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Dipartimento di Statistica e Matematica Applicata all'Economia

Report n.203

Executive Stock Options Evaluation

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Pisa, January 2001

- Stampato in Proprio -

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Abstract

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Keywords:
Classification:

1 Introduction

The role of stock option plans in executive compensation grew up significantly in the last ten years. A large part of executives' compensation is now provided through stock option plans, a phenomenon particularly acute for new (risky) companies, such as new economy companies, see [25]. Through options, executive compensation is linked to the company performance and therefore to the shareholder's wealth in such a way that classical agency problems between shareholders and managers pointed out in [20] result weakened.

A large literature is now available on the topic, we recall the special issue of the *Journal of Financial Economics (2000)* as well as [7, 10, 24, 9, 25, 30]. Two main topics are addressed in the literature: evaluation of the stock option, analysis of the effects of this type of compensation on the management of the company by the executives. With few exceptions, these topics are addressed by using standard option pricing techniques, i.e., Black and Scholes style techniques. The analysis proceeds as follows, the executive stock option is described as a contingent claim contract written on the asset price which is modeled as a stochastic differential equation, then the value of the option is computed according to risk neutral pricing techniques. The effect of this type of compensation on executives' behavior is evaluated by computing the derivative of the value of the contract with respect to the volatility, asset price, dividend ratio, etc.. Following classical option pricing results, the value of most executives' compensation plans is increasing in the stock price and volatility and decreasing in the dividend rate, see [21]. These effects have been confirmed in the empirical analysis, see [10, 24, 9], with some exceptions, e.g. see [30].

This way to evaluate a stock option plan suffers of three main drawbacks: a) executives are risk averse and therefore both the value and the effects of this type of compensation computed according to the risk neutral methodology can be misleading, see [13], b) risk neutral evaluation is based on the assumption that assets can be freely traded, this is not the case of executive options, c) the price dynamics is modeled independently of the executive compensation plan, the price of the company follows a dynamics which does not depend on the executive compensation plan, executives cannot interfere with the asset price evolution.

In this paper we address the evalution of executive stock options by assuming that the executive can manipulate the asset price dynamics. The point we want to make is that executives endowed with an option run the company in order to maximize the expected payoff associated with their option. We simplify the analysis by assuming that executives directly manage the stock price dynamics, and that the asset price is the company value (there is no debt). In this way, we endogenize executives' behavior in the evaluation of their option plan. The market side in determining the asset price evolution is retained in the model by assuming no arbitrage opportunities in the market and evaluating all the compensation plans under the risk neutral martingale measure. To evaluate stock options for executives, we heavily borrow techniques and results from the literature on *passport options*, see [19, 3, 11, 17, 29], and on superhedging contingent claims in incomplete markets, see [4, 12, 27]. The passport option is an option on a traded account which allows the holder to switch during the life of the option among various positions in the underlying asset, typically the payoff is a classical call: the holder gets the value of his trading account if it is positive and zero if the trading account is negative.

We start by considering an executive endowed with an european call option, when he can manipulate the asset price dynamics by controlling the dividend rate and its volatility in order to maximize the value of the option (expected payoff). We show that the optimal strategy coincides with the recommendations obtained according to the Black&Scholes formula: volatility is set to the maximum and the dividend yield to the minimum. This result provides a rationale to what is usually conjectured and to the empirical results. These results do not change if executives also own some shares of the company and if they get a number of options increasing in the asset company price.

The second setting is about an executive endowed with a call option on the asset of the company when he is allowed to switch between two policies: a safe policy with the asset price growing at the (constant) risk free rate and a risky project such that the asset price follows a lognormal diffusion process with the risk free rate as drift. In this setting, the executive option is a passport option of the type studied in the above cited papers: the manager gets the value of an imaginary "trading" account (asset price minus the strike) when it is positive and nothning in case of a negative value. By applying results obtained in the passport options literature, we show that the optimal policy is asymmetric. In general, when the option is out of the money, executives adopt the risky policy, when the option is in the money they adopt the risk free policy. The policy depends strongly on the constraints faced by the manager. The above results are confirmed when the executive is endowed with some shares of the company and/or when he gets an amount of options increasing in the company performance. When dividends are not credited to the executive he has an incentive to set them to the minimum. The analysis is extended to the case of a manager who can decide between two risky policies and a risk free policy. The case of a project with a negative risk premium is also considered. It is shown that the manager will adopt it, this choice is optimal when the option is deeply out of the money. Therefore, not only an option can have a negative welfare effect, but it is also very risky. A bankruptcy condition does not affect considerably the executive's policy; only for a value of the company very close to the bankruptcy level we observe that teh agent does not adopt teh risky project.

We then consider some non-traditional options widely used in executives' comepnsation, see [21]: premium stock option, performance-vested option, repriceable option, purchased option, reload option, indexed option. We evaluate tch executive's policy and teh value of this compensation scheme comapring them to those of the classical stock options and to those obtained in [21] with no control by the executives of the asset price. We show that

The third setting considers the case of an indexed option. Allowing the executive to manage the volatility of the asset, we observe that the volatility is set to the maximum and that the correlation with the index is set to -1.

The analysis described above is characterized by a convex payoff for the executive. Several forms of nonconvexities are introduced in the executive's compensation plans either to penalize the executive in case of bad performance or to place a cap on his compensation. The behaviour of the executive with several nonconvex compensation plans is analyzed by allowing him to switch between the safe policy and the risky policy..

The paper is organized as follows. In Section 2 we address the simplest case of a call option on the asset price with the executive managing the dividend yield and the asset price volatility. In Section 3 we analyze the case of a manager who can decide the percentage of the firm to be run according to a risky policy. In Section 4.6 we analyze the case of an indexed option. In Section 6 we analyze several types of compensation plans with a nonconvex payoff.

2 European Call Option

We consider an economy with two assets: a risk free asset and the company asset. We assume that the asset can be exchanged in the market and therefore there are no arbitrage opportunities. Under the risk neutral probability measure, the stock price of the company evolves according to the equation

$$dS(t) = S(t)(r - \gamma(t))dt + \sigma(t)S(t)dW(t)), \qquad S(0) = s_0, \tag{1}$$

where $W(t)$ is a standard Brownian motion, r is the risk free rate, $\gamma(t)$ is the dividend rate at time t and $\sigma(t)$ is the volatility of the asset. The manager is endowed with an European call option maturing at time T with a strike price K, the payoff is therefore $[S(T) - K]^+$. The manager can control both the dividend rate $\gamma(t)$, $t \in [0,T]$, and the volatility $\sigma(t)$, $t \in [0,T]$. We assume that the manager faces the following constraints: $\gamma(t) \in [0,\Gamma]$ and $\sigma(t) \in [\sigma_1, \sigma_2]$ $\forall t \in [0,T]$, $\Gamma > 0$ and $0 < \sigma_1 < \sigma_2$.

The value of this contract at time t is:

$$V(t,S) = \sup_{\gamma(s),\sigma(s)\ t\leq s\leq T} \left(e^{-r(T-t)} E\left[(S(T)-K)^+ \middle| \mathcal{F}_t \right] \right) \tag{2}$$

subject to $\gamma(s) \in [0,\Gamma]$ and $\sigma(s) \in [\sigma_1,\sigma_2]$. The Hamilton-Jacobi-Bellman equation for $V_{BS}(t,s)$ is

$$\begin{cases} -rV + V_t + rSV_S + \sup\limits_{\gamma(t)\in[0,\Gamma],\ \sigma(t)\in[\sigma_1,\sigma_2]} \left[-\gamma(t)SV_S + \frac{1}{2}\sigma(t)^2 S^2 V_{SS} \right] \\ V(T,S) = (S(T)-K)^+. \end{cases} \tag{3}$$

By employing monotonicity results obtained in [4, 12, 27] on superhedging a contingent claim with uncertain volatility, or simply by assuming $V_S \geq 0, V_{SS} \geq 0$ and then using the verification theorem, we have that the optimal choice for $\gamma(t), \sigma(t)$ is constant and is given by

$$\begin{cases} \gamma^{opt}(t) = 0 \\ \sigma^{opt}(t) = \sigma_2. \end{cases} \tag{4}$$

Given this result, the price of an executive call option is the Black-Scholes price of a call option with null dividend yield and volatility σ_2.

This simple model confirms what is obtained by applying straight on the Black&Scholes formula to the evaluation of executive options: executives are induced to take more risk, i.e., to increase the volatility of the asset, and to cut dividends. This result has been extended to path dependent contingent claims: the price of a path dependent contingent claim contract whose payoff is increasing in the maximum of the price of the asset in the time interval is increasing in its volatility, see [16].

