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1 Introduction

Two important issues of modern economic activity are: *"the private and social incentives to bring about technological innovations"* and *"how already existing innovations are adopted in the marketplace"* (Tirole [38] (1990), II, Ch. 10). In this paper we are concerned with the latter issue.

Modern society has been characterized by a constant supply of technological innovations adoptable in industry, agriculture, services, or other branches of economic activity. Despite that, the rate at which such innovations have been adopted, through a mass production, seems to follow a slower pace[1]. Quoting Tirole ([38] (1990), p. 401): *"Few innovations are adopted instantaneously."* Then the question is: What explains such a delay? The existing theoretical literature on innovation adoption propose several models to justify this phenomenon[2]. These models consider, singly or jointly, various economic aspects such as: a) *market structure* (see, among others, Kamien & Schwartz (1972), Reinganum (1981), Jensen (1982), Mamer & McCardle (1987), Lee & Wilde (1980), Barzel (1968).); b) *externalities*, *spillover effects*, and *learning by doing* (Jovanovic & Lach (1989), Jovanovic & MacDonald (1994), Mason & Weeds (2001)); c) *uncertainty* surrounding the innovation profitability (Rosenberg (1976), Balcer & Lippman (1984), Grenadier & Weiss (1997), Farzin et al. (1998), Bessen (1999), Dosi & Moretto (2000)).

This paper analyzes, in a continuous time setting, the monopolistic choice of innovation[3] adoption as well as the monopolist pricing policy. The firm

[1] For historical accounts of the slow pace of adoption of technology innovations see, for example, Mansifiel (1968).

[2] For a review we may refer to Bridges et al. (1991) and Reinganum (1989).

[3] Here by innovation it is meant *product* innovation.

produces and sells a *durable* good and no potential entrant is threating the monopolist, hence the decision is not affected by strategic considerations. The agent faces an uncertain market demand (this, in turn, implies that the existing product as well as the new one exhibit uncertasin returns) and the decision of adopting is *irreversible*. Moreover, time horizon is infinite and the thechnological change is modeled as a continual process[4].

Our model differs from the existing literature in the following aspects. First, almost no previous work (at least among the once we are aware of) is concerned with the pure monopoly industry innovation adoption issue[5]. The closest work to this one (at least in the model structure) is the papar by Kalish (1985). Nevertheless, the latter is not set in a irreversible investment framework and, moreover, its stochastic structure is far simpler than ours[6]. On the other hand, the state dynamics in that paper is richer than ours.

Second, previous works usually assume future profitability entirely outside the firm control (Cf. Balcer & Lippman (1984), Grenadier & Weiss (1997), Farzin et al. (1998)). In this setting, the policymaker obseves this exogenous evolution of technology profitability and choose the best time to adopt, that is the problem become an optimal stopping problem. In contrast to this point of view, in our model we explicitly consider the adoption strategy followed by the monopolist as a control variable. Moreover, we assume that existing product profitability is influenced by its diffusion which, in turn, is determined by the firms's pricing policy. This point of view has not been considered in previous works.

Finally, as far as we know, in economic literature we have not found a model dealing, in a continuos time stochastic dynamic framework, with the monopolist *innovation adoption & pricing* issue.

Given this setting, we conclude, consistently with empirical evidence, that an adoption delay may take place. However, the motivation that our analysis provides as the major reason for this phenomenon is deeply different from the previous one in that adoption delay is mainly a matter of **residual market demand**. In other words, as long as market demand for the existing product is above a certain level we do not adopt innovation, whereas if market demand falls below this level then innovation is adopted as much as to catch up the aforesaid level. Therefore, our analysis suggests that innovation adoption

[4]Much of the literature models the innovation adoption decision as a once-and-for-all event (Kamien & Schwartz (1972), Reinganum (1981), Jensen (1982), Mamer & McCardle (1987), Lee & Wilde (1980), Grenadier & Weiss (1997),Dosi & Moretto (2000), Mason & Weeds (2001)).

[5]Cf. Reinganum (1981), Jensen (1982), Mamer & McCardle (1987), Lee & Wilde (1980), Balcer & Lippman (1984), Grenadier & Weiss (1997), Farzin et al. (1998).

[6]Kalish himself observe that "the effect of uncertainty is.. to rescale the price axis".

is mainly used to prevent market demand to fall too low regardless when such innovation was already available. Further, the optimal price path is, consistently with the previous literature (cf. [33], [16]), a decreasing function of the product diffusion, that is prices decrease as product diffuses. Moreover, such a decreasing trend reverses when monopolist introduce innovation, that is new products prices are higher than those of the existing ones.

From a mathematical point of view the model is formulated as a continuous time stochastic control problem[7]. In literature the most common approaches used to solve such problems are the Pontryagin's maximum principle and Bellman's dynamic programming (for a thorough analysis of the connection between the two approaches we refer the reader to [39]). We approach the problem by using Bellman's dynamic programming.

In particular, our model falls in the so called *regular-singular* stochastic control (cf. [11], [12], [21], [37], [26]). In fact the control variable represented by the prices acts on the state dynamics in a classical way, that is the cumulative displacement of the state caused by the control is the integral of the control process itself, and so is absolutely continuous with respect to Lebesgue measure; whereas the control variable relative to the effect of innovation adoption on the state dynamics is additive and it is a real finite variation process which may be (and actually the optimal one is) singular with respect to the Lebesgue measure. As a result the HJB equation for the value function is a non-linear free boundary problem (or, equivalently, a variational inequality).

The linear-concave structure of the model would suggest that the so called "principle of smooth fit"[8] (cf. [3], [25]) should apply. Nevertheless, no previous result we are aware of is applicable to our case. Here is our departure from the previous literature. In fact, by using ad hoc methods, we prove that the value function is twice continuously differentiable and, as a result, we show that the principle of smooth fit applies to our case too.

[7]Here, by a stochastic control problem we mean a completely observed control problem with state equation of Itô type and with a cost functional of the Bolza type.

[8]It si well known that the value function which results from absolutely continuous control of a nondegenerate diffusion is twice continuously differentiable. The principle of smooth fit holds that this is also the case in the singular control of a nondegenerate diffusion.

2 The model, first properties, the HJB equation

Suppose at time zero a monopolist has already developed a new (or several new) generation of a *durable* good, that is our firm has the opportunity to introduce a new (or several new) version of an existing product by means of *product innovation*. The question we will address our attention to is: When and how, in a context of uncertainty, is it optimal for a risk neutral agent to adopt innovation? In order to give a reasonable answer to this question we need to formalize the environment in which the whole analysis will be carried out.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space equipped with a filtration $(\mathcal{F}_t)$ satisfying the usual condition of right continuity and augmentation by all $\mathbf{P}$-negligible sets and carrying a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion W_t. Given this setting of uncertainty, which is completely known and observed by the monopolist, we define a *market saturation index*, X_t, whose dynamics evolves over time according to the following stochastic differential equation

$$\begin{cases} X_t = x + \int_0^t f(Q(X_s, p_s))ds + \int_0^t \sigma(A - X_s)dW_s - \xi_t, & t \geq 0 \\ X_{0-} = x \in \mathbb{R} \end{cases} \tag{1}$$

where $x, \sigma \in \mathbb{R}$, $\sigma > 0$, and

$$Q_t = A - X_t - Bp_t, \qquad A, B > 0,$$

represents the market demand rate at time t. The constant A can be interpreted as the *market potential* at time zero, that is A is what the monopolist expects, at the beginning of the analysis, to sell at most. Therefore, from an economic point of view, the analysis is meaningful only for those X_t not exceeding A. Nevertheless, the particular structure of the dynamics enables us to carry out the analysis in the whole real line and then we can show that the constraint on X_t is actually satisfied if the initial condition x is such that $x \leq A$ and the control pair (ξ, p) belongs to a prescribed set. p is the unit price charged by the seller at time t. We will take $p \in \mathcal{P}$, where

$$\mathcal{P} \triangleq \left\{ p : [0, +\infty) \to \mathbb{R} : p_t \quad \begin{array}{l} \text{is } \mathcal{F}_t\text{-progressively measurable,} \\ \mathbf{E}\left[\int_0^t |p_s|^m ds\right] < \infty, \quad \text{for } m = 1, 2, 3, 4. \end{array} \right\}.$$

Moreover, we will define

$$\mathcal{P}_+ \triangleq \{p \in \mathcal{P} \ \text{ such that } p_t \geq 0, \ \forall t \geq 0, \mathbf{P} - a.s.\}.$$

The function f is increasing, positive and such that $f(0) = 0$, and accounts for the effect of past sales on the market saturation index.

The processes ξ_t is an index of the *cumulative innovation adopted* up to time t and we will take $\xi \in \mathcal{V}_+$, where

$$\mathcal{V}_+ \triangleq \left\{ \xi : [0, +\infty) \to \mathbb{R} : \xi_t \begin{array}{l} \text{non-decreasing and} \\ \text{right continuous with left limits} \\ \mathbf{P} - \textit{almost surely}, \\ \mathcal{F}_t\text{-adapted with } \xi_t = 0, \ t < 0 \end{array} \right\}.$$

Finally, for each $x \in \mathbb{R}$

$$\mathcal{A} \triangleq \mathcal{V}_+ \times \mathcal{P}$$

is the set of all *admissible controls*, whereas for each $x \leq A$

$$\mathcal{A}_+ \triangleq \{(\xi, p) \in \mathcal{V}_+ \times \mathcal{P}_+ \ \text{ such that } X_t \leq A, \ \forall t \geq 0, \mathbf{P} - a.s.\}$$

is the set of all *economic admissible controls.*

It is worth noting that the non-decreasing feature of ξ_t has a reasonable economic meaning: it accounts for the fact that once monopolist has adopted innovation he cannot go back on his decision (irreversible decision). The negative effect of ξ_t on X_t means that market enjoys (on average) new energy in adopting innovation.

In the sequel we will assume f to be a linear function of Q, $f \triangleq DQ$, for some $D > 0$ constant, which may be interpreted as a *product diffusion coefficient.* Therefore, the state dynamics evolves over time according to the following stochastic differential equation

$$\begin{cases} X_t = x + \int_0^t D(A - X_s - Bp_s)ds + \int_0^t \sigma(A - X_s)dW_s - \xi_t \\ X_{0-} = x \end{cases}, \qquad (2)$$

which, for any $(\xi, p) \in \mathcal{A}$, admits a unique strong solution[9] (see [32], Ch. V.3, Theorem 7).

It remains to formalize the firm's costs structure. These will be divided into *production costs* and *adoption costs;* the latter are exclusively related to the monopolist innovation adoption policy.

[9] We recall that since ξ_t is right-continuous and $\mathcal{F}_t$-adapted then it is progressively measurable with respect to $(\mathcal{F}_t)$ (see [19], Proposition 1.13 p.5).

