{k=1}^{K} \mu_k = 1$. This expression means that the amount of money invested in the group k of assets is μ_k, that amount of money is then invested inside the group of assets through the weights $\lambda_i,\ i \in G_k$. As above, the expected return is only determined by the weights μ_k.

Proposition 2.2 *Let $(X_i)_{1 \le i \le N}$ be a family of real random variables partially exchangeable with respect to the partition $\mathcal{G} = \{G_1, \ldots, G_K\}$, where $G_1 = \{1, \ldots, N_1\}$, $G_k = \{\sum_{j=1}^{k-1} N_j + 1, \ldots, \sum_{j=1}^{k} N_j\}$, $k = 2, \ldots, K$, and $\sum_{j=1}^{K} N_j = N$. Fixed the expected return, the portfolio with the smallest variance is provided by $\lambda_i = \frac{1}{N_k},\ i \in G_k,\ k = 1, \ldots, K$.*

In the following, for simplicity, we denote $M_{N_1} = M_1, M_{N_1+N_2} = M_2, \ldots, M_{N_1+\cdots+N_K} = M_K$. M_k, defined in (25), represents the wealth-return of the portfolio $(\gamma_1, \gamma_2, \ldots, \gamma_N)$ such that $\gamma_i = \frac{1}{N_k},\ i = G_k$, and zero otherwise: $M_k = \frac{1}{N_k} \sum_{i \in G_k} X_i$.

Propositions 2.1, 2.2 state the following result: assuming asset returns to be partially exchangeable, the returns of the portfolios of the mean variance frontier are spanned by $M_k, k = 1, \ldots, K$, that is by the portfolios made only of the assets belonging to a given group with full diversification (the same weight for each asset). Therefore, the mean variance frontier generated by the N assets coincides with the frontier generated by the portfolios $M_k,\ k = 1, \ldots, K$.

Note that with $K = 1$ (exchangeable random variables), i.e., the case analyzed in [Samuelson, 1956], M_1 belongs to the frontier, for $K = 2$ the portfolios M_k belong to the frontier provided that the two groups of returns are characterized by different expectations. In this case, exploiting the properties of the mean variance frontier, we have that every linear combination of M_1 and M_2 belongs to the frontier, see [Huang and Litzenberger, 1988]. In general, when $K \ge 3$ the portfolios M_k do not belong to the mean variance frontier.

If $E[M_1] = E[M_2] = \ldots = E[M_K]$ and

$$cov(M_j, M_i) = 0 \quad i \neq j = 1, \ldots, K,$$

then the portfolio on the frontier is made up of a positive investment in every portfolio $M_k,\ k = 1, \ldots, K$.

3 K Mutual Funds Separation, Asset Pricing and Diversification

The results established in the last section are about the spanning of the mean variance frontier by the portfolios $M_k,\ k=1,\ldots,K$; nothing is established about the optimal portfolio of an agent maximizing his expected utility.

Thanks to the properties associated with partially exchangeable random variables, we can establish a K funds separation result. The K funds are the random variables $M_k,\ k=1,\ldots,K$. In what follows, we assume $K\geq 2$ and that two groups of exchangeable random variables have different expected returns. First of all, we observe that

$$X_n = M_k + \epsilon_n, \qquad n\in G_k,\quad k=1,\ldots,K, \tag{5}$$

where ϵ_n is a random variable with zero mean, i.e., $E[\epsilon_n]=0,\ n=1,\ldots,N$.

