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On the pseudoaffinity of a class of quadratic fractional functions

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Abstract

Some characterizations of pseudoaffine quadratic fractional functions are studied and it is proved that these functions share a particular structure. The wider class of functions given by the sum of a quadratic fractional function and a linear one is also studied, characterizing their pseudoaffinity by means of simple conditions. The use of pseudoaffine quadratic fractional functions in optimization problems is also deepened on and a simple procedure which checks the pseudoaffinity of these functions is provided.

Keywords Generalized Convexity, Fractional Programming, Quadratic Programming

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1 Introduction

Quadratic fractional programming deals with constrained optimization problems where the objective function is the ratio of a quadratic and an affine one. Due to its importance in application models, this particular class of nonlinear programs has been widely studied both from a theoretical and an algorithmic point of view (see for example [3], [2]).

Many solving algorithms have been given for quadratic fractional problems whose feasible region is a polyhedron. In these cases the generalized

*The paper has been discussed jointly by the authors. In particular, sections 1 and 5 have been developed by Laura Carosi, section 2 and the appendix by Riccardo Cambini, while sections 3 and 4 have been developed jointly by the authors.

convexity of the objective function plays a fundamental role, since it guarantees the global optimality of local optima.

Among generalized convex functions, the pseudoaffine ones are extremely useful since the above nice properties hold for both maximum and minimum problems. We recall that:

> a function $f : A \to \Re$, where $A \subseteq \Re^n$ is a convex set, is said to be *pseudoaffine* if it is both pseudoconcave and pseudoconvex.

It is known (see for all [1]) that when f is a differentiable pseudoaffine function it results that

- f is nonconstant if and only if $\nabla f(x) \neq 0$ for every $x \in A$

while if f is also nonconstant then, given a closed set $X \subseteq A$, the following properties hold:

- there are neither local maxima nor local minima in the interior of X,
- if X is a polyhedral set then the maximum and minimum values are reached on a vertex.

Thanks to their properties, pseudoaffine functions play a key role both in finding optimality conditions and in implementing algorithms for applicative problems.

In this paper we aim to characterize the pseudoaffinity of quadratic fractional functions, looking for necessary and sufficient conditions which can be easily checked. By means of the proposed characterizations we prove that a quadratic fractional function is pseudoaffine if and only if it can be rewritten as the sum of a linear function and a linear fractional one with constant numerator. This result allow us to give conditions characterizing the pseudoaffinity of the wider class of functions given by the sum of a quadratic fractional function and a linear one.

Furthermore we prove that optimization problems, involving a pseudoaffine quadratic fractional function, can be solved through equivalent linear ones.

Finally, we provide an easy procedure which checks the pseudoaffinity of a quadratic fractional function.

2 Preliminary results

A very well known characterization of pseudoaffine functions is the following (see for example [1],[15]).

Theorem 1 *A differentiable function $f : A \to \Re$, $A \subseteq \Re^n$ open convex set, is pseudoaffine if and only if the following implication holds $\forall x \in A$, $\forall v \in \Re^n$, $v \neq 0$, $\forall t \in \Re$ such that $x + tv \in A$:*

$$\nabla f(x)^T v = 0 \quad \Rightarrow \quad \phi_v(t) = f(x + tv) \text{ is constant}$$

In the next section some new characterizations of pseudoaffine quadratic fractional functions will be given using the inertia of symmetric matrices. With this regards, the number of the negative eingenvalues of a symmetric matrix Q is denoted by $\nu_-(Q)$, similarly $\nu_+(Q)$ represents the number of the positive eingenvalues while $\nu_0(Q)$ is the algebraic multiplicity of the 0 eigenvalue. A key tool in our study is the following result given by Crouzeix (see [12]).

Theorem 2 *Let $h \in \Re^n$, $h \neq 0$, and let $Q \in \Re^{n \times n}$ be a symmetric matrix. Then the following implication*

$$h^T v = 0 \quad \Rightarrow \quad v^T Q v \geq 0$$

is verified for every $v \in \Re^n$ if and only if one of the following conditions holds:

i) $\nu_-(Q) = 0$,

ii) $\nu_-(Q) = 1$, $h \in Q(\Re^n)$ *and* $u^T Q u \leq 0$ $\forall u \in \Re^n$ *such that* $Qu = h$.

Starting form the above theorem, we can state the following property which is a key tool in characterizing the pseudoaffinity of quadratic fractional functions.

Corollary 3 *Let $h \in \Re^n$, $h \neq 0$, and let $Q \in \Re^{n \times n}$, $Q \neq 0$, be a symmetric matrix. Then the following implication*

$$h^T v = 0 \quad \Rightarrow \quad v^T Q v = 0 \tag{1}$$

is verified for every $v \in \Re^n$ if and only if one of the following conditions holds:

i) $\nu_0(Q) = n-1$ *(hence Q is positive semidefinite or negative semidefinite) and* $h \in Q(\Re^n)$,

ii) $\nu_-(Q) = \nu_+(Q) = 1$ *(hence Q is indefinite),* $h \in Q(\Re^n)$ *and* $u^T Q u = 0$ $\forall u \in \Re^n$ *such that* $Qu = h$.

Proof. First note that, from Theorem 2,

$$h^T v = 0 \quad \Rightarrow \quad v^T Q v \leq 0$$

is verified for every $v \in \Re^n$ if and only if one of the following conditions holds:

a) $\nu_+(Q) = 0$,

b) $\nu_+(Q) = 1$, $h \in Q(\Re^n)$ and $u^T Q u \geq 0 \ \forall u \in \Re^n$ such that $Qu = h$.

