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A *finite-fuel* singular stochastic control problem

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The competitive choice of innovation adoption: A *finite-fuel* singular stochastic control problem

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1 Introduction

Modern society has been characterized by a constant supply of technological innovations adoptable in industry, agriculture, services, or other branches of economic activity. Despite that, the rate at which such innovations have been adopted, through a mass production, seems to follow a slower pace[1]. Quoting Tirole ([36] (1990), p. 401): *"Few innovations are adopted instantaneously."* Then the question is: What explains such a delay? The existing theoretical literature on innovation adoption propose several models to justify this phenomenon[2]. These models consider, singly or jointly, various economic aspects such as: a) *market structure* (see, among others, Kamien & Schwartz (1972), Reinganum (1981), Jensen (1982), Mamer & McCardle (1987), Lee & Wilde (1980), Barzel (1968).); b) *externalities, spillover effects,* and *learning by doing* (Jovanovic & Lach (1989), Jovanovic & MacDonald (1994), Mason & Weeds (2001)); c) *uncertainty* surrounding the innovation profitability (Rosenberg (1976), Balcer & Lippman (1984), Grenadier & Weiss (1997), Farzin et al. (1998), Bessen (1999), Dosi & Moretto (2000)).

In this paper we analyze, in a continuous time setting, the choice of innovation[3] adoption made by a competitive firm (cf. [25]). The firm produces and sells a *durable* good and no strategic considerations are analyzed. The agent faces an uncertain market demand and the decision of adopting is *irreversible*. Moreover, time horizon is infinite and the technological change is modeled as a continual process[4]. In contrast with [25] here it is assumed a *finite* amount of

[1] For historical accounts of the slow pace of adoption of technology innovations see, for example, Mansifiel (1968).

[2] For a review we may refer to Bridges et al. (1991) and Reinganum (1989).

[3] Here by innovation it is meant *product* innovation.

[4] Much of the literature models the innovation adoption decision as a once-and-for-all event (Kamien & Schwartz (1972), Reinganum (1981), Jensen (1982), Mamer & McCardle (1987), Lee & Wilde (1980), Grenadier & Weiss (1997),Dosi & Moretto (2000), Mason & Weeds (2001)).

innovation available to the firm[5]. This is the essential difference with respect to [25] together with the fact that a competitive firm is price-taker, hence only the singular control comes up. For the rest, the model shares the same mathematical structure as in [25]. As a result, the problem falls in the so called "finite-fuel" singular stochastic control[6]. Despite the presence of the finite-fuel component, the mathematical analysis of the problem is considerably simplified because of the classical control absence. As a matter of fact, previous results on the subject apply to our analysis (see [31], [6]).

From an economic point of view, it is remarkable the fact that, in spite of the limited innovation available, the optimal adoption policy chosen by a competitive firm is, in essence, similar to the one used by a monopolist. In other words, also the competitive firm adopts innovation, at least as long as she runs out of it, mainly to keep market demand above a certain profitable level (cf. [25]).

Before getting into the analysis details, we briefly illustrate the way we are going to follow in the mathematical analysis of the problem. We actually use a device first introduced in [6] (a probabilistic analysis of such device is given in [18]). The idea is that of studing separately the two extreme cases of no innovation at all and the case of infinite innovation available. Once this has been done, it can be shown that the original problem, actually the value function of the original problem, is an appropriate combination of the two extreme value functions.

2 The model, assumptions, and first properties

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space equipped with a filtration $(\mathcal{F}_t)$ satisfying the usual condition of right continuity and augmentation by all $\mathbf{P}$-negligible sets and carrying a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion W_t. Given this setting of uncertainty, which is completely known and observed by the firm, we define the *market saturation index* faced by the firm, X_t, as the unique solution of the following SDE

$$\begin{cases} X_t = x + \int_0^t D(A - X_s)ds + \int_0^t \sigma(A - X_s)dW_s - \xi_t \\ X_{0-} = x, \end{cases} \tag{1}$$

[5] Very likely, the whole innovation available to be adopted by the firm is finite, since, in contrast with the monopolist case, in a competitive market the energies available to be spent by the agent in creating new innovation are far less than the ones available to a monopolist. As a result, a competitive firm has far less innovation readily available to be introduced in the market.

[6] These are control problems in which the total variation (in the whole time interval where the analysis is carried out) of the control process is bounded by a fixed positive constant. Problem of this type arose first in the sixties in order to analyze the best way to control a spacecraft. In these models the control variable is the total amount of fuel used by the spacescraf up to time $t \geq 0$. With this interpretation it is appropriate to impose a constraint on such a control: hence the name finite-fuel. Important contributions on this subject are, among others, [3], [31], [6])

where $x \leq A$, $\sigma > 0$, and $D > 0$. Similar to the monopolistic case, the constant A can be seen as the *relative market potential* at time zero, that is A is what the firm expects, at the beginning of the analysis, to sell at most.