Note that the above results hold for any contingent claim with a convex payoff. Therefore they are confirmed if executives hold not only a call option but also a positive amount of stocks. For example, let us assume that agents hold $\lambda > 0$ of the stock and $\kappa > 0$ stock options, then the payoff will be $\alpha X(T) + \kappa[X(T) - K]^+$ which is a convex function. The same conclusion can be drawn when the number of options of the stock option plan is an increasing function of the asset price performance. Let α be the number of options if $S(T) \geq K_1 > K$ and β ($\beta < \alpha$) be the number of options if $K \leq S(T) \leq K_1$. Then the payoff for the executive is still convex and therefore the above results hold.

By applying results in [27], the above reasoning can be extended to a call option written on a basket of assets.

3 Risky vs. Safe Projects

Let us consider a manager who can adopt a *safe* policy with the asset price growing at the risk free rate or a *risky* policy with the asset price dynamics described by the lognormal diffusion process (1) with constant coefficients and no dividends. The manager is endowed with a call option on the asset price at time T. Following this interpretation, the asset price can be interpreted as an account with the owner trading on the two assets (risk free and risky).

Let $q(t)$ the units of the risky policy adopted by the manager at time t, the asset price evolves as follows:

$$dX(t) = q(t)dS(t) + r\left(X(t) - q(t)S(t)\right)dt = rX(t)dt + q(t)\sigma S(t)dW(t), \; X(0) = x_0. \tag{5}$$

To keep the interpretation, we have normalized the company value be equal to its asset price, there is no debt. The executive is endowed with a call option maturing at time T

written on X with strike price K. Following [29], we impose the following constraint on the executive's strategy: $q(t) \in [\alpha, \beta]$, $\alpha < \beta$, $\forall t \in [0, T]$. For special cases we have different types of options, see [29]:

1. Passport call option: $q(t) \in [-1, 1]$;
2. Vacation call option: $q(t) \in [0, 1]$.

In the first case the executive can either dismiss the risky activity of the company or invest in it. In the second case the executive can only invest in it.

The price of the call option at time t is

$$V(t, S, X) = \sup_{q(s)\in[\alpha,\beta],\ t\leq s\leq T} \left(e^{-r(T-t)}E\left[(X(T) - K)^+ \Big| \mathcal{F}_t\right]\right). \tag{6}$$

Let $Y(t) = X(t) - K$, $Y(t)$ satisfies (26) with initial state $y_0 = x_0 - K$. The price of the contingent claim rewritten in terms of Y, $V(t, S, Y)$ satisfies the Hamilton-Jacobi-Bellman (HJB) equation:

$$-rV + V_t + rYV_Y + rSV_S + \frac{1}{2}\sigma^2 \sup_{q(t)\in[\alpha,\beta]} \left(V_{SS} + V_{YY}q^2(t) + 2q(t)V_{SY}\right) \tag{7}$$

In [29] it is shown that the optimal strategy is

$$q^v(t) = \beta \iff Y(t) < \frac{\alpha+\beta}{2}S(t), \quad q^v(t) = \alpha \iff Y(t) > \frac{\alpha+\beta}{2}S(t). \tag{8}$$

Considering a *Passport call Option* we have that the optimal strategy is

$$q^p(t) = 1 \iff Y(t) \leq 0, \quad q^p(t) = -1 \iff Y(t) > 0.$$

When the price is above the strike price of the option, the executive "dismisses" the risky activity in order to invest in the safe activity; if the price is below the strike price of the option, the executive invests in the risky activity.

Considering a *Vacation call Option* we have that the optimal strategy is

$$q^v(t) = 1 \iff Y(t) \leq \frac{1}{2}S(t), \quad q^v(t) = 0 \iff Y(t) > \frac{1}{2}S(t).$$

When the price minus the strike price is below half the value of the risky activity, the executive adopts the risky activity; if the opposite holds, the executive adopts the safe activity.

These contingent claim contracts induce a behavior different from that predicted by the Black&Scholes paradigm: an executive will adopt a risky activity only if the asset price is below the strike price (passport option) or if it is smaller than the strike price plus half the value of the risky activity (vacation option). Allowing executives to switch between a safe policy and a risky policy, they behave asymmetrically as far as risk is concerned: when things go well they reduce risk; when things do not go well, they increase risk. Let $\alpha = 0$, greater is the percentage of the company that can be managed according to the risky policy, larger is the region $(Y(t) \leq \frac{\beta}{2}S(t))$ characterized by the risky policy, see figure 0. Note that the upperbound of interval $[0, K + \frac{\beta}{2}S]$ charcaterized by the risky project as optimal strategy is an increasing function of β and S, see also figure 3. The reason for this result is due to the fact that a large β and/or S increase the volatility of the company asset when the risky project is chosen: because of the option payoff, a higher volatility of the risky project renders more profitable it and the executive will adopt it for higher values of the asset company. Allowing the executive to manage a large part of the value of the company according to the risky project (high β), he will adopt it until the value of the company has reached a large value.

These results hold true also when the volatility is an increasing function of the asset price, see [17], or stochastic, i.e., the volatility coefficient is driven by a second factor, see [18]. Similar results for the optimal strategy holds when the constraint is on the value invested in the risky activity.

Let us consider the above problem when the executive can also control the volatility of the risky project. Let us consider equation (1) with $\gamma(t) = 0$, $\sigma(t) \in [\sigma_1, \sigma_2]$, $t \in [0, T]$. The value of the option is

$$V(t,S,X) = \sup_{\substack{q(s)\in[\alpha,\beta] \\ \sigma(s)\in[\sigma_1,\sigma_2]}} {}_{t\leq s\leq T} \left(e^{-r(T-t)}E\left[(X(T)-K)^+\middle|\mathcal{F}_t\right]\right). \tag{9}$$

We easily obtain that the strategy for q does not change and volatility is set to the maximum:

$$\begin{cases} \sigma^{opt}(t) = \sigma_2 \\ q^{opt}(t) = \alpha I_{\{Y(t)\geq\frac{\alpha+\beta}{2}S(t)\}} + \beta I_{\{Y(t)<\frac{\alpha+\beta}{2}S(t)\}}. \end{cases} \tag{10}$$

To show this fact, we use the verification Theorem and a result obtained in [16, Lemma 4.3] establishing that $V_{SS} + 2q^{opt}(t)V_{SY} + (q^{opt}(t))^2V_{YY} \geq 0$, where $q^{opt}(t)$ is given by (8) and V is the associated solution. Then it is easy to verify that the candidate solution satisfies the HJB equation associated with problem (9):

$$rV - V_t - rSV_S - rYV_Y = \sup_{q(t)\in[\alpha,\beta],\sigma(t)\in[\sigma_1,\sigma_2]} \frac{1}{2}\sigma^2(t)s^2(V_{SS} + 2q(t)V_{SY} + q^2(t)V_{YY}). \tag{11}$$

Let us now consider the case of an executive endowed with $\kappa > 0$ European options and $\lambda > 0$ stocks of the company. The executive's payoff is $\lambda X(T) + \kappa[X(T) - K]^+$ or, equivalently, $\lambda Y(T) + \kappa Y(T)^+ + \lambda K$. The value of this payoff at time t is:

$$V(t,S,X) = \lambda(y_0 + Ke^{-r(T-t)}) + \kappa e^{-r(T-t)} \sup_{\substack{q(s)\in[\alpha,\beta] \\ \sigma(s)\in[\sigma_1,\sigma_2]}} {}_{t\leq s\leq T} E\left[Y(T)^+\right]. \tag{12}$$

The problem can be reduced to the previous one obtaining the optimal strategy (10). The above result also holds true when the number of options increases with the performance of the company. Let α be the number of options if $S(T) \geq K_1 > K$ and β $(0 < \beta < \alpha)$ be the number of options if $K \leq S(T) \leq K_1$. Then the payoff is still convex and therefore we can apply the results in [14, 29] to show that the optimal policy is the one described above. The same conclusion can be drawn if the manager is compensated through a fixed wage plus a stock option.

When the executive can also manage the instantaneous dividend rate, equation (26) becomes

$$dX(t) = (r - \gamma(t))\, X(t)dt + q(t)\sigma S(t)dW(t), \tag{13}$$

where $\gamma(t)$ is the dividend rate at time t. When dividends are not credited to executives and they manage the dividend rate under the constraint $\gamma(t) \in [0, \Gamma]$, $\forall t \in [0, T]$, the optimal policy is the one obtained above plus $\gamma(t) = 0$, $\forall t \in [0, T]$. This fact is easily shown by means of the verification theorem for the HJB equation (11) with $(r - \gamma(t))YV_Y$ instead of rYV_Y. This result does not hold if the executive is credited for the dividends.