Proposition 3.1 *Let $(X_i)_{1\leq i\leq N}$ be a family of partially exchangeable random variables with respect to $\{G_1,G_2,\ldots,G_K\}$, a mutual funds separation result holds with respect to $M_k,\ k=1,\ldots,K$.*

Proof:

We refer to [Ingersoll, 1987, Ross, 1978], where necessary and sufficient conditions for K mutual funds separation are provided by the fact that the following decomposition holds:

$$X_i = \sum_{k=1}^{K} b_i^k Z_k + \epsilon_i, \quad i=1,\ldots,N, \tag{6}$$

$$E[\epsilon_i|Z_1,Z_2,\ldots,Z_K]=0, \quad i=1,\ldots,N, \tag{7}$$

$$\sum_{i=1}^{N} w_i^k \epsilon_i = 0 \quad k=1,\ldots,K, \tag{8}$$

$$\sum_{i=1}^{N} w_i^k = 1 \quad k=1,\ldots,K, \tag{9}$$

$$rank(A)=K, \tag{10}$$

where w_i^k is the weight of security i in the k-th mutual fund and A is a $K\times K$ matrix such that $a_{mk}=\sum_{i=1}^{N} b_k^i w_i^m,\ k,m=1,\ldots,K$.

Let $b_i^k = 1$ for $i \in G_k$ and $b_i^k = 0$ otherwise, and $Z_k = M_k$, $k = 1, \ldots, K$, ($w_i^k = \frac{1}{N_k}$ for $i \in G_k$ and $w_i^k = 0$ otherwise) then thanks to (5) a decomposition as in (6) is obtained. Conditions (8)-(10) are satisfied by construction. It remains to show that

$$E[\epsilon_i | M_1, M_2, \ldots, M_K] = 0, \quad i = 1, \ldots, N.$$

We prove the result for $K = 2$, the extension to a generic K can be easily obtained. We start by proving that

$$E[\epsilon_i | \mathcal{F}_{N_2}] = 0, \quad i = 1, \ldots, N_1.$$

Fixed j, Proposition A.1 says that for any $1 \le h, k \le N - j + 1$ we have $E[X_h | \mathcal{F}_{j-1}] = E[X_k | \mathcal{F}_{j-1}] = M_{N-j+1}$. Let $j = N_2 + 1$, then for any $i \le N - j + 1 = N_1$ we have

$$E[X_i | \mathcal{F}_{N_2}] = M_1,$$

that is

$$E[\epsilon_i | \mathcal{F}_{N_2}] = E[X_i - M_1 | \mathcal{F}_{N_2}] = 0.$$

The above equality implies

$$E[\epsilon_i | M_1, M_2] = E[E[\epsilon_i | \mathcal{F}_{N_2}] | M_1, M_2] = 0,$$

because M_1, M_2 are $\mathcal{F}_{N_2}$-measurable. In fact, M_1 is measurable by definition and M_2 can be expressed as a function $g(X_1, \ldots X_{N_1}, X_{N_1+1}, \ldots, X_{N_1+N_2})$ which is invariant with respect to any permutation which leaves fixed the last N_2 variables. Therefore, M_2 is $\mathcal{F}_{N_2}$-measurable. □

This result gives us a K mutual funds separation result. A result which is confirmed by [Huberman and Kandel, 1987, Proposition 2 and 3], where it is shown that if M_k, $k = 1, \ldots, K$, span the frontier of X_n, $n = 1, \ldots, N$, and (7) holds, then a K funds separation result holds. For every portfolio of $(X_1, \ldots, X_N)$ with return W^* there exists a portfolio of $(M_1, \ldots, M_K)$ with return W^{**} such that $E[u(W^{**})] \ge E[u(W^*)]$ for every concave monotone utility function u.

When asset returns are partially exchangeable we have a K fund separation result. Note that the portfolios of the mean variance frontier are a subset of the portfolios made up of M_k, $k = 1, \ldots, K$, therefore we can not say that agents hold portfolio on the frontier. This fact implies that the CAPM does not hold. Obviously, the CAPM holds when there are only two groups of exchangeable assets, see [Huang and Litzenberger, 1988]. Partial exchangeability inside two groups of assets is a sufficient condition for the CAPM.