First, as it is usual in this kind of literature (cf. [6], Section 2.4) we will consider the standard linear-quadratic model of the production costs: $\varphi(\cdot) + cQ_t + \gamma Q_t^2$, where $\varphi(\cdot), c, \gamma \geq 0$ but we will take

i) the fixed component of the production costs, $\varphi(\cdot)$, to be a function of the market saturation index, X_t, that is

$$\varphi(X_t) \triangleq F(\alpha - X_t)^2$$

with α, F positive constants;

ii) $c \geq 0$ constant and $\gamma \equiv 0$.

Hence, the profit rate, $\pi(X_t, p_t)$, at time t will be given by:

$$\begin{aligned} \pi(X_t, p_t) &\triangleq (p_t - c)Q_t - F(\alpha - X_t)^2 \\ &= (A + Bc)p_t - Bp_t^2 - X_t p_t + (2\alpha F + c)X_t - FX_t^2 - Ac - \alpha^2 F. \end{aligned}$$

On the parameters A, B, c, F, and α we make the following additional assumption:

$$\text{(A.1)} \qquad 4BF - 1 > 0;$$

$$\text{(A.2)} \qquad A - Bc > 0;$$

$$\text{(A.3)} \qquad 0 < \alpha < A - Bc$$

Assumptions (A.1), implies that $\pi(x,p)$ is *strictly concave* and upper bounded in (x,p).

We assume that if at time t the monopolist adopts innovation $d\xi_t$ she incurs *proportional sunk costs* $Id\xi_t$, with $I > 0$ constant. Therefore, assuming a continuous discount factor $\delta > 0$ and starting at market situation $x \leq A$, monopolist wishes to maximize her expected discounted profits

$$J_{\xi,p}(x) \triangleq \mathbf{E}\left[\int_{[0,\infty)} e^{-\delta t}[\pi(X_t, p_t)dt - Id\xi_t]\right] \tag{3}$$

over all pairs $(\xi, p) \in \mathcal{A}_+$.

This is an infinite horizon autonomous *regular-singular monotone follower* stochastic control. In fact, ξ_t, is a real finite variation process, but may be singular with respect to the Lebesgue measure as a function of time. The *value function* of our problem is given by

$$v(x) \triangleq \sup_{(\xi,p)\in\mathcal{A}_+} J_{\xi,p}(x) \tag{4}$$

In the sequel we will refer to this control problem as Problem (P).

The next assumption will be required to hold throughout the whole analysis

$$\text{(A.4)} \qquad \delta > \sigma^2 + 2D.$$

Moreover, the following condition will turn out to be sufficient to yield the classical decreasing feature of the optimal monopolist pricing path (cf. [33]),

$$\text{(A.5)} \qquad \delta > \sigma^2 + 2D(BF - 1).$$

As we have already pointed out, the analysis will be carried out in the most possible tractable way. As a result (cf. Section 1.4), we will let the state dynamics and the control variable p to move freely in the whole real line, that is the unique strong solution of (2) will be considered with $x \in \mathbb{R}$ and $p \in \mathcal{P}$. The following lemma enables us to prove the equivalence, at least for initial conditions $x \leq A$, between the problem analyzed without any restriction on X and p and the one meaningful from an economic point of view, that is the one where we require $x \leq A$ and $(\xi, p) \in \mathcal{A}_+$.

Lemma 1 *Let $x \leq A$, $(\xi, p) \in \mathcal{V}_+ \times \mathcal{P}_+$, and X_t the unique strong solution of (2). Then*

$$(\xi, p) \in \mathcal{A}_+,$$

that is $\mathcal{A}_+ = \mathcal{V}_+ \times \mathcal{P}_+$.

Proof. See Proof of Lemma 1 in [24]. ∎

By invoking Lemma 1, the following corollary can be easily proved.

Corollary 2 *Let $x \in \mathbb{R}$, and define $v_e : \mathbb{R} \longrightarrow \mathbb{R}$ as follows*

$$v_e(x) \triangleq \sup_{(\xi,p) \in \mathcal{V}_+ \times \mathcal{P}_+} J_{\xi,p}(x), \tag{5}$$

with $J_{\xi,p}(x)$ given by (3). Then for each $x \leq A$

$$v_e(x) = v(x),$$

where v is given by (4).

Proof. It is suffice to observe that for each $x \leq A$ we have $\mathcal{A}_+ = \mathcal{V}_+ \times \mathcal{P}_+$ (see Lemma 1). ∎

The following lemma will be used in the analysis, for its proof we refer to [20] Section 2.5 p. 77, or [9] Appendix B p. 397.

Lemma 3 *Let X_t the unique strong solution of (2) corresponding to the pair of controls $(\xi, p) \in \mathcal{A}$ and initial condition $x \in \mathbb{R}$. Then there exists a constant $C = C(B, D, \sigma)$ such that for each $t \geq 0$*

$$\mathbf{E}\left[|X_t|^m\right] \leq C\,|x|^m + Ct^{(\frac{m}{2}-1)}\mathbf{E}\left[\int_0^t \left(|x|^m + |p_s|^m + |\xi_s|^m\right) ds\right] e^{Ct},$$

with $m = 2, 3, 4$.

Now, the linear concave structure of the problem enables us to prove the following (for the Proof we refer to [24]).

Theorem 4 *The value function $v_e(x)$ defined by (5) is concave and continuous on $\mathbb{R}$. Moreover, there exists a positive constant C such that for each $x \in \mathbb{R}$*

$$-C(1+x^2) \leq v_e(x) \leq C. \tag{6}$$

Proof. Let $x^1, x^2 \in \mathbb{R}$. Consider two policies $(\xi^1, p^1), (\xi^2, p^2) \in \mathcal{V}_+ \times \mathcal{P}_+$ admissible at x^1 and x^2, respectively, and denote by X_t^1, X_t^2 the corresponding solutions of (2). Let $\lambda \in [0,1]$, set $x^\lambda \triangleq \lambda x^1 + (1-\lambda)x^2$, and $(\xi^\lambda, p^\lambda) \triangleq \lambda(\xi^1, p^1) + (1-\lambda)(\xi^2, p^2)$. Then the linearity of equation (2) shows that (ξ^λ, p^λ) is an admissible pair at x^λ and $X_t^\lambda = \lambda X_t^1 + (1-\lambda)X_t^2$ is the solution of (2) with $x = x^\lambda$ and $(\xi, p) = (\xi^\lambda, p^\lambda)$. Now

$$\begin{aligned}
J_{\xi^\lambda, p^\lambda}(x^\lambda) &= \mathbf{E}\int_{[0,\infty)} e^{-\delta t}[\pi(X_t^\lambda, p_t^\lambda)dt - I d\xi_t^\lambda] \\
&\geq \lambda \mathbf{E}\int_{[0,\infty)} e^{-\delta t}[\pi(X_t^1, p_t^1)dt - I d\xi_t^1] + \\
&\quad +(1-\lambda)\mathbf{E}\int_{[0,\infty)} e^{-\delta t}[\pi(X_t^2, p_t^2)dt - I d\xi_t^2] \\
&= \lambda J_{\xi^1, p^1}(x^1) + (1-\lambda) J_{\xi^2, p^2}(x^2).
\end{aligned}$$

Taking the suprema on both sides, we obtain

$$v_e(x^\lambda) \geq \lambda v_e(x^1) + (1-\lambda) v_e(x^2),$$

which proves the concavity of v_e. The continuity is an easy consequence of concavity.

The second inequality in (6) is a direct consequence of the fact that π is upper bounded. On what concerns the lower bound it suffices to prove it for $(\xi, p) \equiv (0,0)$. Now, let X_t^0 the solution of (2) with such a controls, and define $Z_t \triangleq X_t^0 - A$. Then Z_t is the unique solution of the SDE

$$Z_t = (x - A) + \int_0^t -DZ_s ds + \int_0^t -\sigma Z_s dW_s.$$

By first applying Ito's lemma to the function $f(Z_t) = (Z_t)^2$ and then taking the expectation we get

$$\begin{aligned} \mathbf{E}\left[(Z_t)^2\right] &= (x-A)^2 + (\sigma^2 - 2D)\int_0^t \mathbf{E}\left[(Z_s)^2\right] ds \\ &\leq (x-A)^2 + \left|\sigma^2 - 2D\right| \int_0^t \mathbf{E}\left[(Z_s)^2\right] ds \end{aligned}$$

An application of Gronwall inequality yields

$$\mathbf{E}\left[(Z_t)^2\right] \leq (x-A)^2 e^{|\sigma^2 - 2D|t}.$$

Observing that for a constant $C > 0$ sufficiently big

$$\pi(X_t^0, 0) \geq -C(1 + (X_t^0)^2),$$

and that

$$(X_t^0)^2 \leq 2(A - X_t^0)^2 + 2A^2$$

we conclude by using (A.4). In fact, for each $x \in \mathbb{R}$

$$v_e(x) \geq J_{0,0}(x) = \mathbf{E}\left[\int_0^\infty e^{-\delta t}\pi(X_t^0, 0)dt\right] \geq -C(1 + x^2),$$

where, we have used the same letter C to indicate, from time to time, different positive constants. The proof is complete. ∎

In the sequel, the mathematical analysis about v, defined in (4), will be carried out by considering its extension, v_e defined by (5), to the whole real line. Once the results have been obtained, we will check their economic meaningfulness by looking at its restriction on $(-\infty, A]$. Therefore, whenever in the subsequent analysis, we write $v : \mathbb{R} \longrightarrow \mathbb{R}$ we are actually referring to the extension of v, that is we are referring to v_e. We do not adopt different notations in order to avoid useless complications.

2.1 The HJB equation

For the control Problem (P) (see 4) we now wish to derive the *Hamilton-Jacobi-Bellman* (HJB) equation on the whole real line. Based on the *Dynamic Programming Principle* (see [9]), the following heuristic argument motivates the results. For the time being let us assume that $v : \mathbb{R} \longrightarrow \mathbb{R}$ is twice continuously differentiable and consider the following two cases:

First, consider the policy "*do nothing for a little while and then proceed optimally*"; then for $x \in \mathbb{R}$, and every $h > 0$ we have

$$v(x) \geq \mathbf{E}\left[\int_0^h e^{-\delta t}\pi(X_t^{0,p}, p)dt + e^{-\delta h}v(X_h^{0,p})\right],$$

where $X_t^{0,p}$ is the solution of (2) with initial condition x and controls $(\xi_t, p_t) \equiv (0, p)$. Subtracting $e^{-\delta h}v(x)$ from each side, we get

$$v(x)(1 - e^{-\delta h}) \geq \mathbf{E}\left[\int_0^h e^{-\delta t}\pi(X_t^{0,p}, p)dt\right] + e^{-\delta h}\mathbf{E}\left[v(X_h^{0,p}) - v(x)\right].$$

Now, dividing by h and letting $h \downarrow 0$ we obtain (thanks to the C^2 hypothesis made on v)

$$\delta v(x) \geq \frac{1}{2}\sigma^2(A-x)^2 v''(x) + D(A - x - Bp)v'(x) + \pi(x,p), \qquad \forall p \geq 0. \quad (7)$$

Hence,

$$\sup_{p\geq 0}\left\{\mathbb{A}^{(p)}[v](x) + \pi(x,p)\right\} \leq 0, \quad \forall x \in \mathbb{R}$$

where

$$\mathbb{A}^{(p)}[v](x) \triangleq \frac{1}{2}\sigma^2(A-x)^2 v''(x) + D(A - x - Bp)v'(x) - \delta v(x), \qquad (8)$$

where the superscript stands for the operator dependence on p.