The welfare effect of stock option plans is strongly disputed. In particular, an important point is the following: with a stock option, does the executive undertake projects with negative premium, i.e., a drift lower than the risk free rate?

To answer this question we consider the following dynamics for the value of the risky project:

$$dS(t) = S(t)(r + p)dt + \sigma S(t)dW(t)), \tag{14}$$

where p is a constant representing the premium of the project. Note that an executive aiming to maximize the expected value of the asset at time T will never chose the risky project with $p < 0$.

By considering a company whose asset is exchanged in the market we require $e^{-rt}X(t)$ to be a martingale under the risk neutral probability measure. Therefore, changing the measure through the Girsanov theorem, we are led to a new Brownian motion such that the drift for $X(t)$ is r and all the analysis developed above holds. As the option is evaluated under the risk neutral probability measure, the risk premium of the risky project does not affect agent's decisions: the manager will adopt the risky project also with a negative risk premium when the option is out of the money.

Let us consider now a firm whose asset is not exchanged in the market. In this case no change of measure is needed and as a consequence, assuming a risk neutral executive the HJB equation associated with the executive problem (6) becomes

$$\begin{cases} -rV + V_t + (r+p)SV_S + rXV_X + \sup\limits_{q(t)\in[\alpha,\beta]} \left[pq(t)SV_Y + \frac{1}{2}\sigma^2 \left(V_{SS} + V_{YY}q^2(t) + 2q(t)V_{SY}\right)\right] \\ V(T, S, Y) = Y^+. \end{cases} \tag{15}$$

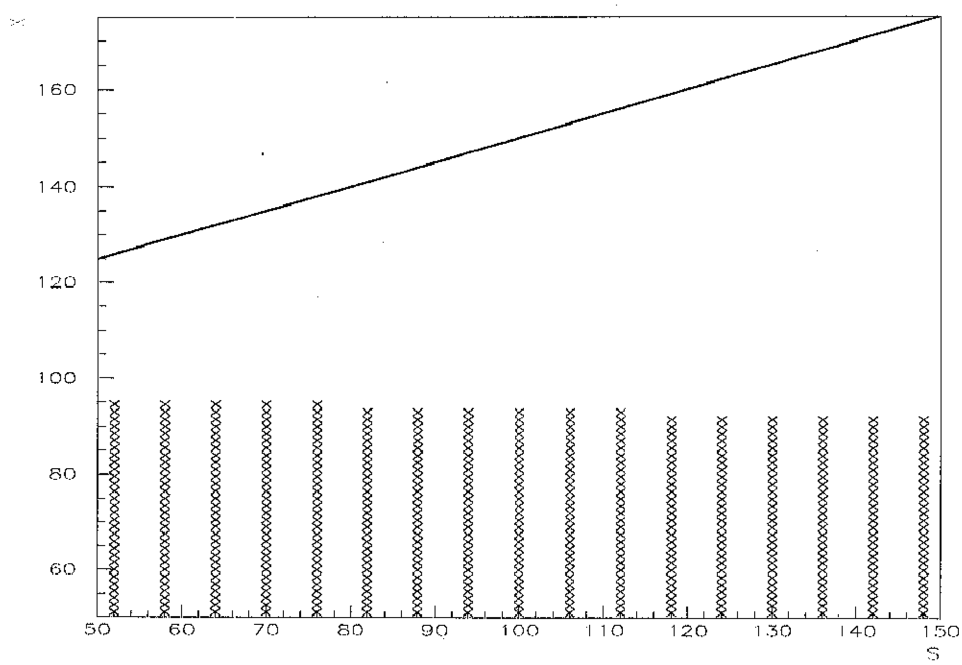

Figure 1: Distribution of the ratio X/S in which $q^{opt} = 1$ for $p = -0.05$ (a) and $p = 0$ (b). The parameters values are $r = 0,\ \sigma = 0.1,\ x_0 = s_0 = 100$.

The analysis of this problem cannot be developed in closed form, we reply upon a numerical analysis as described in Appendix A. The analysis shows that the risky project with $p < 0$ is adopted by the executive endowed with a call option, see figure 1. So an executive option has negative welfare effect: the executive adopts projects with a present value smaller than the risk free policy. As shown in figures 1, the executive adopts the risky project in a region smaller than that associated with a risky project characterized by the risk free rate as drift. The executive adopts the risky project when the ratio X/S is low, i.e., X small and S large. So the executive will adopt the risky project when its variance is high and the company does not perform well. Differently to what is observed with $p = 0$, the region characterized by the risky strategy is decreasing in S. When the project is charcaterized by a drift smaller than the risk free rate, an executive option turns oout to be very risky.

The analysis changes completely when the manager is endowed both with shares and options, i.e., his payoff is $\alpha X(T) + \beta(X(T) - K)^+$ ($\alpha > 0,\ \beta > 0$). In this case it is shown throufggh simulations that the manager never adopts the risky policy if $p < 0$.

Introducing a bankruptcy condition, the analysis does not change too much. Let us assume that when $X = 0$ is reached the company is closed and the manager endowed with a call option on the company asset gets nothing. The problem faced by the manager is the one in (6) with the bankruptcy condition, the HJB equation is (7) with the boundary condition

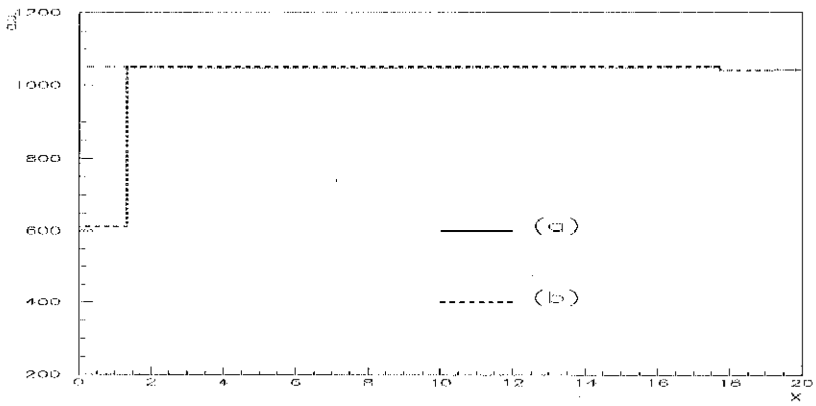

Figure 2: Distribution of the values of X in which $q^{opt} = 1$ when bankruptcy is included (solid line) and when it is not (dashed line) The parameters values are $r = 0,\ \sigma = 0.1,\ x_0 = 10,\ s_0 = 100$.

$V(t, S, 0) = 0$. The option is a *knocked out passport barrier option.* Using a numerical approach, we evaluate the manager strategy in a case in which $X(0) = 10$ and $\sigma = 0.1$. The result is illustrated in Figure 2: only for very low value of X the manager will not adopt the risky incvestment, otherwise the policy is the one depicted in case of the standard option.

4 Non-traditional options

In this section we analyze non-traditional stock options considered in [21]. We will consider a risky project starting from $S(0) = 100$ with a volatility $\sigma = 0.2$ and $r = 0$. The strike price will be set to $K = 100$. The maturity of the granted stock option is ten years. Figure 3 shows the strategy that the executive would follow at $t = 5$ years, as a function of the couple (X, S) if he holds a traditional executive stock option, evaluated with our numerical scheme. In all the options, the possibility of bankruptcy is implied, i.e. if the option price reaches zero the manager does not receive anything.

4.1 Premium stock option

The premium stock option is a traditional stock option with a strike price higher than the asset price at the day of granting, which we set to $K' = 150$. Without bankruptcy, the executive's strategy and the option value can be obtained in a closed form following [29]. We plot the corresponding strategy at $t = 5$ years in Figure 12.