The process ξ_t stand for an index of the cumulative innovation adopted up to time t and we will take $\xi \in \mathcal{V}_+$, where

$$\mathcal{V}_+ := \left\{ \xi : [0, +\infty) \to \mathbb{R} : \xi_t \quad \begin{array}{l} \text{non-decreasing,} \\ \text{right continuous with} \\ \text{left limits, } \mathbf{P} - a.s.; \\ \mathcal{F}_t\text{-adapted with } \xi_t = 0,\ t < 0 \end{array} \right\}$$

It is worth noting that the non-decreasing feature of ξ_t accounts for the fact that once the firm has adopted innovation he cannot go back on his decision (irreversible decision). Moreover the negative effect of ξ on X_t (see (1)) means that market enjoys new energies in adopting innovation.

In the sequel we will assume that the total amount of innovation available to adoption is finite. Formally, the maximization will be done by taking $\xi \in \mathcal{V}_M$, with

$$\mathcal{V}_M := \{\xi \in \mathcal{V}_+ \text{ such that } \xi_\infty \leq M, \mathbf{P} - a.s.\},$$

where the positive constant M stands for the total amount of innovation available. We will refer to $\mathcal{V}_M$ as the *admissible control* set.

It remains to formalize the firm's profits structure. First, since the firm does not control prices then her profit rate π is simply a concave function of the market saturation index, that is $\pi = \pi(X_t)$. In order to avoid usefulness mathematical complications we will take the following profit structure

$$\pi(x) = A^2 - x^2.$$

Observe that once all the potential clients have bought, that is when $x = A$, then profits are zero and the only thing the firm can do is to introduce innovation and start again. Second, the innovation adoption costs are exactly modeled as in the monopolistic case. Therefore, once again, assuming a continuous discount factor $\delta > 0$, the firm wishes to maximize her expected discounted profit

$$J^c_{(\xi)}(x) := \mathbf{E}\left[\int_{[0,\infty)} e^{-\delta t}[\pi(X_t)dt - Id\xi_t]\right] \tag{2}$$

over all $\xi \in \mathcal{V}_M$. On the parameters we make the following assumption

A 1. $\delta > \sigma^2 + 2D$.

As we already said this is an infinite horizon autonomous *singular monotone follower "finite-fuel"* stochastic control. The *value function* of our problem is given by

$$v_c(x) := \sup_{\xi \in \mathcal{V}_M} J^c_{(\xi)}(x) \tag{3}$$

The following lemma shows that no solution of (1) exceeds A.

Lemma 1 *Let $x \leq A$, $\xi \in \mathcal{V}_+$, and X_t the unique strong solution of (1), then*

$$\mathbf{P}\left[X_t \leq A, \quad \forall t \geq 0\right] = 1.$$

Proof. Let X_t^0 the solution of (1) with $\xi = 0$, $\forall t \geq 0$. Then, due to the geometric brownian structure of the equation when $\xi = 0$, it follows

$$\mathbf{P}\left[X_t^0 \leq A, \quad \forall t \geq 0\right] = 1.$$

Next, consider the right-continuous semimartingale $Z_t := X_t - X_t^0$, then Z_t is the unique solution of the following SDE

$$Z_t = \int_0^t -DZ_s ds + \int_0^t -\sigma Z_s dW_s - \xi_t.$$

Arguing as in [13], let φ_n be a sequence of $C^2(\mathbb{R})$ functions[7] such that

$$\begin{cases} \varphi_n(z) = 0, \text{ for } z \leq 0, \\ 0 \leq \varphi_n'(z) \leq 1, \\ \varphi_n(z) \uparrow z^+, \text{ for } n \to \infty, \\ \varphi_n''(z) \leq 2(nz^2)^{-1}. \end{cases}$$

Then, by applying Itô's formula for semimartingales (see [32]) to $\varphi_n(Z_t)$ we obtain

$$\begin{aligned} \varphi_n(Z_t) &= \varphi_n(-\xi_0) + \int_0^t -DZ_s\varphi_n'(Z_s)ds + \frac{1}{2}\int_0^t \sigma^2 (Z_s)^2 \varphi_n''(Z_s)ds + \\ &+ \int_{(0,t]} \varphi_n'(Z_{s-})\, d(-\xi_s) + \int_0^t -\sigma Z_s \varphi_n'(Z_s) dW_s + \\ &+ \sum_{0<s\leq t} \left\{\left[\varphi_n(Z_s) - \varphi_n(Z_{s-})\right] - \varphi_n'(Z_{s-})\,\Delta Z_s\right\} \end{aligned} \tag{4}$$

where, $\Delta Z_s := Z_s - Z_{s-} = -\Delta \xi_s = \xi_s - \xi_{s-}$. Moreover

$$\begin{aligned} &\int_{(0,t]} \varphi_n'(Z_{s-})\, d(-\xi_s) = \\ &= -\int_0^t \varphi_n'(Z_s) d\xi_s^c - \sum_{0<s\leq t} \left\{\varphi_n'(Z_{s-})\,(\Delta\xi_s)\right\} \\ &= -\int_0^t \varphi_n'(Z_s) d\xi_s^c + \sum_{0<s\leq t} \left\{\varphi_n'(Z_{s-})\,\Delta Z_s\right\}, \end{aligned}$$

where ξ_s^c is the continuous part, in the Lebesgue decomposition, of ξ_s. Now, by taking the expectation of both sides of (4) and observing that:

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[7] Such a sequence could be

$$\varphi_n(z) = \begin{cases} ze^{-\frac{1}{nz}}, \text{ for } z > 0 \\ 0, \text{ for } z \leq 0, \qquad n = 1, 2, \ldots \end{cases}$$

i) $\varphi_n(-\xi_0) = 0$, for every n,
ii) $-\int_0^t \varphi_n'(Z_s)d\xi_s^c \leq 0$, for every n,
iii) $\varphi_n(Z_s) - \varphi_n(Z_{s-}) \leq 0$, since φ_n is non decreasing for every n, we get

$$\mathbf{E}[\varphi_n(Z_t)] \leq \mathbf{E}\left[\int_0^t -DZ_s\varphi_n'(Z_s)ds\right] + \frac{1}{2}\mathbf{E}\left[\int_0^t \sigma^2 (Z_s)^2 \varphi_n''(Z_s)ds\right].$$

Now, the first term on the right-hand side above is non-positive since $\varphi_n'(Z_s) = 0$ when $Z_s \leq 0$, and the last is bounded above by $\sigma^2 t/n$. Hence, as $n \to \infty$ we get $\mathbf{E}[(Z_t)^+] \leq 0$, for every $t \geq 0$, and the conclusion follows from the sample path right-continuity of Z_t. ■

Now, in order to handle the finite fuel feature of the problem[8], it simplifies matters if we first study a control problem in a two-dimensional state space and then look at our original problem as a restriction of the latter. Formally: Let's introduce a new state variable, Y_t, as defined: $Y_t = y - \xi_t$, where Y_t represents the remaining innovation at time t if y was available at time $t = 0$; consider the following two-dimensional stochastic differential equation

$$\begin{cases} X_t = x + \int_0^t D(A - X_s)ds + \int_0^t \sigma(A - X_s)dW_s - \xi_t \\ Y_t = y - \xi_t \end{cases}, \tag{5}$$

where $(x, y) \in (-\infty, A] \times \mathbb{R}_+ =: \mathcal{D}$; and maximize

$$J_{(\xi)}(x,y) := \mathbf{E}\left[\int_{[0,\infty)} e^{-\delta t}[\pi(X_t)dt - Id\xi_t]\right] \tag{6}$$

over all pairs $\xi \in \mathcal{V}_y$, where $\mathcal{V}_y$, the set of *admissible control* at $y \in \mathbb{R}_+$, is defined as follows

$$\mathcal{V}_y := \{\xi \in \mathcal{V}_+ \text{ such that } \xi_\infty \leq y, \mathbf{P} - a.s.\}.$$

The *value function* of this problem is given by

$$v(x,y) := \sup_{\xi \in \mathcal{V}_y} J_{(\xi)}(x,y) \tag{7}$$

Of course, $v_c(x) \equiv v(x, M)$ and from now on we will concentrate on the analysis of the two-dimensional problem. First, the linear concave structure of the problem enables us to prove the following

Theorem 2 *The value function $v(x,y)$ is concave and continuous in (x,y) and increasing in y. Moreover, there exists a constant $C > 0$ such that for each $(x,y) \in \mathcal{D}$*

$$-C(1 + x^2) \leq v(x,y) \leq C \tag{8}$$

[8] See [3], [31], [6].

Proof. Continuity and concavity can be proved exactly the same way we used in Chapter 1. Next, fix $x \leq A$ and take $y^1, y^2 \in \mathbb{R}_+$ such that $y^1 < y^2$, then $\mathcal{V}_{y^1} \subset \mathcal{V}_{y^2}$ and

$$\begin{aligned} v(x,y^1) &= \sup_{\xi \in \mathcal{V}_{y^1}} \mathbf{E}\left[\int_{[0,\infty)} e^{-\delta t}[\pi(X_t)dt - Id\xi_t]\right] \\ &\leq \sup_{\xi \in \mathcal{V}_{y^2}} \mathbf{E}\left[\int_{[0,\infty)} e^{-\delta t}[\pi(X_t)dt - Id\xi_t]\right] = v(x,y^2) \end{aligned}$$

which is what we claimed.