Next, let $x \in \mathbb{R}$, $h > 0$, and consider the strategy "*jump immediately from x to $x - h$ and then proceed optimally*"; this yields

$$v(x) \geq -Ih + v(x-h).$$

Subtracting $v(x)$ from each side, dividing by h and letting $h \downarrow 0$, we get

$$-v'(x) - I \leq 0, \quad \forall x \in \mathbb{R} \qquad (9)$$

In view of the above heuristic arguments, we expect that the value function v would satisfy, at least formally, the following conditions:

$$\sup_{p\geq 0}\left\{\mathbb{A}^{(p)}[v](x)+\pi(x,p)\right\} \leq 0, \quad x\in\mathbb{R}, \tag{10}$$

$$-v'(x)-I \leq 0, \quad x\in\mathbb{R} \tag{11}$$

Now, fix $\bar{x}\in\mathbb{R}$ and assume $-v'(\bar{x})-I<0$. Then the strict inequality holds in a whole neighborhood $B_\epsilon(\bar{x})\subset\mathbb{R}$ (recall the assumption $v\in C^2$). Moreover, for every $x\in B_\epsilon(\bar{x})$ there exists $h>0$ sufficiently small, dependent of x, such that $v(x)>-Ih+v(x-h)$ and $v(x-h)\in B_\epsilon(\bar{x})$, but this means that in $B_\epsilon(\bar{x})$ we are better off if we do not adopt innovation (i.e. $d\xi=0$). In turn this implies that in $B_\epsilon(\bar{x})$ the state variable is solely controlled by p and therefore (10) has to hold with equality. Hence, for every $x\in\mathbb{R}$, $v(x)$ should satisfy, in a certain sense, the following variational inequality

$$\max\left\{\sup_{p\geq 0}\left\{\mathbb{A}^{(p)}[v](x)+\pi(x,p)\right\};-v'(x)-I\right\}=0. \tag{12}$$

In the next section we will study the extreme case of pure pricing (i.e. $\xi=0$). The study of this case is justified by the following two reasons. First, from a mathematical point of view, the subsequent analysis will heavily rely, on a comparative way, on the value function of this case in order to prove several features of the value function of the original Problem (P). Second, from an economic point of view, this case has a meaningfulness on its own. In fact, such an analysis shed light on the optimal monopolist pricing under demand uncertainty. We will find, consistently with the previous literature, that the optimal (Markovian) pricing by a monopolist is a decreasing function of the market saturation index. That is, as market saturates prices go down (cf., for instance, [33])

3 Pure pricing

We start by studying the value function of the extreme case $\xi=0$. Also in this case Corollary 2 applies, hence, to begin with, the analysis can be carried out with an arbitrary initial condition $x\in\mathbb{R}$ for the state dynamics. Moreover, we take $p\in\mathcal{P}$. At the end we will verify the economic meaningfulness of the results thus obtained.

For every initial data $x\in\mathbb{R}$, we have the following state dynamics

$$X_t=x+\int_0^t D(A-X_s-Bp_s)ds+\int_0^t \sigma(A-X_s)dW_s, \tag{13}$$

and we wish to maximize the expected total profits

$$J_p^0(x) \triangleq \mathbf{E}\left[\int_0^\infty e^{-\delta t}\pi(X_t, p_t)dt\right].$$

The corresponding value function is

$$u_0(x) \triangleq \sup_{p\in\mathcal{P}} J_p^0(x), \tag{14}$$

and it is concave[10].

Notice that (14) is a classical stochastic control problem (cf. [9]), hence a formal application of the *Dynamic Programming Principle* yields the following HJB equation to be satisfied by $u_0(x)$:

$$\sup_{p\in\mathbb{R}}\left\{\mathbb{A}^{(p)}[u](x) + \pi(x,p)\right\} = 0, \quad \forall x \in \mathbb{R}, \tag{15}$$

with $\mathbb{A}^{(p)}[u]$ defined by (8) Equivalently

$$\begin{aligned}
&\frac{1}{2}\sigma^2(A-x)^2u''(x) + D(A-x)u'(x) - \delta u(x) \\
&+(2\alpha F + c)x - Fx^2 - Ac - \alpha^2 F \\
&+\sup_{p\in\mathbb{R}}\left\{(A + Bc - x - BDu'(x))p - Bp^2\right\} = 0, \quad \forall x \in \mathbb{R}.
\end{aligned} \tag{16}$$

The optimal (stationary) Markov strategy is

$$p(x) = \frac{1}{2B}\left(A + Bc - x - BDu'(x)\right). \tag{17}$$

If we substitute (17) in (16) we obtain

$$\begin{aligned}
&\frac{1}{2}\sigma^2(A-x)^2u''(x) + D(A-x)u'(x) - \delta u(x) \\
&+\frac{1}{4B}\left(A + Bc - x - BDu'(x)\right)^2 + \\
&+(2\alpha F + c)x - Fx^2 - Ac - \alpha^2 F = 0
\end{aligned} \tag{18}$$

to be satisfied by $u_0(x)$ for every $x \in \mathbb{R}$.

To solve (18) using the method of undetermined coefficients, we try a solution of the form

$$u(x) = Hx^2 + Kx + L, \tag{19}$$

[10]This can be proved in the same way as in Theorem 4.

where the values of H, K, and L are to be determined so to satisfy the differential equation (18). An exact solution exists for this problem. The procedure entails substituting (19) in (18), so that the left hand side is expressed in terms of various powers of x. For (18) to be satisfied, the coefficient of each power of x must equal zero, this generates a system of simultaneous equations which, thanks to the concavity of u_0, is uniquely identified and can be solved for H, K, and L. For these coefficients we have:

$$H = \frac{1}{2BD^2}\left(D + \delta - \sigma^2 - \sqrt{(D + \delta - \sigma^2)^2 + D^2(4BF - 1)}\right); \tag{20}$$

$$K = \frac{D^2(4B\alpha F + Bc - A) - (BcD + 2\sigma^2 A - AD)\left(D + \delta - \sigma^2 - \sqrt{(D+\delta-\sigma^2)^2 + D^2(4BF-1)}\right)}{BD^2\left(\delta + \sigma^2 + \sqrt{(D+\delta-\sigma^2)^2 + D^2(4BF-1)}\right)}; \tag{21}$$

$$L = \frac{B^2D^2K^2 + (2ABD - 2B^2cD)K + 4A^2B\sigma^2 H + B^2c^2 - 2ABc + A^2 - 4BF\alpha^2}{4B\delta}, \tag{22}$$

with H and K as in (20) and (21).

Observe that Assumption (A.1) implies $H < 0$ and therefore u, defined by (19), is concave (actually, strictly concave). Now a verification theorem will prove that $u_0(x) = u(x)$, with u_0 given by (14).

Theorem 5 *Let $u(x) = Hx^2 + Kx + L$, then for every $x \in \mathbb{R}$:*
i) $u(x) \geq J_p^0(x)$ for every admissible control process $p \in \mathcal{P}$;
ii) $u(x) = J_{p^*}^0(x)$, where $p_t^* = \frac{1}{2B}(A + Bc - X_t^* - BDu'(X_t^*))$ and X_t^* is the corresponding state dynamics. Therefore,

$$u_0(x) = Hx^2 + Kx + L, \quad \forall x \in \mathbb{R}. \tag{23}$$

Proof. i) Let p_t an admissible control process, we apply the Itô differential rule to $\Phi(t, X_t^p) = e^{-\delta t}u(X_t^p)$, where X_t^p is the unique solution of (13) corresponding to the initial condition x and the control process $p = p_t$. Then

$$e^{-\delta t}u(X_t^p) = u(x) + \int_0^t e^{-\delta s}\mathbb{A}^{(p)}[u](X_s^p)ds + \int_0^t e^{-\delta s}\sigma(A - X_s^p)u'(X_s^p)dW_s.$$

Since $u(x)$ solves (15) then it follows

$$e^{-\delta t}u(X_t^p) \leq u(x) - \int_0^t e^{-\delta s}\pi(X_s^p, p_s)ds + \int_0^t e^{-\delta s}\sigma(A - X_s^p)u'(X_s^p)dW_s,$$

and by taking expectations and by using Lemma 3 we get

$$u(x) \geq \mathbf{E}\left[\int_0^t e^{-\delta s}\pi(X_s^p, p_s)ds\right] + e^{-\delta t}\mathbf{E}\left[u(X_t^p)\right]. \tag{24}$$

We wish to let $t \to \infty$ in order to conclude that

$$u(x) \geq \mathbf{E}\left[\int_0^\infty e^{-\delta t}\pi(X_t^p, p_t)dt\right] \tag{25}$$

To this end we first observe that by assumptions (A.1) there exists $\Pi \in \mathbb{R}$ such that $\pi(x,p) \leq \Pi$ for every $(x,p) \in \mathbb{R}^2$. Therefore (24) can be rewritten as follows

$$u(x) \geq \mathbf{E}\left[\int_0^t e^{-\delta s}\left(\pi(X_s^p, p_s) - \Pi\right)ds + \int_0^t e^{-\delta s}\Pi ds\right] + e^{-\delta t}\mathbf{E}\left[u(X_t^p)\right] \tag{26}$$

If $\mathbf{E}\left[\int_0^\infty e^{-\delta t}(X_t^p)^2 dt\right] = \infty$ then $\mathbf{E}\left[\int_0^\infty e^{-\delta t}\left(\pi(X_t^p, p_t) - \Pi\right)dt\right] = -\infty$ and inequality (25) follows immediately from (26) by applying the monotone convergence theorem and observing that u is upper bounded. On the other hand, if $\mathbf{E}\left[\int_0^\infty e^{-\delta t}(X_t^p)^2 dt\right] < \infty$ then $\liminf_{t\to\infty}\mathbf{E}\left[e^{-\delta t}(X_t^p)^2\right] = 0$ but$|u(x)| \leq a + bx^2$ for some constants a and b and is bounded from above hence also $\limsup_{t\to\infty} e^{-\delta t}\mathbf{E}\left[u(X_t^p)\right] = 0$. Therefore, by letting $t \to \infty$ through a sequence for which $e^{-\delta t}\mathbf{E}\left[u(X_t^p)\right] \to 0$ and using the monotone convergence theorem we obtain

$$\begin{aligned} u(x) &\geq \mathbf{E}\left[\int_0^\infty e^{-\delta t}\left(\pi(X_t^p, p_t) - \Pi\right)dt + \int_0^\infty e^{-\delta t}\Pi dt\right] \\ &= \mathbf{E}\left[\int_0^\infty e^{-\delta s}\pi(X_t^p, p_t)dt\right] = J_p^0(x). \end{aligned} \tag{27}$$