This type of compensation plan induces teh agent to take more risk: he adopts teh risky strategy for prices of the company whcih would not imply teh risky policy with a standard option with a strike price equal to the granting day price. The reason for this outcome can

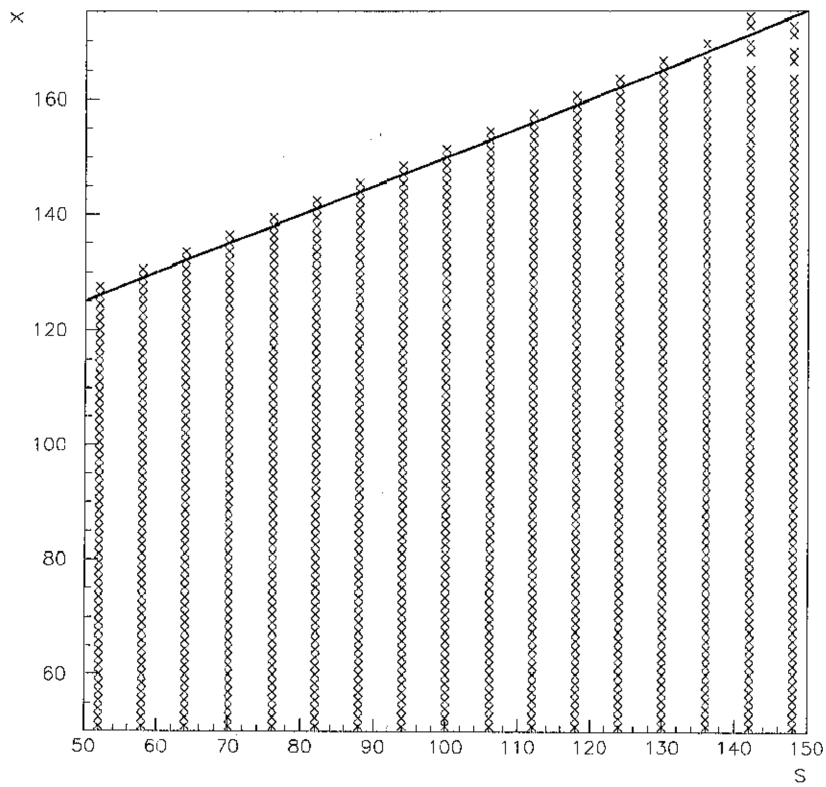

Figure 3: Points (X, S) with $q = 1$ at $t = 5$ years when the executive holds a tradional stock option with strike 150. The solid line represents the teorethical boundary between the the regions in which $q = 1$ and $q = 0$ for the traditional stock option (without bankruptcy).

be traced back to option payoff.

4.2 Performance-vested stock option

The vested option is an up-and-in barrier option; it's a traditional stock option which becomes exercisable only if the stock price hits a prescribed level $\overline{X}$. The equation to be solved numerically is 7 for $V(t, S, X)$ defined in the region $[0, T] \times [0, +\infty] \times [0, \overline{X}]$ with the boundary condition:

$$\begin{cases} V(T, S, X) = 0 \\ V(t, S, \overline{X}) = VC(t, T, \sigma, K, S, \overline{X}) \end{cases} \tag{16}$$

where $VC(t, T, \sigma, K, s_0, x_0)$ denotes the value of a Vacation Call at time t with maturity at T, volatility σ, strike K and strarting values $S(0) = s_0$, $X(0) = x_0$, which can be computed following [29]. The resulting strategy with $\overline{X} = 150$ is to invest in the risky project until the barrier is hit; then the policy of the traditional stock option applies. As a consequence, a

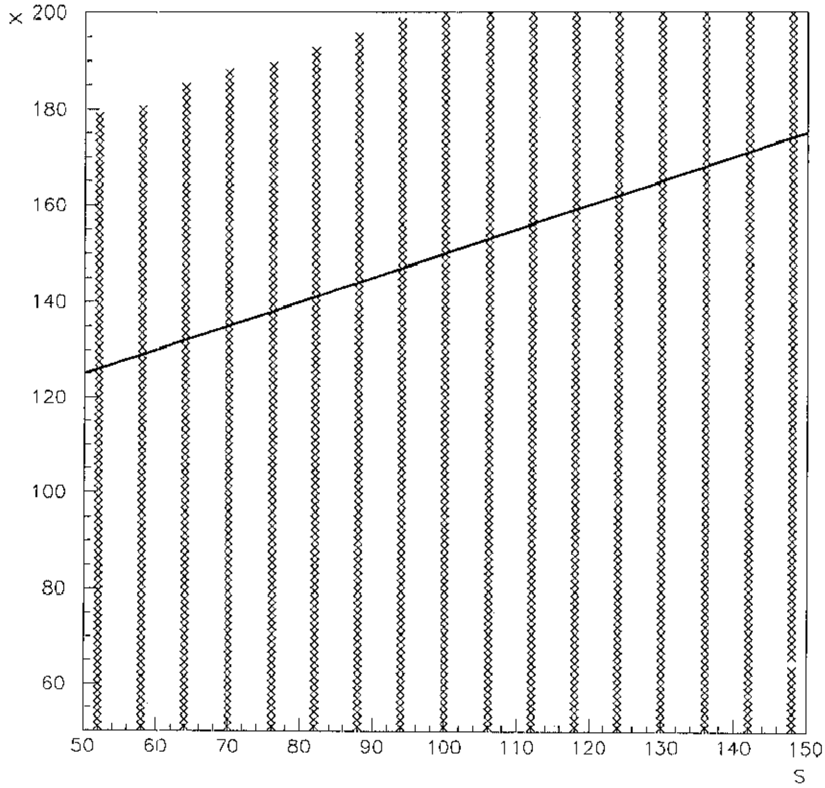

Figure 4: Points (X, S) with $q = 1$ at $t = 5$ years when the executive holds a premium stock option with strike 150. The solid line represents the teorethical boundary between the the regions in which $q = 1$ and $q = 0$ for the traditional stock option.

vested option induces the executive to adopt a very risky behavior before the barrier is hit.

4.3 Repriceable stock option

The repriceable stock option has been created to retain talented executives after a stock price decline. It's like a traditional stock option, but if the price goes under a given level X_{low} the stock price becomes $K' < K$. The equation to be solved is again 7 in the region $[0, T] \times [0, +\infty) \times [X_{low}, +\infty)$ with the boundary conditions:

$$\begin{cases} V(T, S, X) = (X - K)^+ \\ V(t, S, X_{low}) = VC(t, T, \sigma, K', S, X_{low}) \end{cases} \tag{17}$$

The resulting strategy with $X_{low} = 50,\ K' = 50$ is shown in figure 5.

The figure shows that teh manger is induced to adopt a policy riskier than the one associated with teh standard option. The rationale for this result is very simple: as there is an opportunity of repricing the option the manager does not care of a bad performance.

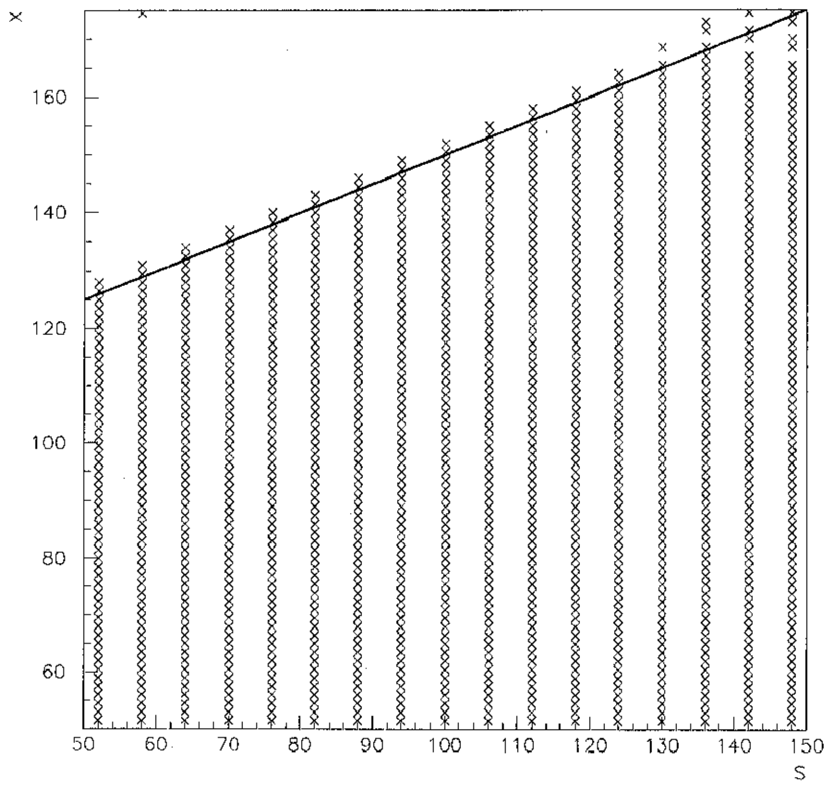

Figure 5: Points (X, S) with $q = 1$ at $t = 5$ years when the executive holds a repriceable stock option with the strike resetted to 50 if the stock price falls to 50. The solid line represents the teorethical boundary between the the regions in which $q = 1$ and $q = 0$ for the traditional stock option.

4.4 Purchased stock option

In the purchased stock option, the executive has to prepay a fraction α of the strike price if the option expires out of the money, the executive looses this amount of money. The evaluation of this stock option plan can be addressed trough equation 7 with the final payoff given by:

$$V(T, S, X) = \begin{cases} X - K & for\ X \geq K \\ -\alpha K & for\ X < K \end{cases} \tag{18}$$

The resulting strategy with $\alpha = 0.1$ is shown in figure 6

In this context teh executive will play a less aggressive policy. The region characterized by the risky strategy is smaller than the one associated with the classical option.