In [Grinblatt and Titman, 1987, Huberman and Kandel, 1987, Huberman et al., 1987], it is shown that the fact that a portfolio of M_k, $k = 1, \ldots, K$, (different from the globally mean-variance portfolio) belongs to the mean variance frontier is equivalent to exact arbitrage pricing. They observe that the mutual fund property is stronger than exact arbitrage pricing relative to the separating funds. The condition is obviously satisfied in our setting and therefore for any portfolio $\gamma_1, \ldots, \gamma_N$ we have that

$$E[\sum_{n=1}^{N} \gamma_n X_n] = \sum_{k=1}^{K} \sum_{n \in G_k} \gamma_n E[M_k].$$

Absence of arbitrage opportunities implies the expected return of a portfolio be equal to the corresponding weighted sum of the expectation of M_k.

As shown in [Huberman and Kandel, 1987], exact arbitrage pricing also holds with a risk free rate. In that case, exact arbitrage is equivalent to the fact that the minimum variance frontier made up of $M_1, \ldots, M_K$ intersects the minimum variance frontier of $X_1, \ldots, X_N$, $M_1, \ldots, M_K$ and the risk free rate. This condition is easily verified as the tangent portfolio belongs to the two frontiers.

The separation result provides us with a diversification result. An agent characterized by a monotone and concave utility function holds a portfolio which is made up of the portfolios M_k, $k = 1, \ldots, K$, that is a portfolio with full diversification inside each group of exchangeable assets. This result generalizes the classical one obtained in [Samuelson, 1956]. A portfolio which does not provide the same weight for the assets of a group is dominated in the sense of second order stochastic dominance (SSD) by another portfolio made up of M_k, $k = 1, \ldots, K$.

As a corollary of the separation result, we have that M_k dominates in sense of second order stochastic dominance X_n, $n \in G_k$, $k = 1, \ldots, K$. Some other diversification results can be obtained exploiting the properties of partial exchangeability and some results established in the literature.

If $E[X_i]$ is constant, $i = 1, \ldots, N$, X_j SSD X_i, $\forall j \in G_u$, $\forall i \in G_v$ with $1 < v < u < K$, X_j and X_i, $\forall j \in G_u$, $\forall i \in G_v$, $\forall v, u = 1, \ldots, K$, are independent random variables and have a common range then a portfolio with positive weights for every M_k dominates in the sense of second order stochastic dominance M_k, $k = 1, \ldots, K$. This result can be established by exploiting [Hadar, et al., 1977, Theorem 3] and by observing that if X_j SSD X_i, $\forall j \in G_u$, $\forall i \in G_v$ with $1 < v < u < K$ then M_v SSD M_u, see the Appendix.

Let us assume that there are two groups of exchangeable random variables. If X_j and X_i, $\forall j \in G_1, \forall i \in G_2$, are independent random variables with the same mean and positive variance then for each monotone and concave utility function there exists a diversified portfolio (positive weights vector) of M_1 and M_2 which is optimal, see [Hadar and Russel, 1971, Theorem 6]. M_1 and M_2 are in fact independent, see the Appendix. The result can be generalized to the case of $K > 2$ groups, see [Samuelson, 1956, Theorem 3].

Dropping independence, diversification still can be established following [Hadar and Russel, 1974 Theorem 9]. Let us consider two groups G_1 and G_2, if the joint probability distribution of (X_i, X_j), $\forall j \in G_1, \forall i \in G_2$, dominates in the sense of first order stochastic dominance the product of the two marginal probability distribution then for each monotone and concave utility function there exists a diversified portfolio (positive weights vector) of M_1 and M_2 which is optimal, see the Appendix. A necessary condition for diversification is negative covariance between two random variables belonging to G_1 and G_2.