Since $p \in \mathcal{P}$ is an arbitrary control process the proof of part i) is complete.

ii) If we fix an initial condition $x \in \mathbb{R}$ and substitute p^* then (13) has a unique strong solution X_t^*. In fact the function p^* is clearly Lipschitz continuous as a function of x. Now we apply once again the Itô differential rule to $\Phi(t, X_t^*) = e^{-\delta t}u(X_t^*)$ and we get

$$u(x) = \mathbf{E}\left[\int_0^t e^{-\delta s}\pi(X_s^*, p_s^*)ds\right] + e^{-\delta t}\mathbf{E}\left[u(X_t^*)\right], \tag{28}$$

with equality since p^* is the maximizer in (15). Again we need to pass to the limit as $t \to \infty$. We proceed as in part i) but this time is sufficient to observe that, since $u(x)$ is upper bounded, $\liminf_{t\to\infty} e^{-\delta t}\mathbf{E}\left[u(X_t^*)\right] \leq 0$. Hence, if in (28) we let $t \to \infty$ through a sequence for which $e^{-\delta t}\mathbf{E}\left[u(X_t^*)\right] \to L \le 0$ we get

$$u(x) \le \mathbf{E}\left[\int_0^\infty e^{-\delta t}\pi(X_t^*, p_t^*)dt\right] = J_{p^*}^0(x). \tag{29}$$

Finally, inequality (27) together with (29) yields the desired result. ■

Remark 6 *We observe that the above theorem implies that the solution (19) that we found by simply using the method of undetermined coefficients is the unique solution of equation (15) in the class of quadratically growing and concave functions. In other words, this means that every other solution of (15) is either not concave or not quadratically growing.*

The values of H, K, and L given by (20), (21), and (22), respectively, yield the following optimal Markov price path:

$$p(x) = \tfrac{A+Bc-BDK}{2B} - \left(\tfrac{2D+\delta-\sigma^2-\sqrt{(D+\delta-\sigma^2)^2+D^2(4BF-1)}}{2BD}\right)x. \tag{30}$$

Now, if we ignore for a while the assumption (A.5), which has not been really used in the previous analysis, then it can be seen that the parameters B, D, F, δ, and σ can determine two different price paths. In fact, two cases can be single out:

Case a) $\delta > 2D(BF-1)+\sigma^2$, then $\frac{dp}{dx} < 0$ and prices decrease as the product diffuses;

Case b) $\delta < 2D(BF-1)+\sigma^2$, then $\frac{dp}{dx} > 0$ and prices increase as the product diffuses.

Of course, if $\delta = 2D(BF-1)+\sigma^2$ then prices are kept constant independently of product diffusion.

Remark 7 *In this setting, the optimal price path is sensible to the market structure. For example, take two markets and fix for both of them the same δ, B, D, and F such that (A.1) holds, and assume that the first market has a sufficiently low variance whereas the second one does not. Then the first market would exhibit a classical decreasing price path (cf. [33]), whereas the agent operating in the second one would be better off by increasing prices as the product diffuses (see Figure 2).*

We now need to check the economic admissibility of the solution when $x \le A$. To do that it suffices to prove that $p_t^* \ge 0$, $\forall t > 0$. Then, invoking Lemma 1 we may conclude that $X_t^* \le A$, $\forall t \ge 0$, almost surely in $\mathbf{P}$, and therefore Corollary 2 applies. To proceed we require assumption (A.5) to hold.

The Markovian structure of the optimal price path p^* (see 17) implies that we need to check that $p(x)$ given by (30) is in fact non negative for $x \leq A$.

First of all, due to assumption (A.5), prices are decreasing in x and

$$p(A) = \frac{2DF(A-\alpha)+c\delta}{\delta+\sigma^2+\sqrt{(D+\delta-\sigma^2)^2+D^2(4BF-1)}} > 0$$

by assumption (A.3). Therefore, the optimal Markov price $p(x)$ given by (30) is positive on $(-\infty, A]$. Hence, invoking Lemma 1 and Corollary 2, we conclude

$$\sup_{p\in\mathcal{P}_+} J_p^0(x) = u_0(x) \qquad \forall x \leq A,$$

which is exactly the economic admissibility we were looking for.

Moreover, assumptions (A.1), (A.2), and (A.3) imply $p(0) < \frac{A}{B}$ (see Figure 1). In fact,

$$p(0) - \tfrac{A}{B} = \frac{\sigma^2 A\left(D+\delta-\sigma^2-\sqrt{(D+\delta-\sigma^2)^2+D^2(4BF-1)}\right)-D\delta(A-Bc)-2BD^2F\alpha}{BD\left(\delta+\sigma^2+\sqrt{(D+\delta-\sigma^2)^2+D^2(4BF-1)}\right)} < 0.$$

Remark 8 *Since we are working in an uncertain environment, it is interesting to study the behavior of $p'(x)$ and $p(x)$ with respect to σ^2. In fact, the sensitivity of the price path and its slope (this can be viewed as the price reaction to the market saturation) with respect to different volatility conditions of the market is certainly important from an economic point of view. For $p'(x)$ we have*

$$\frac{d}{d\sigma^2}p'(x) = -\frac{DH}{\sqrt{(D+\delta-\sigma^2)^2+D^2(4BF-1)}} > 0.$$

Therefore, as the market volatility increases the optimal price path slope becomes smaller and smaller, in absolute value, although its pattern is decreasing in x. Next,

$$\frac{d}{d\sigma^2}p(x) = \left(\frac{2DF(A-\alpha)+c\delta}{\left(\delta+\sigma^2+\sqrt{(D+\delta-\sigma^2)^2+D^2(4BF-1)}\right)^2} + \frac{(A-x)}{2BD}\right)\frac{2BD^2H}{\sqrt{(D+\delta-\sigma^2)^2+D^2(4BF-1)}},$$

which is negative for every $x \leq A$ since H is negative. Hence, as market variance increases the optimal price path exhibits smaller prices (see Figure 2). We finally observe that all these features are consistent with the previous literature (cf. [16], [33])

Figure 1: Optimal price path under assumption (A.5)

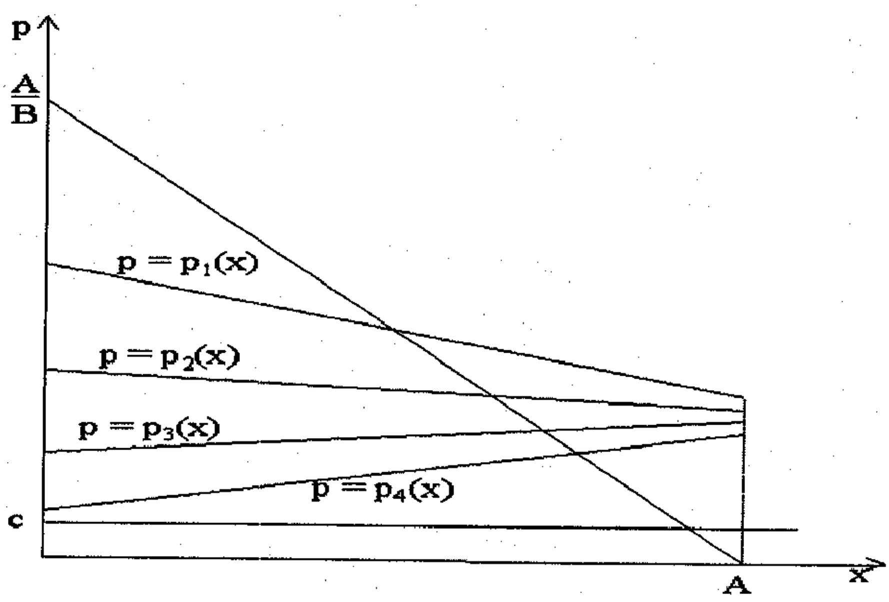

Figure 2: Optimal price paths with $\sigma_1 < \sigma_2 < \delta - 2D(BF-1) < \sigma_3 < \sigma_4$

Remark 9 *Notice that $H < 0$ implies*

$$p'(x) = -\frac{1}{2B}(1 + 2BDH) > -\frac{1}{B},$$

and this may account for a mean-reverting feature for the optimal state dynamics (see Figures 1 and 3). In fact, let $\underline{x}$ such that

$$p(\underline{x}) = \frac{A - \underline{x}}{B}.$$

Then the drift coefficient is positive if $x < \underline{x}$, hence pushing upwards the diffusion; whereas when $x > \underline{x}$ this coefficient becomes negative, hence pushing downwards the diffusion.

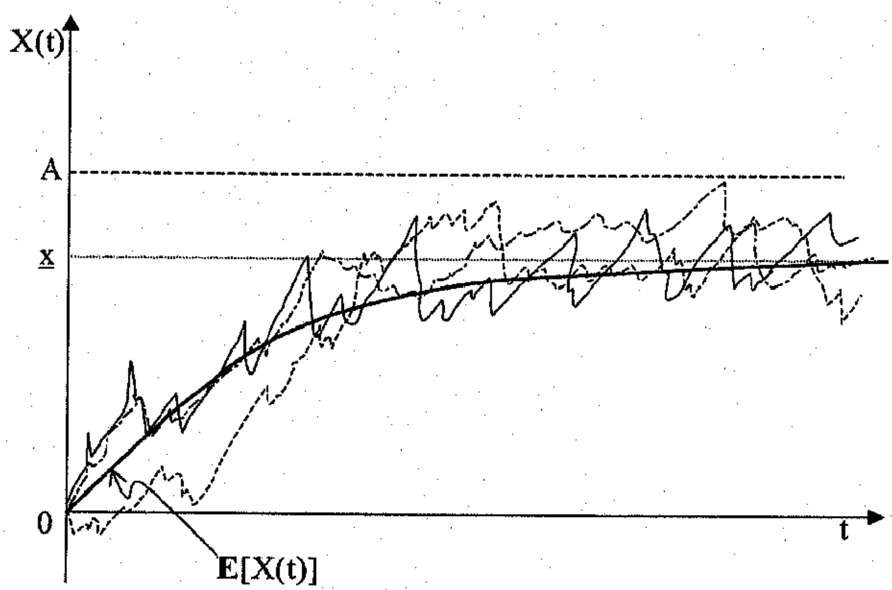

Figure 3: Mean-reverting effect (cf. Figure 1.1)

4 The case of non-zero innovation and general classical controls

We now turn to the general case. Consistently with the previous analysis, we first work without any restriction on the state dynamics and on the control variable p, that is we will take $p \in \mathcal{P}$.