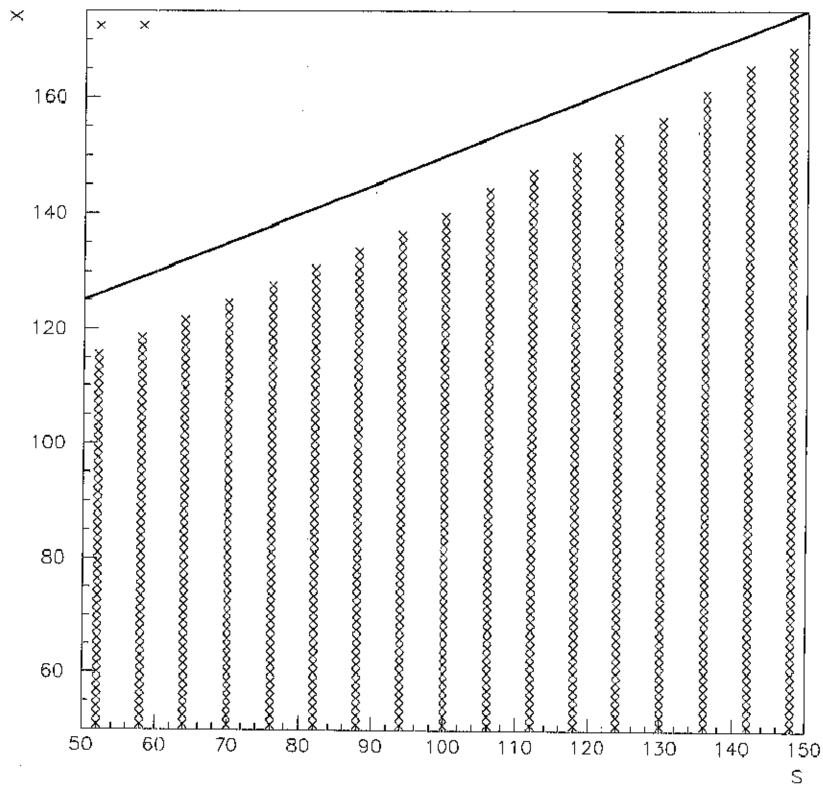

Figure 6: Points (X, S) with $q = 1$ at $t = 5$ years when the executive holds a purchased stock option with a prepay of 10% of the strike price. The solid line represents the teorethical boundary between the the regions in which $q = 1$ and $q = 0$ for the traditional stock option.

4.5 Reload stock option

With the reload stock option the manager has the right, after exercising the option, to "reload" the option paying it with shares of stock (...). We make the simplyfing hyphotesys that the manager can exercitate early only at a pre-fixed date $T_1 < T$. The equation to be solved is 7 in the region $[0, T_1] \times [0, +\infty], [0, +\infty]$ with the boundary condition:

$$V(T_1, S, X) = \begin{cases} VC(T_1, T, \sigma, K, S, X) & for\ X < K \\ X - K + \frac{K}{X} VC(T_1, T, \sigma, X, S, X) & for\ X \geq K \end{cases} \tag{19}$$

The resulting strategy with $T_1 = 5$ years is shown in figure 7.

4.6 Indexed Option

In the setting described in Section 2, we consider an indexed option of the type analyzed in [22]. The payoff of the contract at time T is

$$[S(T) - \lambda Z(T)]^+, \ \lambda > 0,$$

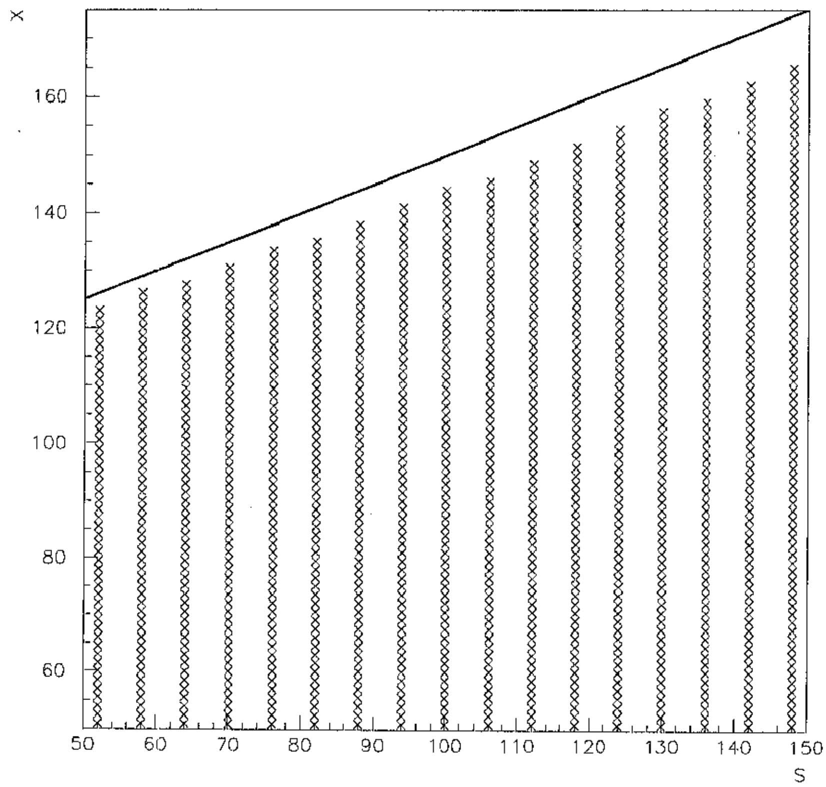

Figure 7: Points (X, S) with $q = 1$ at $t = 2.5$ years when the executive holds a reload stock option, with the grant to "reload" the option at $t = 5$ years. The solid line represents the theoretical boundary between the the regions in which $q = 1$ and $q = 0$ for the traditional stock option.

where $S(t)$ is the company asset price and $Z(t)$ is the benchmark of the manager's compensation (index), the two state variables evolve according to the following system of stochastic differential equations:

$$\begin{aligned} dS(t) &= rS(t)dt + S(t)(\sigma_{11}(t)dW_1(t) + \sigma_{12}(t)dW_2(t)) \\ dZ(t) &= rZ(t)dt + Z(t)(\sigma_{21}(t)dW_1(t) + \sigma_{22}(t)dW_2(t)). \end{aligned} \tag{20}$$

The executive is endowed with an indexed option and can manage the volatility of the stock price $(\sigma_{11}(t),\ \sigma_{12}(t))$, under some constraints. The executive maximizes the value of the contingent claim contract. In [27] it is shown that the executive attains his goal by setting $\sigma_{11}(t) + \sigma_{12}(t)$ to the maximum under the constraint. Therefore the results obtained in Section 2 for a classical call option are confirmed. In the above paper it is shown that when the executive can manage the correlation between the index and the stock of the company, the maximum of the value of the contract is attained by setting the correlation between $S(t)$ and $Z(t)$ constant and equal to -1. When the executive is remunerated by the performance of the asset price with respect to a benchmark, he has an incentive to have an asset negatively correlated with the benchmark. Moreover, it is easy to verify through the verification theorem for the HJB equation associated to the problem that the dividend rate should be set to be equal to the minimum allowed by the constraints.

Let us consider now the case of an executive who can manage the value of the company $X(t)$ by adopting a risk free policy or a risky policy such that the value of the company follows the index evolution. The value of the company evolves as

$$dX(t) = q(t)dZ(t) + r\left(X(t) - q(t)Z(t)\right)dt = rX(t)dt + q(t)\sigma Z(t)dW(t), \;\; X(0) = x_0 \quad (21)$$

where

$$dZ(t) = rZ(t)dt + \sigma Z(t)dW(t)$$

and the payoff is

$$[X(T) - Z(T)]^+.$$

As above we assume that $q(t) \in [\alpha, \beta]$. Set $Y(t) = X(t) - Z(t)$, it is easy to show that the optimal management of the company can be reduced to that of the passport option yielding the optimal policy described in (8). As a consequence, when $\alpha = 0$ we have

$$q^{opt} = 0 \iff X(t) - Z(t) > \frac{\beta}{2}Z(t), \;\; q^{opt} = \beta \iff X(t) - Z(t) \leq \frac{\beta}{2}Z(t).$$

If the asset of the company is well above the benchmark, the executive manages the company adopting the risk free policy, otherwise he follows the index amplifying his dynamics. The interval $[0, Z(1 + \beta/2)]$ describing the region for which it is optimal to follow teh index is increasing in β and Z, see figure... As in the non indexed option, this phenomenon is due to the fact that the option payoff is asymmetric. As in the non indexed otpion, below the benchmark it is always optimal to adopt the risky policy; the region above the benchmark characterized by the risky policy is increasing in Z and β.