4 Ordering Risks

In [Goovaerts, et al., 1989, Goovaerts, et al., 1990] it is shown an interesting result for the sum of independent and identically distributed risks. Let X_i, $i = 1, \ldots, N$, and Y_i, $i = 1, \ldots, N$, two vectors of positive random variables identically and independently distributed and $\tilde{x}, \tilde{y}$ two generic random variables of the two vectors. If

$$y \geq_{SL} x,$$

that is

$$E[(x-d)_+] \leq E[(y-d)_+] \;\forall d \in \Re^+, \tag{11}$$

then

$$Y \geq_{SL} X, \tag{12}$$

where $Y = \sum_{i=1}^{N} Y_i$ and $X = \sum_{i=1}^{N} X_i$. Given two random variables ordered by the SL criterion, the order is maintained for the sum of N iid random variables of the first type and of the second type. This result does not hold in case of exchangeable random variables.

Limiting our attention to digital risks, i.e., risk taking only two possible values, it is possible to establish a condition such that the above order is maintained under addition for exchangeable random variables. On conditions ensuring that (12) holds in a general setting see also[Muller, 1997].

Let the distribution of the digital random variable x be $[1, 0], [\xi_1, \xi_0]$, where ξ_1, ξ_0 are respectively the probabilities that the insured event with damage 1 happens or not and let the distribution of the digital random variable y be $[a, 0], [\omega_1, \omega_0]$ with $a \in [0, 1]$.

For an insurer employing the premium principle based on a fixed loading of the expected value of the risk, insuring one of the two sets of risks allow the same expected gain if and only if x and y have the same expected value; from which the following condition

$$\omega_1 = \frac{\xi_1}{a}. \tag{13}$$

In order to have $\omega_1 \in [0,1]$, we impose $a \geq \xi_1$. Then the risk y is preferred by the SL criterion to the risk x. As a matter of fact (11) holds, in particular if $a < 1$, then (11) holds with strict inequality $\forall d \in (0,1)$.

Let $\omega_{N,k}$ and $\xi_{N,k}$, $k = 0,1,...,N$, the probabilities that the insured event happens for k risks respectively of the set X_i, $i = 1,\ldots,N$, and Y_i, $i = 1,\ldots,N$. The distribution for the random variable X is $X = k$ with probability $\xi_{N,k}$, $k = 0,1,...,N$ and for the random variable Y is $Y = ka$ with probability $\omega_{N,k}$, $k = 0,1,...,N$. We want now to determine necessary and/or sufficient conditions on $\omega_{N,k}$ and $\xi_{N,k}$, $k = 0,1,...,N$, for (12) to hold.

Indicating with F_X and F_Y the distribution functions respectively of X and Y, we have

$$\varphi(d) = \int_d^{+\infty} (1 - F_X(t))\, dt = \tag{14}$$

$$= \begin{cases} E[X] - d & d \leq 0 \\ \sum_{j=h+1}^{N} j\xi_{N,j} - d\sum_{j=h+1}^{N} \xi_{N,j} & h < d \leq h+1,\ h = 0,1,...,N-1 \\ 0 & d > N \end{cases}$$

and

$$\phi(d) = \int_d^{+\infty} (1 - F_Y(t))\, dt = \tag{15}$$

$$= \begin{cases} E[Y] - d & d \leq 0 \\ a\sum_{j=h+1}^{N} j\omega_{N,j} - b\sum_{j=h+1}^{N} \omega_{N,j} & ha < d \leq (h+1)a,\ h = 0,1,...,N-1 \\ 0 & d > Na \end{cases}$$

If (13) holds true (note $E[X] = \varphi(0) = \phi(0) = E[Y]$) it is trivial to observe that $\varphi(d) \geq \phi(d)$ holds for every $d \in (-\infty, 0] \cup [Na, +\infty)$, and so for (12) to hold, it is necessary and sufficient to show that

$$\varphi(d) \geq \phi(d) \quad \forall d \in (0, Na). \tag{16}$$

In the interval $[0, Na]$ both $\varphi(d)$ and $\phi(d)$ are continuous, not increasing, piece-wise linear and convex functions and therefore a necessary and sufficient condition for (16) is

$$\varphi(d) \geq \phi(d) \quad \begin{matrix} d = 1,2,...,\lfloor Na \rfloor & if\ Na \notin N, \\ d = 1,2,...,Na-1 & if\ Na \in N, \end{matrix} \tag{17}$$

where $\lfloor \bullet \rfloor$ stands for the integer part of $\bullet$.