So let x an arbitrary initial condition in $\mathbb{R}$ and consider the general state dynamics given by the SDE

$$X_t = x + \int_0^t D(A - X_s - Bp_s)ds + \int_0^t \sigma(A - X_s)dW_s - \xi_t. \tag{31}$$

We want to maximize the functional

$$J^{\infty}_{\xi,p}(x) \triangleq \mathbf{E}\left[\int_{[0,\infty)} e^{-\delta t}[\pi(X_t, p_t)dt - Id\xi_t]\right] \tag{32}$$

over all pairs $(\xi, p) \in \mathcal{A}$. The value function is

$$v_\infty(x) \triangleq \sup_{(\xi,p)\in\mathcal{A}} J^{\infty}_{\xi,p}(x). \tag{33}$$

In the sequel we will refer to this control problem as Problem (P_∞).

We start by assuming that $v_\infty \in C^2(\mathbb{R})$, then a formal application of the *Dynamic Programming Principle* yields the following HJB equation to be satisfied by $v_\infty(x)$ for every $x \in \mathbb{R}$:

$$\begin{cases} \sup_{p\in\mathbb{R}} \{\mathbb{A}^p[u](x) + \pi(x,p)\} \leq 0 \\ -I - u'(x) \leq 0 \\ \left(\sup_{p\in\mathbb{R}} \{\mathbb{A}^p[u](x) + \pi(x,p)\}\right)(-I - u'(x)) = 0 \end{cases} \tag{34}$$

with $\mathbb{A}^p[u](x)$ defined by (8).

4.1 Preliminary considerations

We first observe that for each $x \in \mathbb{R}$

$$u_0(x) \leq v_\infty(x) \leq C, \tag{35}$$

where u_0, v_∞ are respectively defined by (23) and (33), and $C > 0$ constant. In fact, in (35), the upper bound is a direct consequence of the upper bound of π (cf. Theorem 4), whereas the lower bound is true by construction since $\mathcal{P} \times \{0\} \subset \mathcal{A}$. Moreover, it can be proved that v_∞ is concave (cf. Theorem 4) and therefore, as it is typical in singular stochastic control problems, we expect v_∞ to be a straight line with slope $-I$ in a certain interval $[z^*, \infty)$, where the "free boundary" z^* is to be determined. Hence $(34)_2$ holds with equality on $[z^*, +\infty)$, whereas $(34)_1$ holds with equality on $(-\infty, z^*]$. In order to get some idea about the nature of the optimal policies as well as about the derivation of (34), let us consider a restricted class of adoption policies in which ξ is constraint to be absolutely continuous, as a function of time, with bounded derivative, that is

$$\xi_t = \int_0^t \dot{\xi}_s ds,$$

where $0 \le \dot{\xi}_t \le \frac{1}{\epsilon}$ for almost every $t \ge 0$, with $\epsilon > 0$ constant. Call the set of all such a controls $\mathcal{V}_\epsilon$. Then, equation (31) becomes

$$X_t = x + \int_0^t \left[D(A - X_s - Bp_s) - \dot{\xi}_s \right] ds + \int_0^t \sigma(A - X_s) dW_s,$$

and (32) changes into

$$J^{\epsilon}_{\xi,p}(x) \triangleq \mathbf{E}\left[\int_0^{\infty} e^{-\delta t} [\pi(X_t, p_t) - I\dot{\xi}_t] dt \right].$$

The corresponding value function is

$$v_\epsilon(x) \triangleq \sup_{(\dot{\xi},p)\in\mathcal{V}_\epsilon\times\mathcal{P}} J^{\epsilon}_{\xi,p}(x).$$

The HJB equation, to be solved for the value function v_ϵ, is

$$\sup_{p\in\mathbb{R}} \{\mathbb{A}^p[u](x) + \pi(x,p)\} + \sup_{0\le\dot{\xi}_t\le\frac{1}{\epsilon}} \left\{ (-u'(x) - I)\dot{\xi} \right\} = 0.$$

The maximum in the $\dot{\xi}$ control variable is then

$$\begin{aligned} \dot{\xi} &= 0, \quad \text{if } (-u'(x) - I) \le 0 \\ \dot{\xi} &= \frac{1}{\epsilon}, \quad \text{if } (-u'(x) - I) > 0. \end{aligned}$$

Hence, the HJB equation becomes

$$\sup_{p\in\mathbb{R}} \{\mathbb{A}^p[u](x) + \pi(x,p)\} + \frac{1}{\epsilon}(-u'(x) - I)^+ = 0 \tag{36}$$

where, for $x \in \mathbb{R}$, $(x)^+$ indicates the positive part of x. This indicates that the optimal innovation adoption policy is of *bang-bang* type, that is adoption take place at maximum rate or not at all. Therefore, the real line is split up into the following two regions

$$\begin{aligned} R_N &\triangleq \{x \in \mathbb{R} : \; -u'(x) - I \le 0\} \\ R_A &\triangleq \{x \in \mathbb{R} : \; -u'(x) - I > 0\}, \end{aligned}$$

where R_A, R_N stand for *adoption* and *no-adoption* region, respectively. Since $v_\epsilon(x)$ is concave[11], then we expect the two regions to be separated. In other words, we expect the existence of a point z_ϵ^* such that

$$\begin{aligned} \dot{\xi} &= 0, \quad \text{if } x \leq z_\epsilon^* \\ \dot{\xi} &= \frac{1}{\epsilon}, \quad \text{if } x > z_\epsilon^*. \end{aligned}$$

Clearly, for $x \in (-\infty, z_\epsilon^*]$ equation (36) becomes

$$\sup_{p \in \mathbb{R}} \{\mathbb{A}^p[u](x) + \pi(x,p)\} = 0.$$

Since this is true for every $\epsilon > 0$ then the same conclusions may be heuristically drawn after passing to the limit as $\epsilon \downarrow 0$. But, in the limit we formally need $-u'(x) - I \leq 0$ in order to give meaning to the equation (36). The latter inequality together with concavity of v_∞ yield (34) to be satisfied by v_∞ itself. Finally, on what concern the optimal adoption policy, if $x > z^*$ then adoption take place at maximum, i.e. infinite, speed, which implies that the monopolist will adopt an instantaneous amount of innovation in order to reach immediately z^*. After this initial adoption, all further adoptions take place when the state dynamics hits z^*, and this suggests an optimal adoption policy of "local time" type.

4.2 The smooth fit conditions: The Problem (IVP$_*$)

In the previous subsection we saw, although in a heuristic way, that the control ξ is not employed in the interval $(-\infty, z^*)$, where z^* is the "free boundary" to be determined. Such a consideration imply that, in the interval $(-\infty, z^*]$, the value function is a solution of equation of $(34)_1$ with equality and a certain boundary condition at z^*. Since $p \in \mathcal{P}$ then the maximizer in $(34)_1$ is given by $p(x) = \frac{1}{2B}(A + Bc - x - BDu'(x))$ and therefore $(34)_1$ with equality becomes (18) on $(-\infty, z^*]$. Now, relying on the fact that $v_\infty \in C^2(\mathbb{R})$, we are looking for a solution of the following Problem:

$$\left[\begin{array}{c} \text{there exists } z^* \in \mathbb{R} \text{ and } u_* \in C^2((-\infty, z^*]), \\ \text{concave solution of (18) such that} \\ u_*'(z^*) = -I, \qquad u_*''(z^*) = 0, \text{ and} \\ u_0(x) \leq u_*(x) \leq C \quad \text{on } (-\infty, z^*], \end{array}\right] \qquad \text{(SFC)}$$

[11] Again this is an immediate consequence of the linear-concave structure of the problem.

where the bounds for u_* are the ones required for v_∞ and are explicitly given by (35). Now, assume that the pair (z^*, u_*) is a solution of (SFC) such that $z^* < A$. Then, substituting u_* in (18) and evaluating at z^*, we have

$$\begin{array}{c} -DI(A-z^*) - \delta u_*(z^*) + \frac{1}{4B}\left(A + Bc - z^* + BDI\right)^2 \\ +(2\alpha F + c)z^* - F(z^*)^2 - Ac - \alpha^2 F = 0, \end{array}$$

that is

$$u_*(z^*) = g(z^*),$$

where

$$g(x) \triangleq \frac{(A+Bc-x+BDI)^2}{4B\delta} + \frac{(2\alpha F+c)x - Fx^2 - Ac - \alpha^2 F - DI(A-x)}{\delta}. \tag{37}$$

Now consider the Problem

$$\left[\begin{array}{c} \text{there exists } z^* < A \text{ and } u_* \in C^2((-\infty, z^*]), \\ \text{concave solution of (18) such that} \\ u_*(z^*) = g(z^*), \qquad u_*'(z^*) = -I, \text{ and} \\ u_0(x) \leq u_*(x) \leq C \quad \text{on } (-\infty, z^*]. \end{array}\right] \tag{IVP$_*$}$$

It is easily seen that if (z^*, u_*) is a solution of (IVP$_*$) then

$$u_*''(z^*) = 0,$$

and, as a result, (z^*, u_*) is also a solution of (SFC). Hence, under the assumption $z^* < A$ the two problems (SFC) and (IVP$_*$) are equivalent.

In the sequel the analysis will be aimed at looking for a solution of Problem (IVP$_*$).

4.3 A family of auxiliary regular control problems v_z

In order to solve (IVP$_*$) we introduce a family of regular control problems indexed by $z < A$. For these problems we have a state dynamics driven by the usual SDE

$$X_t = x + \int_0^t D(A - X_s - Bp_s)ds + \int_0^t \sigma(A - X_s)dW_s, \tag{38}$$

with $x \in (-\infty, z]$, and we maximize the functional

$$J_p^z(x) \triangleq \mathbf{E}\left[\int_0^{T_z} e^{-\delta t}\pi(X_t, p_t)dt + e^{-\delta T_z}\chi_{\{T_z < \infty\}} g(X_{T_z})\right] \tag{39}$$

where g is given by (37), X_t is the unique solution of (38) (with $p \in \mathcal{P}$), and the stopping time T_z is defined by

$$T_z \triangleq \inf \{t \geq 0 : X_t = z\}. \tag{40}$$

The value function is

$$v_z(x) \triangleq \sup_{p \in \mathcal{P}} J_p^z(x), \tag{41}$$

and we refer to this control problem as Problem (P_z).