The second inequality in (8) is a direct consequence of the boundedness of π. To prove the first inequality it is suffices to prove it for $y = 0$. Now, let X_t^0 the solution of (1) with $\xi = 0$ and define $Z_t := X_t^0 - A$. Then Z_t is the unique solution of the SDE

$$Z_t = (x - A) + \int_0^t -DZ_s ds + \int_0^t -\sigma Z_s dW_s.$$

By first applying Ito's lemma to the function $f(Z_t) = (Z_t)^2$ and then taking the expectation we get

$$\begin{aligned} \mathbf{E}\left[(Z_t)^2\right] &= (x-A)^2 + (\sigma^2 - 2D)\int_0^t \mathbf{E}\left[(Z_s)^2\right] ds \\ &\leq (x-A)^2 + \left|\sigma^2 - 2D\right| \int_0^t \mathbf{E}\left[(Z_s)^2\right] ds \end{aligned}$$

An application of Gronwall inequality yields

$$\mathbf{E}\left[(Z_t)^2\right] \leq (x-A)^2 e^{|\sigma^2 - 2D|t}.$$

Observing that

$$\pi(X_t^0) = A^2 - \left(X_t^0\right)^2 \geq -2Z_t^2 - A^2,$$

we conclude by using A 2

$$\begin{aligned} v(x,y) &\geq v(x,0) = \mathbf{E}\left[\int_0^\infty e^{-\delta t}\pi(X_t)dt\right] \geq \mathbf{E}\left[\int_0^\infty e^{-\delta t}(-2Z_t^2 - A^2)dt\right] \\ &\geq -C(1+x^2). \end{aligned}$$

■

2.1 Some heuristics: The HJB equation

We now derive, though in a heuristic way, the HJB equation for $v(x,y)$. Invoking the *Dynamic Programming Principle*, and assuming that $v \in C^2(\mathcal{D})$, the following heuristic arguments motivate the conclusions.

First, consider the policy "*do nothing for a little while and then proceed optimally*"; then for $x \leq A$, $y \in \mathbb{R}_+$, and every $h > 0$ we have

$$v(x,y) \geq \mathbf{E}\left[\int_0^h e^{-\delta t}\pi(X_t^0)dt + e^{-\delta h}v(X_h^0, y)\right],$$

where (X_t^0, y) is the solution of system (5) with initial condition (x, y) and controls $\xi_t \equiv 0$. Subtracting $e^{-\delta h}v(x,y)$ from each side, we get

$$v(x,y)(1-e^{-\delta h}) \geq \mathbf{E}\left[\int_0^h e^{-\delta t}\pi(X_t^0)dt\right] + e^{-\delta h}\mathbf{E}\left[v(X_h^0, y) - v(x,y)\right].$$

Dividing by h and letting $h \downarrow 0$ we obtain

$$\delta v(x,y) \geq \frac{1}{2}\sigma^2(A-x)^2 v_{xx}(x,y) + D(A-x)v_x(x,y) + \pi(x). \tag{9}$$

Hence,

$$\mathbb{D}[v](x,y) + \pi(x) \leq 0, \quad \forall (x,y) \in \mathcal{D}$$

where

$$\mathbb{D}[v](x,y) \triangleq \frac{1}{2}\sigma^2(A-x)^2 v_{xx}(x,y) + D(A-x)v_x(x,y) - \delta v(x,y), \tag{10}$$

here subscripts indicate the partial derivatives.

Next, let $x \leq A$, $y \in \mathbb{R}_+$, and $0 < h \leq y$ and consider the strategy "*jump immediately from x to $x-h$ and then proceed optimally*"; this yields

$$v(x,y) \geq -Ih + v(x-h, y-h).$$

Subtracting $v(x,y)$ from each side, dividing by h and letting $h \downarrow 0$, we get

$$-v_x(x,y) - v_y(x,y) - I \leq 0, \quad \forall (x,y) \in (-\infty, A] \times (0, +\infty). \tag{11}$$

In view of the above heuristic arguments, we expect that the value function v should satisfy, at least formally, the following conditions:

$$\mathbb{D}[v](x,y) + \pi(x) \leq 0, \quad (x,y) \in (-\infty, A] \times [0, +\infty), \tag{12}$$

$$-v_x(x,y) - v_y(x,y) - I \leq 0, \quad (x,y) \in (-\infty, A] \times (0, +\infty). \tag{13}$$