Suppose now that the two sets $\{X_1, X_2, ..., X_N\}$ and $\{Y_1, Y_2, ..., Y_N\}$, with the generic element described respectively by the two digital random variables x and y, consist of risks that are partially exchangeable random variables, i.e., for the two sets of risks there are Q sets $\left\{X_{1q}, X_{2q}, ..., X_{n_q q}\right\}$ and $\left\{Y_{1q}, Y_{2q}, ..., Y_{n_q q}\right\}$ $(q = 1, 2, ..., Q)$ such that $\sum_{q=1}^{Q} n_q = N$ consisting of exchangeable risks. If the generic elements $X_{iq'}$ and $Y_{iq'}$ $\left(i \in \left\{1, 2, ..., n_{q'}\right\}\right)$ of a q' set are not correlated respectively with the generic elements $X_{jq''}$ and $Y_{jq''}$ $\left(j \in \left\{1, 2, ..., n_{q''}\right\}\right)$ of a q'' set $\left(q', q'' \in \{1, 2, ..., Q\} \quad q' \neq q''\right)$, then exploiting the well-known property which says that SL ordering is preserved under convolution of independent risks, see e.g. [Goovaerts, et al., 1989 Goovaerts, et al., 1990], we have that (17) with $n_q^* a$ instead of Na ,where $n_q^* = \max\{n_q, q = 1, 2, ..., Q$ is a sufficient condition for (12) to hold. If $Q = \frac{n}{2}$ and $n_q = 2,\ q = 1, 2, ..., Q$, this result is a particular case of the result shown in [Dhaene and Goovaerts, 1996].

5 Ordering Risks and Frechèt bounds

Let F_1, F_2, ..., F_N be the univariate distribution functions (c.d.f.'s in short) of the non-negative random variables $X_1, X_2, ..., X_N$ and consider the Frechèt space $\Re_N (F_1, F_2, ..., F_N)$ made up of all N-dimensional c.d.f.'s F^N having $F_1, F_2, ..., F_N$ as marginals. We have that $\forall F^N \in \Re_N (F_1, F_2, ..., F_N)$ the following inequality holds

(18) $$M_N(x) \leq F^N(x) \leq W_N(x) \quad \forall x = (x_1, x_2, ..., x_N) \in \Re_+^N,$$

where W_N is usually referred to as Frechèt upper bound of $\Re_N (F_1, F_2, ..., F_N)$ and is defined as

(19) $$W_N(x) = \min\{F_1(x_1), F_2(x_2), ..., F_N(x_N)\}$$

and M_N is usually referred to as the Frechèt lower bound of $\Re_N (F_1, F_2, ..., F_N)$ and is defined as

(20) $$M_N(x) = \max\left\{\sum_{i=1}^{N} F_i(x_i) - N + 1, 0\right\}.$$

If we consider the exchangeable random variables $X_i,\ i = 1, 2, ..., N$, considered in the above Section (they have the same distribution F), then (19) becomes

$$W_N(x) = F(\min x_i),\ i = 1, 2, ..., N,$$

and therefore

$$W_N(x) = \begin{cases} \xi_0 & if \;\; \exists i : x_i < 1 \\ 1 & if \;\; \forall i : x_i \geq 1 \end{cases}$$

and (20) becomes

$$M_N(x) = h\xi_0 - h + 1 \quad where \quad h = \#\{i = 1, 2, ..., N : x_i < 1\}.$$

While W_N is always reacheable in $\Re_N(F_1, F_2, ..., F_N)$, for a necessary and sufficient condition for M_N to be a c.d.f. in $\Re_N(F_1, F_2, ..., F_N)$ see e.g. [Joe, 1997].