Let us consider now the case of an executive who can adopt a risky activity ($S(t)$) which is not perfectly correlated to the benchmark. We assume that the benchmark evolves as:

$$dZ(t) = rZ(t)dt + \overline{\sigma}Z(t)dW_1(t).$$

and the risky activity evolves as:

$$dS(t) = rS(t)dt + \rho\sigma S(t)dW_1 + \sqrt{1-\rho^2}\sigma S(t)dW_2(t), \quad (22)$$

where W_1, W_2 are two independent Brownian motions. The company price evolves as follows:

$$\begin{aligned} &dX(t) = q(t)dS(t) + r\left(X(t) - q(t)S(t)\right)dt = \\ &rX(t)dt + q(t)[\rho\sigma S(t)dW_1(t) + \sqrt{1-\rho^2}\sigma S(t)dW_2(t)], \;\; X(0) = x_0, \end{aligned} \quad (23)$$

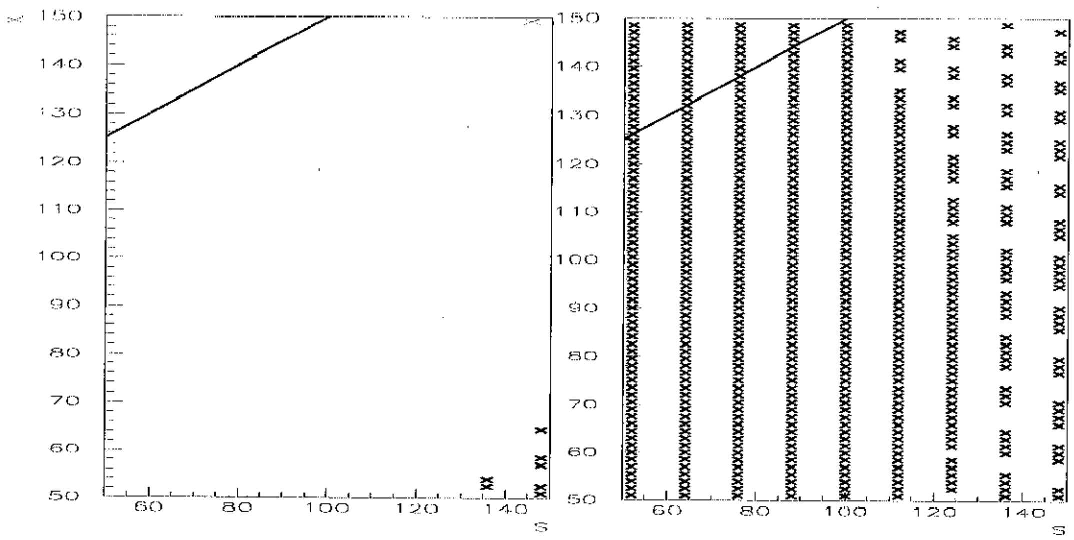

Figure 8: Points (X, S) with $q = 1$ at $t = 5$ years and $Z = 100$ when the executive holds an indexed stock option with $\rho = 0.75$ (left) and $\rho = -0.75$ (right). The solid line represents the teorethical boundary between the the regions in which $q = 1$ and $q = 0$ for the traditional stock option.

the constraint on the startegy is $q(t) \in [\alpha, \beta]$, the executive's payoff is as above and the HJB equation reads:

$$-rV + V_t + rZV_Z + rSV_S + rXV_X + \tfrac{1}{2}Z^2\overline{\sigma}^2V_{ZZ} + \tfrac{1}{2}\sigma^2S^2V_{SS} + \rho\sigma\overline{\sigma}SZV_{SZ} + \\ + \sup_{q(t)\in[\alpha,\beta]}\left(\frac{1}{2}\sigma^2S^2V_{XX} + q\sigma^2S^2V_{SX} + q\rho\sigma\overline{\sigma}SZV_{ZX}\right) = 0. \quad (24)$$

Figure 8 shows the strategy of the executive in the cases in which the index and the risky activity are correlated and anti-correlated, showing that the manager adopt the risky activity when the correlation is negative. This result confirms teh above findings.

4.7 A Comparison

We can follow [21]

We show value comparison between different options in figure 13

5 The Digital Option

A less sophisticated compensation scheme for the manager is provided by the so called "bonus":

$$\begin{cases} \alpha_1, & X(T) \geq K \\ 0, & X(T) < K. \end{cases} \quad (25)$$

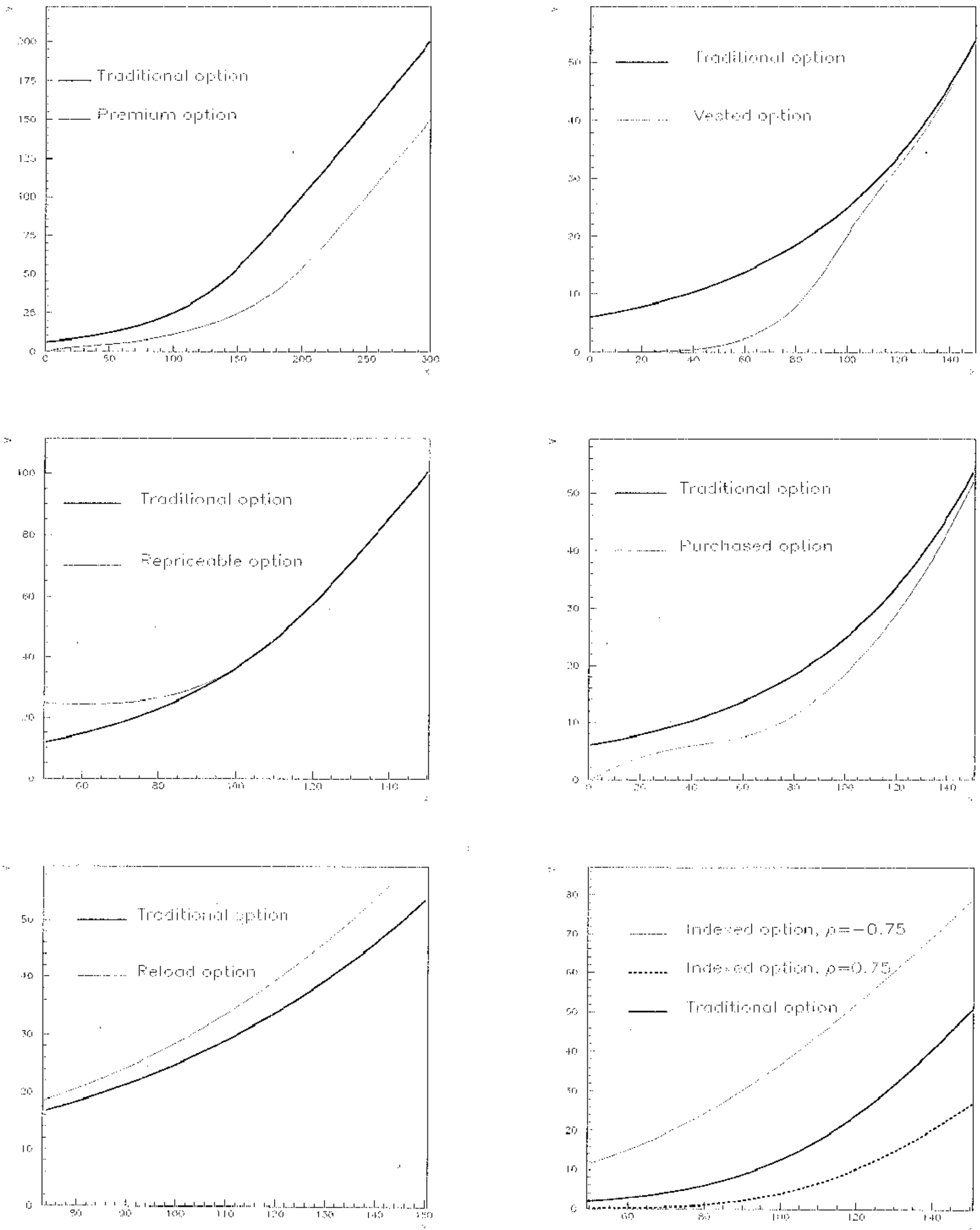

Figure 9: Valori delle opzioni in confronto al valore dell'opzione tradizionale

The manager receives the bonus (fixed amount of money) $\alpha_1 > 0$ only if the value-price of the company is above a certain value, otherwise the manager gets nothing. Note that the payoff is not convex, therefore results used in Section 2 do not apply.