Now, fix $(\bar{x}, \bar{y}) \in (-\infty, A] \times (0, +\infty)$ and assume $-v_x(\bar{x},\bar{y}) - v_y(\bar{x},\bar{y}) - I < 0$. Then (13) holds with the strict inequality in a whole neighborhood $B_\epsilon(\bar{x},\bar{y}) \subset (-\infty, A] \times (0, +\infty)$. Moreover, for every $(x,y) \in B_\epsilon(\bar{x},\bar{y})$ there exists $h > 0$ sufficiently small, dependent of (x,y), such that $v(x,y) > -Ih + v(x-h, y-h)$ and $v(x-h, y-h) \in B_\epsilon(\bar{x},\bar{y})$ but this means that in $B_\epsilon(\bar{x},\bar{y})$ we are better off if we do not adopt innovation (i.e. $d\xi = 0$). In turn this implies that in $B_\epsilon(\bar{x},\bar{y})$

the state variable evolves freely and therefore (12) holds with equality. Hence, $v(x, y)$ should satisfy, in some sense, the following variational inequality

$$\max\{\mathbb{D}[v](x,y) + \pi(x); -v_x(x,y) - v_y(x,y) - I\} = 0, \tag{14}$$

for every $(x, y) \in (-\infty, A] \times (0, +\infty)$. An appropriate boundary condition for (14) is

$$v(x, 0) = u^0(x), \quad \forall x \leq A, \tag{15}$$

where $u^0(x)$ is the value function corresponding to the problem with $\xi_t \equiv 0$.

In the next two sections we are going to analyze the two extreme cases of no innovation at all and the case of infinite innovation available. Once this has been done we will see how the value function (7) can be obtained as an appropriate combination of the extreme cases value functions.

3 Extreme case: no innovation

When no innovation is available the state dynamics became the following uncontrolled diffusion

$$X_t^0 = x + \int_0^t D(A - X_s^0)ds + \int_0^t \sigma(A - X_s^0)dW_s,$$

with initial condition $x \leq A$. The value function is simply

$$u^0(x) \triangleq \mathbf{E}\left[\int_0^\infty e^{-\delta t}\pi(X_t^0)dt\right] = \mathbf{E}\left[\int_0^\infty e^{-\delta t}(A^2 - (X_t^0)^2)dt\right],$$

where we clearly expect $u^0(A) = 0$. Therefore, applying the dynamic programming principle, we deduce that, for each $x \leq A$, u^0 should satisfy the following HJB equation

$$\begin{cases} \frac{1}{2}\sigma^2(A-x)^2v''(x) + D(A-x)v'(x) - \delta v(x) + \pi(x) = 0 \\ v(A) = 0 \end{cases} \tag{16}$$

The differential equation in $(16)_1$ is a second order linear differential equation of Euler type; therefore it can be integrated by quadrature. The general solution is

$$v(x) = C_1(A-x)^{\lambda_1} + C_2(A-x)^{\lambda_2} + H(A-x)^2 + K(A-x), \tag{17}$$

where C_1, C_2 are two arbitrary real constants,

$$H = \frac{1}{\sigma^2 - \delta - 2D}, \qquad K = \frac{2A}{D + \delta}, \tag{18}$$

and λ_1, λ_2 are the two real distinct roots of

$$\frac{1}{2}\sigma^2\lambda^2 - \left(\frac{1}{2}\sigma^2 + D\right)\lambda - \delta = 0.$$

For λ_1, λ_2 we have

$$\lambda_1 = \sigma^{-2}\left(\frac{1}{2}\sigma^2 + D - \sqrt{\left(\frac{1}{2}\sigma^2 + D\right)^2 + 2\sigma^2\delta}\right), \tag{19}$$

$$\lambda_2 = \sigma^{-2}\left(\frac{1}{2}\sigma^2 + D + \sqrt{\left(\frac{1}{2}\sigma^2 + D\right)^2 + 2\sigma^2\delta}\right). \tag{20}$$

Now, $\lambda_1 < 0$ and Assumption (A.1) implies $H < 0$ and $\lambda_2 > 2$. Therefore, since u^0 has to satisfy (8) and $(16)_2$, we conclude that $C_1 = C_2 = 0$ and

$$u^0(x) = H(A-x)^2 + K(A-x), \tag{21}$$

for each $x \leq A$. Hence, u^0 is concave, although this feature could have been seen immediately.

4 Extreme case: infinite innovation

If the firm has infinite innovation available to be introduced into the market then the state dynamics is

$$X_t = x + \int_0^t D(A - X_s)ds + \int_0^t \sigma(A - X_s)dW_s - \xi_t, \tag{22}$$

where the initial condition is such that $x \leq A$, and $\xi \in \mathcal{V}_+$. Hence, we maximize

$$J_\xi^\infty(x) \triangleq \mathbf{E}\left[\int_{[0,\infty)} e^{-\delta t}[\pi(X_t)dt - Id\xi_t]\right], \tag{23}$$

over all $\xi \in \mathcal{V}_+$. For each $x \leq A$, the value function becomes

$$v_\infty(x) \triangleq \sup_{\xi \in \mathcal{V}_+} J_\xi^\infty(x), \tag{24}$$

and it is concave with growth condition given by (8).