Once again, assuming that $v_z \in C^2((-\infty, z])$, an application of the *Dynamic Programming Principle* yields the following HJB equation to be satisfied by v_z

$$\sup_{p \in \mathbb{R}} \left\{ \mathbb{A}^{(p)}[u](x) + \pi(x, p) \right\} = 0, \quad x < z, \tag{42}$$

with the following one side boundary condition

$$u(z) = g(z), \tag{43}$$

where $\mathbb{A}^{(p)}[u]$ is defined by (8). Notice that, by definition, $v_z'(z)$ and $v_z''(z)$ are assumed to be equal to the limits of $v_z'(x)$ and $v_z''(x)$ as $x \uparrow z$.

For every $z < A$, we now analyze the Problem (P_z). In particular, we are interested in checking whether $v_z \in C^2((-\infty, z])$ or not and if it is the unique classical solution of (42)-(43) in the class of all concave functions with at most quadratically growth as $x \to -\infty$.

4.3.1 Basic properties of the auxiliary control problem.

We start with same basic properties of the value function v_z defined by (41). For tractability, because of the number of parameters, we repeatedly need to restrict the admissible parameters set in order to obtain sufficient conditions for the solution of (IVP_*).

First, the *Dynamic Programming Principle* holds for v_z, that is the following theorem can be proved (for a proof see [9], section V.2).

Theorem 10 (DPP) *If θ is an $\mathcal{F}_t$-stopping time then, for every $x \in (-\infty, z]$,*

$$v_z(x) = \sup_{p \in \mathcal{P}} \mathbf{E} \left[\int_0^{T_z \wedge \theta} e^{-\delta t} \pi(X_t, p_t) dt + e^{-\delta (T_z \wedge \theta)} v_z(X_{T_z \wedge \theta}) \right].$$

Proposition 11 *Define*

$$z_4 \triangleq \max\left\{z < A : \max_{p\in\mathbb{R}}\{\pi(z,p) - \delta g(z)\} \geq 0\right\}, \tag{44}$$

that is $z_4 = A - Bc - \frac{1}{2}BDI$*, where* g *is defined by (37). Then for every* $z \in (-\infty, z_4]$ *the value function* $v_z(x)$ *is continuous and concave on* $(-\infty, z]$ *with* $v_z(z) = g(z)$.

Proof. See Proof of proposition 7 in [24]. ■

Being v_z concave, it is also locally Lipschitz. In the following proposition (for its Proof see [24], Proposition 8) we actually show that $v_z(x)$ is upper bounded, grows at most quadratically as $x \to -\infty$, and its increments grow at most linearly.

Proposition 12 *For every* $z \leq z_4$ *there exists a constant* $C_z > 0$ *such that for* $x \leq z$ *we have*

$$-C_z(1 + x^2) \leq v_z(x) \leq C_z \tag{45}$$

and

$$-C_z \leq \frac{v_z(x+h) - v_z(x)}{h} \leq C_z(1 + |x|), \tag{46}$$

for $h \in \mathbb{R}$ *such that* $|h| \leq 1$ *and* $x + h \leq z$.

4.3.2 C²-regularity of v_z.

Now, we want to show that v_z is twice continuously differentiable on $(-\infty, z]$. We proceed by proving that $v_z \in C^2([-k, z])$ for every $k \in \mathbb{N}$ such that $-k < z$. To this purpose consider a family of *truncated problems* (TP_k), indexed by k, of exit time type with terminal cost exactly equal to v_z. The state dynamics of this problem is as in (38) with initial condition $x \in [-k, z]$, but this time we maximize the functional

$$J_p^k(x) \triangleq \mathbf{E}\left[\int_0^{T_k} e^{-\delta t}\pi(X_t, p_t)dt + e^{-\delta T_k}v_z(X_{T_k})\right],$$

where

$$T_k \triangleq \inf\{t \geq 0 : X_t \notin (-k, z)\}. \tag{47}$$

Then the value function for (TP_k) is given by

$$v_k(x) \triangleq \sup_{p\in\mathcal{P}} J_p^k(x), \quad x \in [-k, z], \tag{TP_k}$$

The HJB equation for v_k is

$$\begin{cases} \sup_{p\in\mathbb{R}} \{\mathbb{A}^p[v_k](x) + \pi(x,p)\} = 0 \\ v_k(-k) = v_z(-k), \quad v_k(z) = v_z(z), \end{cases} \tag{48}$$

with $x \in [-k, z]$ and $\mathbb{A}^p[u]$ defined by (8).

A straightforward application of the *Dynamic Programming Principle* for v_z, yields the following

Lemma 13 *For every $x \in [-k, z]$, $v_z(x) = v_k(x)$.*

Proof. Since $(-k, z) \subset (-\infty, z)$ then $T_k \leq T_z$, $\mathbf{P} - a.s.$ Now, applying the *Dynamic Programming Principle* to v_z with the stopping time T_k we get

$$\begin{aligned} v_z(x) &= \sup_{p\in\mathcal{P}} \mathbf{E}\left[\int_0^{T_z\wedge T_k} e^{-\delta t}\pi(X_t, p_t)dt + e^{-\delta(T_z\wedge T_k)} v_z(X_{T_z\wedge T_k})\right] \\ &= \sup_{p\in\mathcal{P}} \mathbf{E}\left[\int_0^{T_k} e^{-\delta t}\pi(X_t, p_t)dt + e^{-\delta T_k} v_z(X_{T_k})\right] = v_k(x). \end{aligned}$$

■

Therefore, we need only to prove that $v_k \in C^2([-k, z])$ for each k sufficiently big. The domain of (TP_k) is compact but the control space is not. Nevertheless, since $v_z(x) = v_k(x)$ on $[-k, z]$ then v_k is Lipschitz-continuous in its domain. We do not know yet that v_k is C^2, but we certainly know that $\Delta v_z^h(x)$ (see (??)) is bounded in $[-k, z]$ by a constant L_k, for each k. Hence, by looking at (17) (with v_k instead of u) we can guess that the optimum is obtained on the set of all controls $p \in \mathcal{P}$ such that $|p_t| \leq P_k$, with

$$P_k > \max_{x\in[-k,z]} \frac{1}{2B}\left(|A + Bc - x| + BDL_k\right),$$

therefore we now concentrate on the new problem

$$v_{k,P_k}(x) \triangleq \sup_{p\in\mathcal{P}_k} \mathbf{E}\left[\int_0^{T_k} e^{-\delta t}\pi(X_t, p_t)dt + e^{-\delta T_k} v_z(X_{T_k})\right], \tag{ATP_k}$$

where $\mathcal{P}_k \triangleq \{p \in \mathcal{P} : |p_t| \leq P_k, \ \ \forall t \geq 0, \mathbf{P} - a.s.\}$, and X_t is the unique solution of (38) with $x \in [-k, z]$ and T_k is defined in (47), and we show that, in fact, $v_{k,P_k}(x) = v_k(x)$ on $[-k, z]$.

Theorem 14 *i) $v_{k,P_k}(x)$ with its first and second derivatives is continuous on $[-k,z]$, and $v''_{k,P_k}(x)$ satisfies a Lipschitz condition on $[-k,z]$. Moreover, v_{k,P_k} is the unique solution of*

$$\begin{cases} \sup_{|p|\leq P_k} \left\{ \mathbb{A}^{(p)}[v_{k,P_k}](x) + \pi(x,p) \right\} = 0 \\ v_{k,P_k}(-k) = v_z(-k), \quad v_{k,P_k}(z) = v_z(z), \end{cases} \tag{49}$$

on $[-k,z]$.

ii) For each $x \in [-k,z]$, $v_{k,P_k}(x) = v_k(x)$.

Proof. i) This is a classical result in stochastic control theory. For its proof we refer to [20] Theorem 1.4.5 p. 24.

ii) By construction we have $v_{k,P_k}(x) \leq v_k(x)$. In order to prove the reverse inequality, we first observe that $v_{k,P_k}(x)$ is a C^2 solution of (48). In fact, clearly the boundary conditions are satisfied. Moreover, if we substitute $v_{k,P_k}(x)$ in the equation appearing in the first line of (48) then we can see that the (Markovian) maximizer on the left hand side of such equation is

$$p_M(x) = \frac{1}{2B}[A + Bc - x - BDv'_{k,P_k}(x)] \tag{50}$$

but, since $\left|v'_{k,P_k}(x)\right| \leq L_k$ for each $x \in [-k,z]$, then for every $x \in [-k,z]$

$$\left| \frac{1}{2B} \left(A + Bc - x - BDv'_{k,P_k}(x) \right) \right| < P_k.$$

Therefore, for each $x \in [-k,z]$

$$\sup_{|p|\leq P_k} \left\{ \mathbb{A}^{(p)}[v_{k,P_k}](x) + \pi(x,p) \right\} = \sup_{p\in\mathbb{R}} \left\{ \mathbb{A}^{(p)}[v_{k,P_k}](x) + \pi(x,p) \right\},$$

for any $p \in \mathbb{R}$ such that $\mathbb{A}^{(p)}[v_{k,P_k}](x) \leq -\pi(x,p)$ and by an application of Itô's formula to the process $e^{-\delta t}v_{k,P_k}(X_t^{p_M})$ we conclude that $v_{k,P_k}(x) \geq v_k(x)$ (i.e. $v_{k,P_k}(x)$ is a superharmonic function on $[-k,z]$ with respect to equation (48)). ■

Remark 15 *We first observe that the proof of point ii) of Theorem 14 does not rely on the existence of optimal controls for the problem (ATP_k), although a standard verification theorem would verify such an existence.*

Theorem 16 *i) $v_z(x)$ is C^2 on $(-\infty,z]$;*

ii) v_z is a solution of (42)-(43).