If the dependence structure of the risks is given by

$$(21) \qquad \begin{cases} \Pr(X_i = 0 \;\; \forall i = 1, 2, ...N) = \xi_0 \\ \Pr(X_i = 1 \;\; \forall i = 1, 2, ...N) = \xi_1 \\ \Pr(\exists i, j : X_i = 0 \;\; and \;\; X_j = 1) = 0 \end{cases}$$

then the c.d.f. of the aggregate claim X is given by

$$(22) \qquad F_X(t) = \begin{cases} 0 & for \; t < 0 \\ \xi_0 & for \; 0 \leq t < N \\ 1 & for \; N \leq t \end{cases}$$

Given the expected value of the aggregate claim, it is easy to check, see e.g. [Dhaene and Goovaerts, : that the distribution of the aggregate claim described in (22) is the riskiest with the SL criterion. Notice that the set of risks with the dependence structure given by (21) is the Frechèt upper bound in the space $\Re_N(F, F, ..., F)$.

Let $m = \lfloor N\xi_1 \rfloor$. If the dependence structure for the risks is given by

$$(23) \qquad \begin{cases} 0 \leq \Pr(\#\{i = 1, 2, ..., n : X_i = 1\} = m) = \xi_{N,m} \leq 1 \\ 0 \leq \Pr(\#\{i = 1, 2, ..., n : X_i = 1\} = m + 1) = \xi_{N,m+1} \leq 1 \\ \Pr(\#\{i = 1, 2, ..., N : X_i = 1\} = h) = 0 \qquad \forall h = 1, 2, ..., m - 1, m + 2, ..., N \end{cases}$$

then the c.d.f. of the aggregate claim X is given by

$$(24) \qquad F_X(t) = \begin{cases} 0 & for \;\; t < m \\ \xi_{N,m} & for \;\; m \leq t < m + 1 \\ 1 & for \;\; m + 1 \leq t \end{cases}$$

Given the expected value of the aggregate claim, it is easy to check that the distribution of the aggregate claim described in (24) is the safest with the SL criterion. Notice that in case $m = 0$ or $m = N - 1$ the set of risks with the dependence structure given by (23) is the Frechèt lower bound in the space $\Re_N(F, F, ..., F)$. To show this when $m = 0$ it is sufficient to exploit the inconsistence of the events $X_i = 1$ and $X_j = 1$ while, when $m = N - 1$ it is sufficient to exploit the inconsistence of the events $X_i = 0$ and $X_j = 0$.

Notice that to obtain the results showed in this section the hypotesis of exchangeability is used only in the sense of random variables with the same distribution, so the results obtained in this section hold also for the sets of partial exchangeable risks considered in Section 4.

6 Conclusions

We have analyzed the classical portfolio selection problem and asset pricing theory when returns are a vector of partially exchangeable random variables, that is the assets can be ordered in K groups in such a way that inside each group the distribution of the random variables is exchangeable. A particular role in our analysis is played by the portfolios characterized by full diversification inside each group. We have shown that these portfolios span the mean variance portfolio frontier, that a mutual funds separation result with respect to these portfolios holds as well as exact arbitrage pricing. Further we have analyzed the problem of ordering sets of exchangeable and partially exchangeable digital risks with SL criterion and we have shown the relationship between the safest and the riskiest dependence structures among a set of risks and the Frechèt bounds.

A Preliminary results

In this Appendix we present some properties of partially exchangeable random variables useful for the analysis presented above.

We define the sequence $(M_j)_{1\leq j\leq N}$ in the following manner

$$M_j = |G_j|^{-1} \sum_{i\in G_j} X_i \tag{25}$$

where G_j is the intersection of $\{1,\ldots,j\}$ with the unique element of $\mathcal{G}$ which contains j and $|G_j|$ is the cardinality of G_j. In particular if $\mathcal{G}$ is the trivial partition, then M_j is equal to $(X_1+\cdots+X_j)/j$.