Again, assuming that $v_\infty \in C^2((-\infty, A])$, a formal application of the *Dynamic Programming Principle* yields, for each $x \leq A$, the following HJB for v_∞

$$\begin{cases} \frac{1}{2}\sigma^2(A-x)^2 v''(x) + D(A-x)v'(x) - \delta v(x) + A^2 - x^2 \leq 0 \\ -v'(x) - I \leq 0 \\ \left(\frac{1}{2}\sigma^2(A-x)^2 v''(x) + D(A-x)v'(x) - \delta v(x) + A^2 - x^2\right)(-v'(x) - I) = 0. \end{cases} \tag{25}$$

This is a free boundary problem. We are going to solve it by direct computations. In other words we will impose the *smooth fit* conditions[9] at the unknown

[9] See [3], [18].

boundary, and, thanks to the particular simple structure of the equation, we will be able to explicitly find out the free boundary and the value function. Therefore, since the general solution of $(25)_1$ is given by (17), we are looking for a point $x^* < A$ and a pair of real constants C_1, C_2 such that

$$\begin{cases} v'(x^*) = -I \\ v''(x^*) = 0 \end{cases} . \tag{26}$$

First of all, the growth condition (8), required for v_∞, implies $C_2 = 0$ since $\lambda_2 > 2$. Hence, conditions (26) become the following non-linear algebraic system

$$\begin{cases} v'(x^*) = -\lambda_1 C_1 (A - x^*)^{\lambda_1 - 1} - 2H(A - x^*) - K = -I \\ v''(x^*) = \lambda_1(\lambda_1 - 1) C_1 (A - x^*)^{\lambda_1 - 2} + 2H = 0, \end{cases}$$

to be solved for x^* and C_1. Such a solution is explicitly given by

$$x^* = A + \frac{I - K}{2H(\frac{2-\lambda_1}{\lambda_1 - 1})}, \tag{27}$$

$$C_1 = \frac{-2H}{\lambda_1(\lambda_1 - 1)(\frac{K-I}{2H(\frac{2-\lambda_1}{\lambda_1-1})})^{\lambda_1 - 2}}, \tag{28}$$

where H, K, and λ_1 are respectively defined in (18) and (19). We first recall that $\lambda_1 < 0$ and then observe that if $K > I$ then $C_1 > 0$. Now, two cases can be single out: *Case a)* $I \geq K$; *Case b)* $I < K$.

Case a) $I \geq K$. In this case x^* should be greater than or equal to A, but a straightforward computation would prove that the function u^0, defined in (21) is a solution of (25). In other words the free boundary problem disappear (see Figure 1).

For *Case b)* (see Figure 2) we prove the following

Theorem 3 *Let $I < K$ and consider x^*, C_1 as in (27) and (28). Then*
i) $x^* < A$ and $C_1 > 0$;
ii) the function

$$\bar{v}(x) = \begin{cases} C_1(A - x)^{\lambda_1} + H(A - x)^2 + K(A - x), & x \in (-\infty, x^*] \\ C_1(A - x^*)^{\lambda_1} + H(A - x^*)^2 + K(A - x^*) - I(x - x^*), & x \in [x^*, A] \end{cases}, \tag{29}$$

is a two times continuous differentiable concave solution of (25) satisfying the bounds in (8).

Proof. i) This is obvious.
ii) $\bar{v} \in C^2((-\infty, A])$ by construction. Moreover, for each $x \leq x^*$ the function $v''(x) = \lambda_1(\lambda_1 - 1)C_1(A - x)^{\lambda_1 - 2} + 2H$ is increasing, hence it reaches the

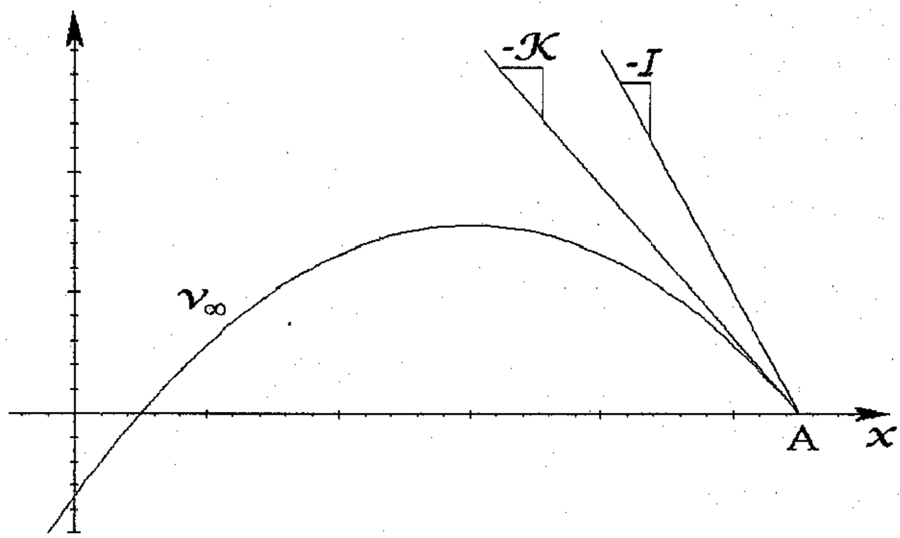

Figure 1: *Case a)* $I \geq K$

maximum at x^*, and $v''(x^*) = 0$. This imply the desired concavity of $\bar{v}$. The bounds in (8) are easily seen by noting that $C_1 > 0$, and

$$u^{0\prime}(x^*) = -2H(A - x^*) - K = \frac{I + K - \lambda_1 I}{\lambda_1 - 2} < -I.$$

Finally, we need to prove that $\bar{v}$ is a solution of (25). If $x \leq x^*$ then $(25)_1$ is satisfied with the equality and $(25)_3$ follows immediately. To verify $(25)_2$ it suffices to notice that $\bar{v}$ is concave.