Proof. i) The proof follows easily by using a contradiction argument.

ii) By construction we easily see that v_z satisfies (43). Next, define the $\mathcal{F}_t$-stopping time $\tau = T_z \wedge \frac{1}{n}$, then the dynamic programming principle yields

$$v_z(x) = \sup_{p\in\mathcal{P}} \mathbf{E}\left[\int_0^\tau e^{-\delta t}\pi(X_t^p, p_t)dt + e^{-\delta\tau}v_z(X_\tau^p)\right], \tag{51}$$

where, for every $p \in \mathcal{P}$, X_t^p is the unique solution of (38) with initial state condition $x \le z$ and control variable p. On the other hand, by applying Itô's differential rule, integrated from 0 to τ, to the twice continuous differentiable function $\Phi(t, X_t^p) = e^{-\delta t}v_z(X_t^p)$ we get

$$e^{-\delta\tau}v_z(X_\tau^p) = v_z(x) + \int_0^\tau e^{-\delta s}\mathbb{A}^{(p)}[v_z](X_t^p)dt + \int_0^\tau e^{-\delta t}\sigma(A - X_t^p)v_z'(X_t^p)dW_t. \tag{52}$$

Now by substituting (52) into (51) and recalling Lemma 3 we get

$$0 = \sup_{p\in\mathcal{P}} \mathbf{E}\left[\int_0^\tau e^{-\delta t}\left[\mathbb{A}^{(p)}[v_z](X_t^p) + \pi(X_t^p, p_t)\right]dt\right].$$

Dividing both sides by $\mathbf{E}[\tau]$ and passing to the limit as $n \to \infty$ yields (42). The proof is complete. ■

Remark 17 *Finally, we need to note that nothing has been said about uniqueness of solution of (42)-(43). This will be our next task.*

Before attacking the uniqueness question we conclude this subsection by proving the following simple

Lemma 18 *Let $z \le z_4$ and $g(z) > Hz^2 + Kz + L$, where $H, K,$ and L are respectively defined by (20), (21), and (22). Then for each $x \le z$*

$$v_z(x) > Hx^2 + Kx + L. \tag{53}$$

Proof. Let z be such that $g(z) > Hz^2 + Kz + L =: g_1$ and consider a control problem similar to (39) but with terminal reward g_1 and state dynamics given by (38); then, with $x \le z$, set

$$v_1(x) \triangleq \sup_{p\in\mathcal{P}} \mathbf{E}\left[\int_0^{T_z} e^{-\delta t}\pi(X_t, p_t)dt + e^{-\delta T_z}\chi_{\{T_z<\infty\}}g_1\right],$$

where T_z is defined by (40). Then $v_z(x) \in C^2((-\infty, z])$ by Theorem 16, and it can be shown that $v_z(x) \ge v_1(x)$. The two problems have the same HJB equation except for the boundary conditions. On the other hand the application of a verification theorem similar to Theorem 5 implies that $v_1(x) = Hx^2 + Kx + L$ for each $x \leq z$. Therefore we now have two classical solution, $v_z(x)$ and $v_1(x)$, of (42) such that $v_z(x) \geq v_1(x)$ for each $x \leq z$.

Notice that if there were a point $x_0 < z$ where $v_z(x_0) = v_1(x_0)$, then from $v_z(x) \geq v_1(x)$ would follow that x_0 is a maximum point of $v_1 - v_z$, hence $v_z'(x_0) = v_1'(x_0)$. But then uniqueness of solutions would imply $v_z(x) = v_2(x)$ for every $x \leq z$. This is clearly a contradiction since at the boundary $v_z(z) = g(z) > Hz^2 + Kz + L = v_1(z)$. ■

4.4 Existence and uniqueness of the solution to (IVP$_*$)

We now prove uniqueness of a pair (z^*, u_*) solution of (IVP$_*$). In fact, we will show that u_* coincides with v_{z^*} for an appropriate $z^* < A$.

By using comparison results for ordinary differential equations (cf. [31]) we show that, for every $z < A$, v_z is the unique concave solution of (42)-(43) with $u_0(x) \leq v_z(x) \leq C$ for $x \leq z$, where u_0 and C are defined in (IVP$_*$). For a complete proof of the following theorem we refer to [24]

Theorem 19 *Let Assumptions (A.1), (A.3), (A.4), and (A.5) hold and further assume*

i) $\sigma^2 < D < 2\sigma^2$;

ii) c *and* I *are sufficiently small.*

Then there exists a unique pair (z^*, u_*) *solution of (IVP$_*$).*

4.5 Optimal controls and verification theorem

We now show that u_* defined in Theorem 19 can be extended from $(-\infty, z^*]$ to $\mathbb{R}$ to provide a solution $v \in C^2(\mathbb{R})$ of the variational inequality (34) in the class of concave and upper bounded functions $u \in C^2(\mathbb{R})$ with growth condition

$$C \geq u(x) \geq u_0(x), \quad \forall x \in \mathbb{R} \tag{54}$$

where $u_0(x)$ is given by (23) and $C > 0$ constant (see Figure 4).

Theorem 20 *Under the conditions of Theorem 19, if* $v_* : \mathbb{R} \to \mathbb{R}$ *is defined by*

$$v_*(x) \triangleq \begin{cases} u_*(x), & \text{if } x \leq z^* \\ u_*(z^*) - I(x - z^*), & \text{if } x \geq z^*, \end{cases} \tag{55}$$

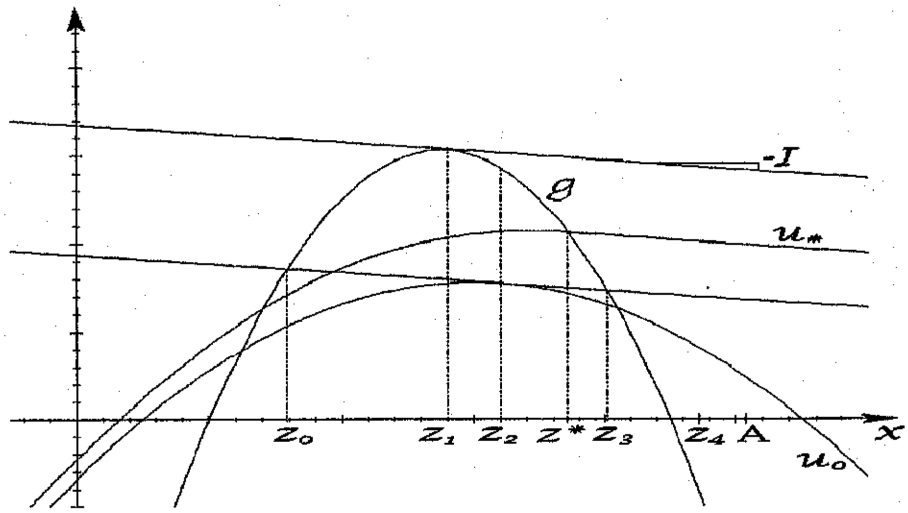

Figure 4:

where z^*, u_* are as in Theorem 19. Then:

i) $v_* \in C^2(\mathbb{R})$;

ii) v_* is the unique solution of (34) in the class of all concave and upper bounded functions which satisfy the growth condition (54).

Proof. i) This is true by construction.

ii) Consider $x \leq z^*$ then $(34)_1$ holds with equality. This implies that also $(34)_3$ is satisfied. $(34)_2$ follows by concavity of v_*.

Next, if $x \geq z^*$ then $(34)_2$ and $(34)_3$ are readily satisfied. It remains to prove that v_* satisfies $(34)_1$. In $[z^*, \infty)$, we have $v_*(x) = u_*(z^*) - I(x - z^*)$. Hence by substituting into the differential equation $(34)_1$, we only need to check that

$$(\frac{1}{2}(DI + c) - \frac{A}{2B} + 2F\alpha + \delta I) + (\frac{1}{4B} - F)z^* + (\frac{1}{4B} - F)x \leq 0$$

for each $x \geq z^*$. But the latter inequality is certainly true if

$$z^* \geq \frac{4F\alpha B + 2\delta IB + cB + DIB - A}{4FB - 1} = z_1,$$

and this is the case since $z_1 < z_2 < z^*$.

The lower bound represented by u_0 (cf. (23)) is certainly true for $x \leq z^*$. For $x \geq z^*$ it suffices to observe that $u_0(z^*) < -I$. ∎

We now have the following version of the *Skorohod problem.*

Theorem 21 *Let z^*, v_* as in Theorem 20. Then, for each initial condition $x \in \mathbb{R}$ of (31), there exists a unique pair $(\xi^*, p^*) \in \mathcal{A}$ such that with X_t^* the corresponding solution (cf. (31)), the following conditions hold almost surely:*
i) if $x > z^*$, then $X_0^* = z^*$;
ii) $X_t^* \leq z^*, \quad \forall t \geq 0$;
iii) $\xi_t^* = \int_{[0,t]} \chi_{\{X_s^* : X_s^* = z^*\}} d\xi_s^*, \quad \forall t \geq 0$;
iv) $p_t^* = (2B)^{-1}[A + Bc - X_t^* - BDv_*'(X_t^*)], \quad \forall t \geq 0$.

Notice that local Lipschitz continuity of v_* and non degeneracy of X_t^*, at least for $x \leq z^* < A$ are sufficient conditions for a unique solution to the *Skorohod problem.* For a complete proof we refer to [23]. Besides we make the following

Remark 22 *ξ_t^* is right-continuous, actually continuous except at $t = 0$ where it can exhibit a jump of size $(x - z^*)^+$ with probability one, where $(x - z^*)^+$ is the positive part of $(x - z^*)$. Moreover, ξ_t^* is singular with respect to Lebesgue measure. In fact, ξ_t^* is the local time of the semimartingale Y_t at $y = z^*$, where*

$$Y_t = y + \int_0^t D(A - Y_s - Bp_s^*)ds + \int_0^t \sigma(A - Y_s)dW_s.$$

We now are ready to prove the following

Theorem 23 *Let v_* as in Theorem 20. Then*
i) for every admissible pair $(\xi, p) \in \mathcal{A}$ and for every $x \in \mathbb{R}$

$$v_*(x) \geq J_{\xi,p}^{\infty}(x),$$

with $J_{\xi,p}^{\infty}(x)$ defined by (32);
ii) for each $x \in \mathbb{R}$ and the pair (ξ^*, p^*) defined in Theorem 21 we have

$$v_*(x) = J_{\xi^*,p^*}^{\infty}(x).$$

In particular $v_* = v_\infty$ (cf. 33) and (ξ^*, p^*) is optimal for Problem (P_∞).

Proof. Fix $(\xi, p) \in \mathcal{A}$ and apply Ito's formula for semimartingale (see [30] pp. 278-301) to $\Phi(t, X_t^{p,\xi}) = e^{-\delta t} v_*(X_t^{p,\xi})$, where $X_t^{p,\xi}$ is the unique solution of (31) relative to the initial condition $x \in \mathbb{R}$ and the control pair (p, ξ). Then

$$\begin{aligned} e^{-\delta t} v_*(X_t^{p,\xi}) &= v_*(x) + \int_0^t e^{-\delta s} \mathbb{A}^{(p)}[v_*](X_s^{p,\xi}) ds \\ &\quad - \int_{[0,t]} e^{-\delta s} v_*'(X_{s-}^{p,\xi}) d\xi_s + \int_0^t e^{-\delta s} \sigma(A - X_s^{p,\xi}) v_*'(X_s^{p,\xi}) dW_s \\ &\quad + \sum_{0 \le s \le t} e^{-\delta s} \left[v_*(X_s^{p,\xi}) - v_*(X_{s-}^{p,\xi}) - v_*'(X_{s-}^{p,\xi}) \Delta X_s^{p,\xi} \right], \end{aligned}$$

where $\Delta X_s^{p,\xi} \triangleq X_s^{p,\xi} - X_{s-}^{p,\xi} = -\Delta \xi_s$. Also

$$\begin{aligned} \int_{[0,t]} e^{-\delta s} v_*'(X_{s-}^{p,\xi}) d\xi_s &= \int_0^t e^{-\delta s} v_*'(X_{s-}^{p,\xi}) d\xi_s^c + \sum_{0 \le s \le t} e^{-\delta s} \left[v_*'(X_{s-}^{p,\xi}) \Delta \xi_s \right] \\ &= \int_0^t e^{-\delta s} v_*'(X_{s-}^{p,\xi}) d\xi_s^c + \\ &\quad - \sum_{0 \le s \le t} e^{-\delta s} \left[v_*'(X_{s-}^{p,\xi}) \Delta X_s^{p,\xi} \right], \end{aligned} \tag{56}$$

where ξ^c is the continuous part of ξ in the Lebesgue decomposition. By rearranging terms one gets

$$\begin{aligned} v_*(x) &= - \int_0^t e^{-\delta s} \mathbb{A}^{(p)}[v_*](X_s^{p,\xi}) ds + e^{-\delta t} v_*(X_t^{p,\xi}) \\ &\quad - \int_0^t e^{-\delta s} \sigma(A - X_s^{p,\xi}) v_*'(X_s^{p,\xi}) dW_s \\ &\quad + \int_0^t e^{-\delta s} v_*'(X_{s-}^{p,\xi}) d\xi_s^c - \sum_{0 \le s \le t} e^{-\delta s} \left[v_*(X_s^{p,\xi}) - v_*(X_{s-}^{p,\xi}) \right] \end{aligned} \tag{57}$$