We consider the setting as in Section 3 working under the risk neutral probability measure. Let $f(t)$ the value of the company run at time t according to the risky policy (i.e., $f(t) =$

$q(t)S(t)$), the asset price evolves as follows:

$$dX(t) = rX(t)dt + f(t)\sigma dW(t), \ X(0) = x_0, \tag{26}$$

under the constraint that $X = 0$ is an absorbing barrier, as in the last section. The goal of the manager (maximize the value of the compensation plan) is equivalent to maximize the prabability of reaching α_1 by time T. This type of problem has been addressed in [23, 6].

The goal of the manager is

$$\sup_{f(s)\ t\leq s\leq T} \mathbf{Pr}\left[X(T) \geq K|\mathcal{F}_t\right]. \tag{27}$$

Let $\Phi(\cdot)$ and $\phi(\cdot)$ be the cumulative distribution function and the probability density function of a standard normal variate respectively, the optimal startegy for the manager is

$$f(t) = \frac{1}{\sigma\sqrt{T-t}} Ke^{-r(T-t)}\phi(\Phi^{-1}(\frac{X(t)}{K}e^{r(T-t)})). \tag{28}$$

Being $\Phi^{-1}(u) = \infty$ for $u > 1$ we have that $f(t) = 0$ if $x(t) > Ke^{r(t-T)}$. If the executive can reach K safely he does not invest in the risky project. Thwe value of the bonus is

$$\begin{cases} Ke^{-r(t-T)}, & X(t) \geq Ke^{r(t-T)} \\ x(t), & X(t) \leq Ke^{r(t-T)}. \end{cases} \tag{29}$$

The numerical solution for the strategy with $K = 100, \alpha_1 = 30$ is plotted in figure 10.

6 Non Convex Payoff

The stock option plans analyzed in the above sections are characterized by a convex payoff. Nonconvexities are introduced in the executive's payoff to penalize him for a bad performance or to limit his compensation from above.

The second goal can be pursued by means of the following stock options plan:

$$\begin{cases} [X(T) - K]^+, & X(T) \leq \overline{X} \\ \overline{X} - K, & X(T) > \overline{X}, \end{cases} \tag{30}$$

where X is the asset price and $0 < K < \overline{X}$. This type of payoff is built to provide an incentive for the executive up to $\overline{X}$.

Again we consider the case of a manager who can decide a safe riskless policy and a risky policy evolving as in Section 3. The constraint on the policy is $[0, 1]$. Being the payoff no convex, results in [29] cannot be applied. We tackle this problem by formulating an implicit solution, then evaluating the solution numerically.

Analyzing the region characterized by adoption of the risky project, we have that it is contained in the one associated with a classical option. Obviously when $X \geq \overline{X}$ the agent does not take any risk.

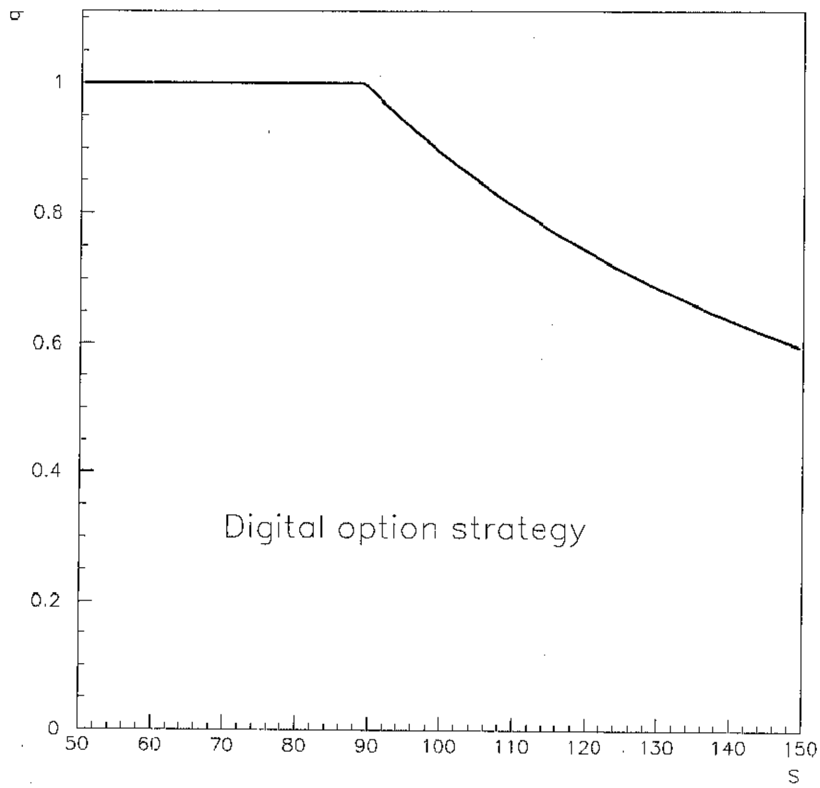

Figure 10: Strategy q as a function of S when $X < K$ for a digital option. When $X > K$ we have $q = 0$

The HJB equation is given by (7) with $Y = X$ and the terminal condition

$$V(T,S,X) = \begin{cases} (X-K)^+ & X \leq \overline{X} \\ \overline{X} - K & X > \overline{X} \end{cases} \tag{31}$$

Using the verification theorem, we have that the solution $V(t,S,X)$ to the problem (6) satisfies the PDE

$$rV - V_t - rsV_S - ryV_X = \frac{1}{2}\sigma^2 s^2 (V_{SS} + 2q_{opt}(t)V_{SX} + q_{opt}^2(t)V_{XX}), \tag{32}$$

with the terminal condition (31). The optimal strategy is given by:

$$q_{opt}(t) = \begin{cases} 1 & if\ V_{XX} > 0, V_{SX} > 0\ or \\ & V_{XX} > 0, V_{SX} < 0, V_{XX} > -2V_{SX}\ or \\ & V_{XX} < 0, V_{SX} > 0, -V_{XX} < V_{SX} \\ 0 & if\ V_{XX} < 0, V_{SX} < 0\ or \\ & V_{XX} > 0, V_{SX} < 0, V_{XX} < -2V_{SX} \\ -\frac{V_{SX}}{V_{XX}} & if\ V_{XX} < 0, V_{SX} > 0, -V_{XX} > V_{SX}. \end{cases} \tag{33}$$

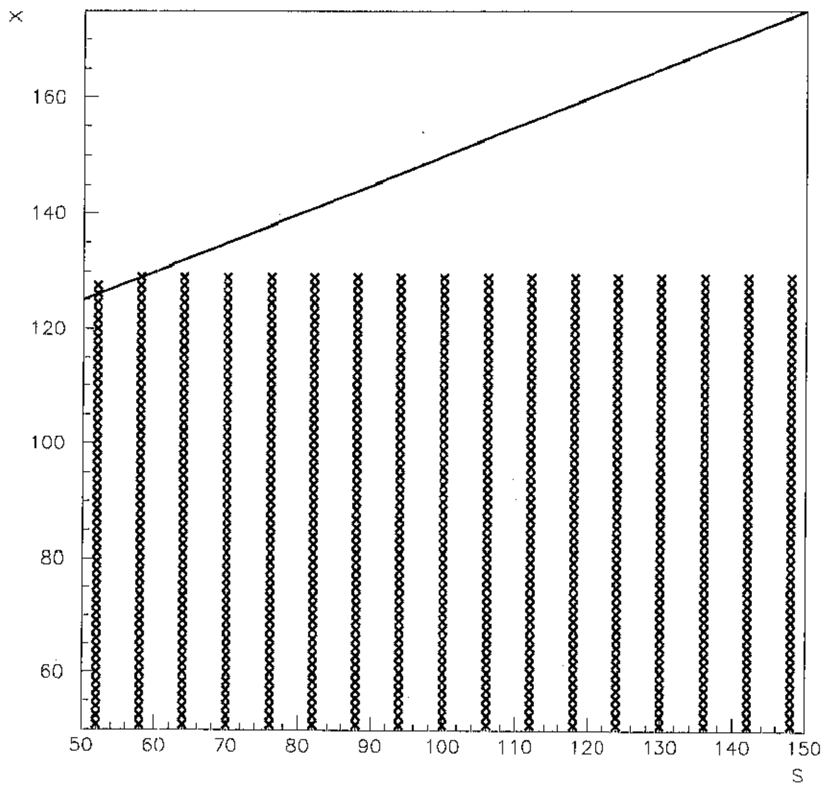

Figure 11: Points (X, S) with $q = 1$ at $t = 5$ years when the executive holds a call spread stock option with strike $\overline{X} = 150$. The solid line represents the teorethical boundary between the the regions in which $q = 1$ and $q = 0$ for the traditional stock option (without bankruptcy).