We now turn to the interval $[x^*, A]$. Clearly $(25)_2$ and $(25)_3$ are satisfied with equality. It remains to prove $(25)_1$. Since $x^* < A$ and

$$\begin{cases} \frac{1}{2}\sigma^2(A - x^*)^2\bar{v}''(x^*) + D(A - x^*)\bar{v}'(x^*) - \delta\bar{v}(x^*) + A^2 - (x^*)^2 = 0 \\ \bar{v}'(x^*) = -I \\ \bar{v}''(x^*) = 0 \end{cases},$$

it follows that

$$\bar{v}(x^*) = \frac{-DI(A - x^*) + A^2 - (x^*)^2}{\delta}.$$

Therefore, for each $x \in [x^*, A]$, we have

$$\bar{v}(x) = \frac{-DI(A - x^*) + A^2 - (x^*)^2}{\delta} - I(x - x^*), \tag{30}$$

with $\bar{v}'(x) = -I$, and $\bar{v}''(x) = 0$. If we substitute $\bar{v}$ into $(25)_1$ we get

$$(x - x^*)(DI - x^* - x + \delta I) \leq 0.$$

First, the last inequality is certainly true for $x = x^*$. For $x \in (x^*, A]$ we may divide both sides by $(x - x^*)$ and we get

$$DI - x^* - x + \delta I \leq 0,$$

to hold for each $x \in [x^*, A]$. Now, the expression on the left hand side of the last inequality attains its maximum on $[x^*, A]$ at $x = x^*$. Therefore, if x^* is such that

$$x^* \geq \frac{DI + \delta I}{2} \tag{31}$$

then $\bar{v}(x)$ defined in (30) solves $(25)_1$ on $[x^*, A]$. Now, by using concavity of $\bar{v}$ on $(-\infty, x^*]$ we can show that x^* actually satisfies (31). In fact, $\bar{v}$ satisfies $(25)_1$ with equality on $(-\infty, x^*]$, and by deriving one more time we get

$$\frac{1}{2}\sigma^2(A - x)^2\bar{v}'''(x) + \left(D(A - x) - \sigma^2(A - x)\right)\bar{v}''(x) - (D + \delta)\bar{v}'(x) - 2x = 0.$$

At $x = x^*$ we have

$$\frac{1}{2}\sigma^2(A - x^*)^2\bar{v}'''(x^*) + (D + \delta)I - 2x^* = 0.$$

Now, since $\bar{v}$ is concave in a left neighborhood of x^* then at x^* we must have

$$\frac{1}{2}\sigma^2(A - x^*)^2\bar{v}'''(x^*) \geq 0.$$

Equivalently, x^* must be such that

$$(D + \delta)I - 2x^* \leq 0.$$

Therefore $\bar{v}$ satisfies $(25)_1$ also on $[x^*, A]$. The proof is complete. ■

We now state theorems on the existence of optimal controls and on the equivalence of v_∞ defined in (24) with $\bar{v}$ defined in (29). We need to consider separately the two cases aforesaid.

First, we deal with *Case a)*, that is we assume $I \geq K$. For the proof of the following theorem we refer to [24], Theorem 32.

Theorem 4 *Assume $I \geq K$. Let u^0 and $J_\xi^\infty(x)$ defined, respectively in (21) and (23). Then*