But $v_*(x)$ is a solution of (34), hence

$$- \int_0^t e^{-\delta s} \mathbb{A}^{(p)}[v_*](X_s^{p,\xi}) ds \ge \int_0^t e^{-\delta s} \pi(X_s^{p,\xi}, p_s) ds.$$

Then, $(34)_2$, the mean value theorem, and the fact that $X_s^{p,\xi} - X_{s-}^{p,\xi} = -\Delta\xi_s$ yield

$$\begin{aligned} &\int_0^t e^{-\delta s} v_*'(X_{s-}^{p,\xi}) d\xi_s^c - \sum_{0\le s\le t} e^{-\delta s}\left[v_*(X_s^{p,\xi}) - v_*(X_{s-}^{p,\xi})\right] \\ = &\int_0^t e^{-\delta s} v_*'(X_{s-}^{p,\xi}) d\xi_s^c + \sum_{0\le s\le t} e^{-\delta s} v_*'(\bar{x})\Delta\xi_s \\ \ge &\int_0^t e^{-\delta s}(-I) d\xi_s^c + \sum_{0\le s\le t} e^{-\delta s}(-I)\Delta\xi_s = \int_{[0,t]} e^{-\delta s}(-I) d\xi_s, \end{aligned}$$

where $\bar{x}$ is a point such that $X_s^{p,\xi} \le \bar{x} \le X_{s-}^{p,\xi}$ for each $s \in [0,t]$ and $X_s^{p,\xi} < \bar{x} < X_{s-}^{p,\xi}$ for those s where $X_s^{p,\xi}$ has a jump. Therefore, we get

$$\begin{aligned} v_*(x) \ \ge\ &\int_{[0,t]} e^{-\delta t}[\pi(X_s^{p,\xi}, p_s)dt - I d\xi_s] \\ &- \int_0^t e^{-\delta s}\sigma(A - X_s^{p,\xi}) v_*'(X_s^{p,\xi}) dW_s + e^{-\delta t} v_*(X_t^{p,\xi}) \end{aligned}$$

Now, by taking expectations, the second integral vanishes (cf. Lemma 3) and we obtain

$$v_*(x) \ge \mathbf{E}\left[\int_{[0,t]} e^{-\delta t}[\pi(X_s^{p,\xi}, p_s)dt - I d\xi_s]\right] + \mathbf{E}\left[e^{-\delta t} v_*(X_t^{p,\xi})\right].$$

Finally, we pass to the limit as $t \to \infty$ and we conclude that

$$v_*(x) \ge \mathbf{E}\left[\int_{[0,\infty)} e^{-\delta t}[\pi(X_t^{p,\xi}, p_t)dt - I d\xi_t]\right] = J_{\xi,p}^{\infty}(x),$$

by arguments similar to those used in the proof of Theorem 5, part i).

ii) Now call X_t^* the unique solution of (31) corresponding to the initial condition $x \in \mathbb{R}$ and the control pair (ξ^*, p^*). We consider two cases: *case a)* $x \le z^*$, and *case b)* $x > z^*$.

In *case a)* we start from (57) and observe that point ii) of Theorem 21 and $(34)_2$, $(34)_2$ imply

$$-\int_0^t e^{-\delta s}\mathbb{A}^{(p^*)}[v_*](X_s^*) ds = \int_0^t e^{-\delta s}\pi(X_s^*, p_s^*) ds. \tag{58}$$

Also, since $x \le z^*$, then from Theorem 21 and Remark 22 follows that X_t^* is continuous on $[0,t]$ and therefore

$$\sum_{0\le s\le t} e^{-\delta s}\left[v_*(X_s^*) - v_*(X_{s-}^*)\right] = 0.$$

Next, from Theorem 21, points ii) and iii), and (56) follows that

$$\begin{aligned}\int_0^t e^{-\delta s} v_*'(X_{s-}^*) d\xi_s^{*c} &= \int_{[0,t]} e^{-\delta s} v_*'(X_{s-}^*) d\xi_s^* \\ &= \int_{[0,t]} e^{-\delta s} v_*'(X_{s-}^*) \chi_{\{X_s^* : X_s^* = z^*\}} d\xi_s^* \\ &= \int_{[0,t]} e^{-\delta s} (-I) d\xi_s^*.\end{aligned}$$

After arranging terms in (57) we get

$$\begin{aligned}v_*(x) &= \int_{[0,t]} e^{-\delta t} [\pi(X_s^*, p_s^*) dt - I d\xi_s^*] \\ &\quad - \int_0^t e^{-\delta s} \sigma(A - X_s^*) v_*'(X_s^*) dW_s + e^{-\delta t} v_*(X_t^*). \qquad (59)\end{aligned}$$

Now by taking expectations and then passing to the limit as $t \to \infty$ we obtain the desired result.

In *case b)* we start from (57) and we observe that equality (58) still holds, since $X_t^* \leq z^*$ for each $t \geq 0$. Next, we have

$$\begin{aligned}\int_0^t e^{-\delta s} v_*'(X_{s-}^*) d\xi_s^{*c} &= \int_{[0,t]} e^{-\delta s} v_*'(X_{s-}^*) d\xi_s^* - \sum_{0 \leq s \leq t} e^{-\delta s} \left[v_*'(X_{s-}^*) \Delta \xi_s^*\right] \\ &= \int_{[0]} e^{-\delta s} v_*'(X_{s-}^*) d\xi_s^* + \int_{(0,t]} e^{-\delta s} v_*'(X_{s-}^*) d\xi_s^* - v_*'(x) \Delta \xi_0^* \\ &= \int_{(0,t]} e^{-\delta s} v_*'(X_{s-}^*) d\xi_s^* \\ &= \int_{(0,t]} e^{-\delta s} v_*'(X_{s-}^*) \chi_{\{X_s^* : X_s^* = z^*\}} d\xi_s^* \\ &= \int_{(0,t]} e^{-\delta s} (-I) d\xi_s^*.\end{aligned}$$

Now, by recalling that $X_0^* = x - \xi_0^* = z^*$ and that $v_*(x)$ is a straight line with slope equal to $-I$ on $[z^*, +\infty)$ (hence $v'(x) = -I$ on $[z^*, +\infty)$), we obtain

$$\begin{aligned}\sum_{0 \leq s \leq t} e^{-\delta s} \left[v_*(X_s^*) - v_*(X_{s-}^*)\right] &= v_*(x - \xi_0^*) - v_*(x) = v_*(z^*) - v_*(x) \\ &= v_*'(x)(z^* - x) = -v_*'(x) \Delta \xi_0^* = I \Delta \xi_0^*.\end{aligned}$$

Therefore,

$$\int_0^t e^{-\delta s} v_*'(X_{s-}^*) d\xi_s^{*c} - \sum_{0 \le s \le t} e^{-\delta s} \left[v_*(X_s^*) - v_*(X_{s-}^*) \right]$$
$$= \int_{(0,t]} e^{-\delta s}(-I) d\xi_s^* + (-I)\Delta\xi_0^* = \int_{[0,t]} e^{-\delta s}(-I) d\xi_s^*.$$

Finally, collecting terms in (57) we get again equality (59). The desired result is obtained by proceeding as in *case a*). The proof is now complete. ■.

5 The original problem (P)

We now turn to the question of economic admissibility of the mathematical solution figure out in Section 4. To this end it suffices to verify that the optimal price path is non-negative. Then, by invoking Lemma 1 and Corollary 2, the desired admissibility follows.

Recall that the optimal price path is Markovian and given by

$$p^*(X_t^*) = \frac{A + Bc - X_t^* - BDv'(X_t^*)}{2B},$$

hence if we compute the derivative of p^* with respect to X^* we get

$$\frac{dp^*}{dX^*} = -\frac{1}{2B} \left(1 + BDv''(X^*)\right).$$

In Section 3, where we analyzed the problem with zero innovation, we had that assumption (A.5) implies $-\frac{1}{2B}(1 + 2BDH) < 0$. Now, it is possible to show (see [24]) that $2H \le v''(X^*) \le 0$ and hence

$$\frac{dp^*}{dX^*} < 0.$$

Thus, optimal prices decrease as the market saturates and, since X^* never exceeds z^*, we conclude that the minimum price is reached at z^*. As a result, it suffices to prove the non-negativity of $p^*(z^*)$. This follows from $z^* < A$, the definition of p^*, and the fact that $v_*'(z^*) = -I$.

In other words, if (A.5) holds then, by invoking Lemma 1 and Corollary 2, the original Problem (P) and the Problem (P_∞) (i.e. the one studied in Section 1.4) are equivalent in that $v_\infty(x) = v(x)$ for each $x \le A$, where v_∞ and v are respectively defined by (33) and (4).

6 Conclusions

We now investigate a bit closer the optimal control pair (p^*, ξ^*). As we have already point out in Remark 22, ξ^* is a local time and, essentially, it is merely used to prevent the state dynamics X^* to exceed z^*. Let us look at this phenomenon from an economic point of view. We defined the state dynamics X as a *market saturation index*, that is X is an index of the residual market demand faced by our agent who covers the whole market supply. As X increases, the market gets more and more saturated and less people buy the product. Hence, our analysis suggests how and when to adopt new products that are already available. It turns out that innovation is best adopted when it is used to prevent the market to saturate too much. In other words, there is a critical level of market saturation (i.e. z^*, which depends on the variables, included the proportional sunk costs I the monopolist incur when introduces into the market a new product) above which it becomes profitable to innovate.

On what concern p^* we have seen that, consistently with both previous literature (cf. [33], [16]) and empirically evidence, optimal prices decrease as the market saturates.

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