We can introduce nonconvexity in the stock option plan by penalizing the executive in case of bad performance and cutting the fixed wage when the performance is not satisfactory (). The following payoff is typically obtained:

$$\begin{cases} [X(T) - K]^+, & X(T) \geq K_1 \\ B, & X(T) < K_1, \end{cases} \tag{34}$$

where B is a negative constant. As for the previous payoff, we evaluated the solution numerically with $B = 10, \alpha = 0, K = 100, K_1 = 130$. The HJB equation is (32)-(33) with the above final condition. This solution is plotted in figure ??.

The value of the option (30) is plotted in figure ?? as a function of the initial value of the asset $X(0)$, with the parameter values indicated in the figure's caption.

7 Conclusion

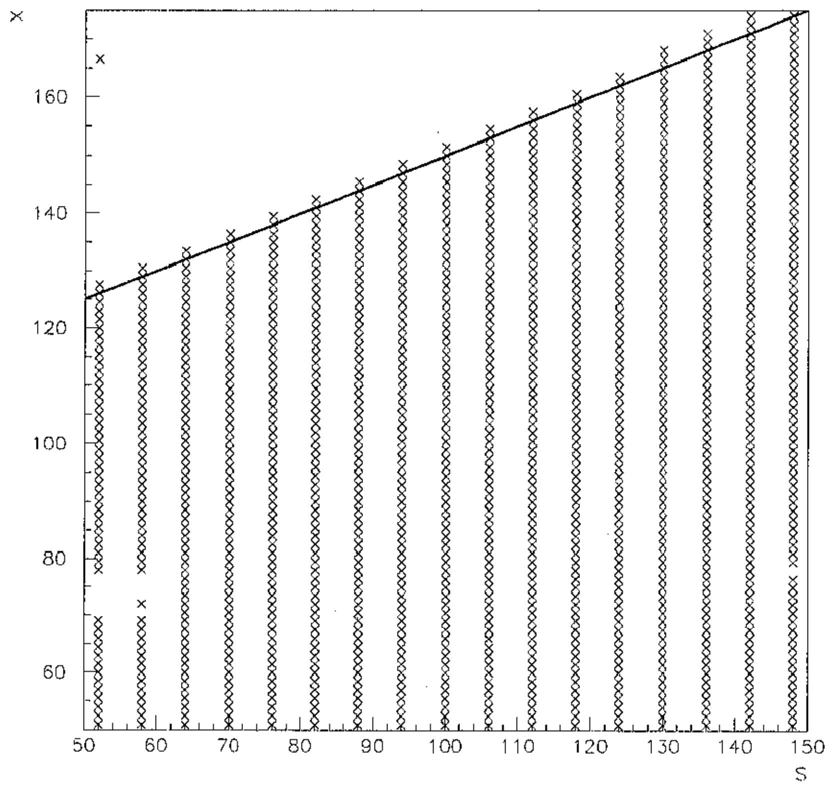

Figure 12: Points (X, S) with $q = 1$ at $t = 5$ years when the executive holds a penalty stock option with $B = -10$, $K_1 = 70$. The solid line represents the teorethical boundary between the the regions in which $q = 1$ and $q = 0$ for the traditional stock option (without bankruptcy).

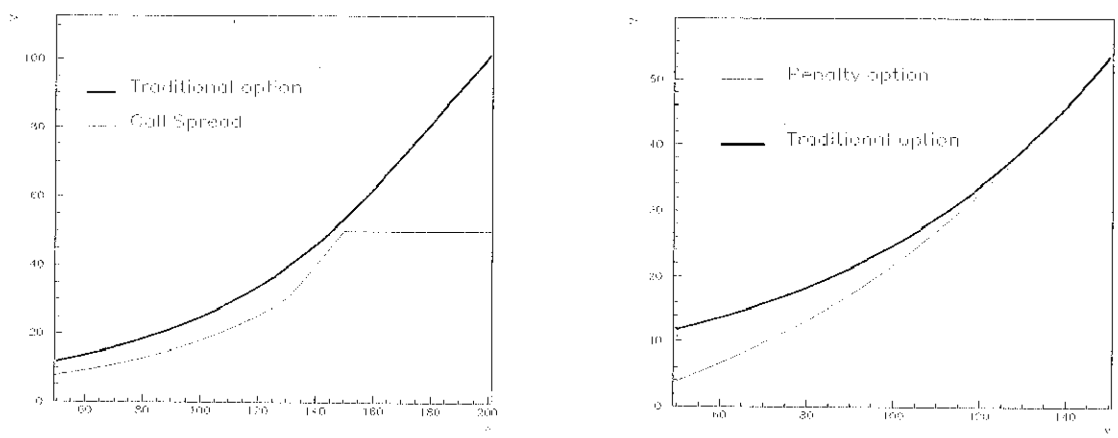

Figure 13: Valori delle opzioni non-convex tradizionale

A Appendix

To handle the PDE (32) with the optimal solution (33) we define a uniform mesh:

$$t_k = k\Delta_t, \qquad k = 0, \ldots, N$$
$$s_i = i\Delta_S, \qquad i = 0, \ldots, M$$
$$x_j = x_0 + j\Delta_X, \qquad j = 0, \ldots, L$$

where $t_N = T$, $S_M = S(0) + e^{3\sigma_2\sqrt{T-t}}$, $x_{0,L} = X(0) \mp e^{4\sigma_2\sqrt{T-t}}$. We will use the shorthand notation $V_{i,j,k} = V(s_i, x_j, t_k)$ as well as $q_{i,j,k} = q(s_i, x_j, t_k)$.

When discretizing equation (32) we have to deal with a boundary condition (31) which is not differentiable in 0. This implies that using an explicit finite-difference method leads to instability near 0. On the other hand, using a mixed implict/explicit finite-difference discretization as in [3], we would obtain a band-diagonal algebric system of $(M+1)(L+1)$ equations which implies a considerable burden of numerical computation. In order to avoid these difficulties, we will adopt an implicit discretization *a-là* Crank-Nicholson (see [28]) when computing V_{XX} and an explicit discretization when computing the other derivatives. We discretize (32) as:

$$\begin{aligned}
&rV_{i,j,k} - \frac{V_{i,j,k+1} - V_{i,j,k}}{\Delta_t} - rS\frac{1}{2\Delta_S}(V_{i+1,j,k+1} - V_{i-1,j,k+1}) \\
&-rX\frac{1}{2\Delta_X}(V_{i,j+1,k+1} - V_{i,j-1,k+1}) \\
&= \frac{1}{2}\sigma^2_{i,j,k}s^2\Big[\frac{1}{\Delta_S^2}(V_{i+1,j,k+1} - 2V_{i,j,k+1} + V_{i-1,j,k+1}) \\
&+\frac{1}{2\Delta_X^2}q^2_{i,j,k}(V_{i,j+1,k} - 2V_{i,j,k} + V_{i,j-1,k} + V_{i,j+1,k+1} - 2V_{i,j,k+1} + V_{i,j-1,k+1}) \\
&+\frac{1}{2\Delta_S\Delta_X}q_{i,j,k}(V_{i+1,j+1,k+1} - V_{i+1,j-1,k+1} - V_{i-1,j+1,k+1} + V_{i-1,j-1,k+1})\Big]
\end{aligned} \tag{35}$$

In the equation (35) the terms with the index $k+1$ are known, and those with index k are unknown; (35) is a so-called tri-diagonal system of linear equations, which can be solved with standard procedures, e.g. following [26]. The discretization (35) is complemented by setting proper boundary conditions for V and by the following algorithm to find the optimal strategy $q_{i,j,k}$ at the $k-$th step.

1. Set $q_{i,j,k} = q_{i,j,k+1}$. This is only a guess.
2. Solve the tridiagonal system (35) and compute the derivatives of V at $t = t_k$. At this step, impose proper boundary conditions to V.
3. Compute $q_{i,j,k}$ using (33). Come back to Step 2 until q remains unchanged by Step 3 (with a tolerance of 0.1), for a maximum of 5 iterations, then accept the last value of q.

To evaluate the reliability of this algorithm, we use it to evaluate a vacation call, whose value can be computed with a closed formula given in [29]. We choose $X(0) = S(0) = K = 100$, $\sigma = 0.1$, $r = 0$ and we obtain a theoretical value of 3.987. Using a mesh with $N = 50$, $M = 50$, $L = 200$ we obtain a value of 3.962. We judge this precision to be sufficient for our purposes, and in the following we will adopt this mesh.

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