i) for every $\xi \in \mathcal{V}_+$ and for each $x \leq A$

$$u^0(x) \geq J_\xi^\infty(x);$$

ii) for $\xi_t \equiv 0$ and $x \leq A$

$$u^0(x) = J_0^\infty(x).$$

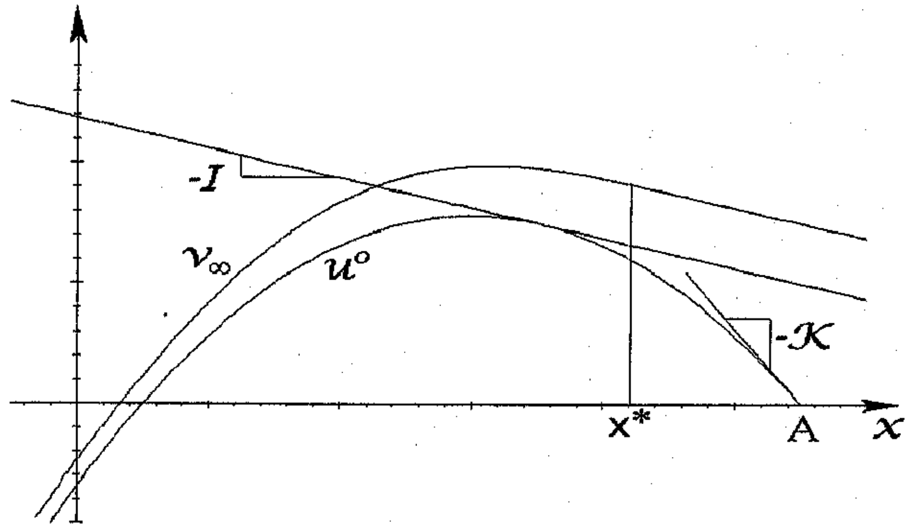

Figure 2: *Case b) $I < K$*

Remark 5 *As a result, when $I \geq K$, we have $u^0(x) = v_\infty(x)$ for each $x \leq A$.*

In *Case b)*, that is when $I < K$, the following theorem states the existence of a particular admissible control that later on will turn out to be optimal.

Theorem 6 *Let x^* as in (27). Then for each $x \leq A$, there exists a unique $\xi^* \in \mathcal{V}_+$ such that with X_t^* defined by (22), the following conditions hold almost surely:*
i) if $x \in (x^*, A]$, then $X_0^* = x^*$;
ii) $X_t^* \leq x^*, \quad \forall t \geq 0$;
iii) $\xi_t^* = \int_{[0,t]} \chi_{\{X_s^*: X_s^* = x^*\}} d\xi_s^*, \quad \forall t \geq 0$.

This is again a *Skorohod problem.* As we have seen in Chapter 1, the non degeneracy of X_t^*, at least for $x \leq x^* < A$ (which is actually our case), is a sufficient condition for the existence of a unique solution. For a complete proof we refer to [23].

We finally state the following verification theorem (for its proof we refer to Theorem 28 in [24]) that, for $I < K$, guaranties

$$\bar{v}(x) = v_\infty(x), \qquad x \leq A,$$

with $\bar{v}$ and v_∞ respectively defined by (29) and (24).

Theorem 7 *Let x^* be as in (27) and let $\bar{v}$ be defined (29). Then*
i) for every $\xi \in \mathcal{V}_+$

$$\bar{v}(x) \geq J_\xi^\infty(x), \qquad x \leq A;$$

ii) for ξ_t^* defined in Theorem 6

$$\bar{v}(x) = J_{\xi^*}^\infty(x), \qquad x \leq A.$$

5 The original problem

We now turn to problem (7). For it, the following theorem can be proved (for the proof we refer the reader to [31], Theorem 3.3, p.796 and [6], Theorem 4.2, p. 887).

Theorem 8 *Assume (A.1) and $I < K$. Then, for every $y > 0$, the value function $v(x, y)$ of the finite-fuel monotone stochastic control problem (7) can be decomposed as*

$$v(x, y) = v_\infty(x) + u^0(x - y) - v_\infty(x - y), \qquad \forall x \leq A, \tag{32}$$

where u^0 is the expected total profit in the case of zero innovation (cf. (21), and v_∞ is the value function of the problem (24) with infinite innovation.

Again referring to [31], Theorem 3.4, p.797 (see also [18], Theorem 1, p. 5580, for a probabilistic approach), it can be proved the following

Theorem 9 *Assume (A.1) and $I < K$. Let $\xi^* \in \mathcal{V}_+$ be the optimal control process for the infinite innovation problem (24), and define*

$$T(y) \triangleq \inf\{t \geq 0 : \xi_t^* \geq y\}.$$

Then, the control process $\xi^y \in \mathcal{V}_+$, given by

$$\xi_t^y \triangleq \begin{cases} \xi_t^*, & 0 \leq t \leq T(y) \\ y & t \geq T(y) \end{cases} \tag{33}$$

is optimal for the control problem (7); that is

$$v(x, y) = J_{\xi^y}(x, y),$$

where $J_{\xi^y}(x, y)$ is defined by (6).

Remark 10 *We point out that according to (33), in the case of a finite amount y of available innovation, it is optimal to adopt innovation as if one possessed an infinite amount of it, until the supply is exhausted.*

6 Conclusions

Comparing the results between the monopolistic case studied in [25] and the competitive one here studied, we conclude that, despite the fact that a competitive firm has a finite amount of adoptable innovation, she employes such a resource in exactly the same way as the monopolist does. That is the adoption policy does not depend of the amount of innovation available and this turn out to be a quite astonishing conclusion.

In addition to that and in contrast with the previous literature, our model motivates the, empirically evident, innovation adoption delay mainly as a matter of market demand and product diffusion on the market itself (that is market saturation).

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