We define then the following filtration of σ-algebras associated with $(X_i)_{1\leq i\leq N}$. The filtration $(\mathcal{F}_j)_{0\leq j\leq N}$ is defined in the following way: $\mathcal{F}_j$ is the σ-algebra generated by the random variables $f(X_1,\ldots,X_N)$ where f is a Borel function on $\mathcal{R}^N$, invariant with respect to the permutations σ, compatible with $\mathcal{G}$, which leave fixed the last j arguments.

We remark that $\mathcal{F}_0$ coincides with the σ-algebra $\mathcal{G}$-symmetric and that $\mathcal{F}_N$ and $\mathcal{F}_{N-1}$ coincide with the σ-algebra generated by the X_i.

The following result holds, see [Letta and Pratelli, 1998, Mancino and Pratelli, 2000].

Proposition A.1 *Let $(X_i)_{1\leq i\leq N}$ be a family of real random variables partially exchangeable with respect to $\mathcal{G}$. Then for any j $1\leq j\leq N$ the random variable X_{N-j+1} is measurable with respect to $\mathcal{F}_j$ and M_{N-j+1} is a version of its conditional expectation with respect to $\mathcal{F}_{j-1}$.*

Proof:

By the definition of the filtration $(\mathcal{F}_j)$ it is immediate that the random variable X_{N-j+1} is measurable with respect to $\mathcal{F}_j$. Because the sequence $(X_j)_{1\leq j\leq N}$ is partially exchangeable with respect to the partition $\mathcal{G}$, the conditional expectations

$$E[X_h|\mathcal{F}_{j-1}]$$

with $1\leq h\leq N-j+1$ coincide. Therefore

$$M_{N-j+1} = E[X_h|\mathcal{F}_{j-1}]. \tag{26}$$

□

For any finite sequence of real constants $(a_j)_{1\leq j\leq N}$ consider the finite sequence

$$(a_j(X_{N-j+1} - M_{N-j+1}))_{1\leq j\leq N}. \tag{27}$$

By the Proposition it follows that the above sequence is a "suite centrée" with respect to the filtration $(\mathcal{F}_j)$, that is the random variables (27) are measurable with respect to $\mathcal{F}_j$ and

$$E[a_j(X_{N-j+1} - M_{N-j+1})|\mathcal{F}_{j-1}] = 0. \tag{28}$$

We observe that the covariance between M_j and M_i is equal to the covariance between two (any) random variables of group G_j and group G_i. Let us consider the case of two groups G_1 and G_2, we have that $E[X_iX_j]$ is constant $\forall i \in G_1$ and $\forall j \in G_2$ because the random variables belonging to the same group are identically distributed, then

$$E[X_iX_j] = \frac{1}{N_1}\frac{1}{N_2}\sum_{i\in G_1}\sum_{j\in G_2} E[X_iX_j] = \frac{1}{N_1}\sum_{i\in G_1} E[X_iM_2] = E[M_1M_2].$$

This implies that

$$cov(M_1, M_2) = cov(X_i, X_j), \quad \forall i \in G_1, \ \forall j \in G_2.$$

The same reasoning applies when there are more than two partially exchangeable groups of random variables.

If $(X_1, \ldots, X_{N_1})$ is independent of $(X_{N_1+1}, \ldots, X_N)$ then M_1 and M_2 being functions respectively of the first vector of random variables and of the second vector are independent. The same reuslt holds when there are K groups.

If $X_i =^d X_j + \epsilon_j$, $\forall X_i \in G_u$ and $\forall X_j \in G_v$ with $E[\epsilon_j|X_j] = 0$, then there exists a random variable ζ such that $M_u =^d M_v + \zeta$ and $E[\zeta|M_v] = 0$. If X_i and X_j are independent, then also M_u and M_v are independent. $=^d$ denotes equivalence in distribution.

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