$\Leftarrow$*)* If *i)* holds and Q is positive semidefinite then $\nu_+(Q) = 1$, $\nu_-(Q) = 0$, $h \in Q(\Re^n)$ and $u^T Q u \geq 0 \ \forall u \in \Re^n$. Thus *i)* of Theorem 2 and condition *b)* hold, hence $h^T v = 0$ implies $v^T Q v \geq 0$ and $v^T Q v \leq 0$, so that Condition (1) holds. The case Q negative semidefinite can be proved with the same arguments. If *ii)* holds then both conditions *b)* and *ii)* of Theorem 2 are verified; again $h^T v = 0$ implies $v^T Q v \geq 0$ and $v^T Q v \leq 0$.

$\Rightarrow$*)* First note that Condition (1) holds if and only if

$$\{h^T v = 0 \quad \Rightarrow \quad v^T Q v \geq 0\} \quad \text{and} \quad \{h^T v = 0 \quad \Rightarrow \quad v^T Q v \leq 0\}$$

and this happens if and only if one of conditions *i)* and *ii)* of Theorem 2 holds together with one of conditions *a)* and *b)*. Observe that conditions *a)* and *i)* of Theorem 2 imply $Q = 0$ which is a contradiction. Conditions *a)* and *ii)* of Theorem 2 imply condition *i)* and the same happens if *b)* and *i)* of Theorem 2 hold. If otherwise conditions *b)* and *ii)* of Theorem 2 are verified then condition *ii)* follows immediately. The result is then proved since all the possible exaustive cases have been considered. ■

3 Pseudoaffinity of quadratic fractional functions

In this section we are going to characterize the pseudoaffinity of quadratic fractional functions of the following kind ([1]):

$$f(x) = \frac{\frac{1}{2}x^T Q x + q^T x + q_0}{b^T x + b_0} \tag{2}$$

defined on the set $X = \{x \in \Re^n : b^T x + b_0 > 0\}$, where $Q \neq 0$ is a $n \times n$ symmetric matrix, $q, x, b \in \Re^n$, $b \neq 0$, and $q_0, b_0 \in \Re$. Note that being Q symmetric, it is $Q \neq 0$ if and only if $\nu_0(Q) \leq n - 1$.

[1]Note that the pseudoconvexity of this class of function has been recently studied in [6] and [9].

Remark 4 It is important to point out that function f in (2) is not constant. Suppose by contradiction that f is constant, that is $f(x) = k$ and $\nabla f(x) = \dfrac{Qx + q - f(x)b}{b^T x + b_0} = 0\ \forall x \in X$. Consider an arbitrary $x_1 \in X$ and let $\alpha \in \Re$ be such that $\alpha \neq 0$ and $\alpha x_1 \in X$; it results $Qx_1 + q - kb = Q\alpha x_1 + q - kb$ and hence $Qx_1 = \alpha Q x_1$ which implies $Qx_1 = 0$, i.e. $Qx = 0\ \forall x \in X$. Since X is an n-dimensional halfspace it must be $Q = 0$ which contradicts the definition of (2).

The next theorem gives a new characterization of the pseudoaffinity of f based on the behavior of Q along the directions orthogonal to $\nabla f(x)$.

Theorem 5 *Function f defined in (2) is pseudoaffine if and only if the following implication holds $\forall x \in X$, $\forall v \in \Re^n$, $v \neq 0$, $\forall t \in \Re$ such that $x + tv \in X$:*

$$\nabla f(x)^T v = 0 \quad \Rightarrow \quad v^T Q v = 0$$

Proof. From Theorem 1 f is pseudoaffine if and only if the following implication holds $\forall x \in X$, $\forall v \in \Re^n$, $v \neq 0$:

$$\nabla f(x)^T v = 0 \quad \Rightarrow \quad \phi_v(t) = f(x + tv) \text{ is constant}$$

By means of simple calculations we have that:

$$\nabla f(x) = \frac{Qx + q - f(x)b}{b^T x + b_0}$$
$$\phi_v(t) = f(x + tv) = f(x) + \frac{\frac{1}{2}t^2 v^T Q v + t(b^T x + b_0)\nabla f(x)^T v}{b^T x + b_0 + t b^T v}$$

When $\nabla f(x)^T v = 0$ then $\phi_v(t) = f(x) + \frac{\frac{1}{2}t^2 v^T Q v}{b^T x + b_0 + t b^T v}$ and this restriction comes out to be constant if and only if $v^T Q v = 0$. ■

The following further characterization, based on the inertia of Q, can now be proved.

Corollary 6 *Function f defined in (2) is pseudoaffine if and only if one of the following conditions holds:*

i) $\nu_0(Q) = n-1$ *(hence Q is positive semidefinite or negative semidefinite) and* $\nabla f(x) \in Q(\Re^n)$, $\nabla f(x) \neq 0$, $\forall x \in X$,

ii) $\nu_-(Q) = \nu_+(Q) = 1$ *(hence Q is indefinite)*, $\nabla f(x) \in Q(\Re^n)$, $\nabla f(x) \neq 0$, $\forall x \in X$ *and* $u^T Q u = 0\ \forall u \in \Re^n$ *such that* $Qu = \nabla f(x)$.

Proof. $\Leftarrow$ Follows directly from Theorem 5 and Corollary 3.
$\Rightarrow$ Suppose by contradiction that there exists $x \in X$ such that $\nabla f(x) = 0$. Since f is pseudoaffine $\nabla f(x) = 0$ implies that f is constant and this contradicts (2) as it has been already pointed out in Remark 4. Since $\nabla f(x) \neq 0$ $\forall x \in X$, the result follows directly from Theorem 5 and Corollary 3. ∎

Now we aim to prove that all the pseudoaffine quadratic fractional functions can be rewritten in the same way. For this reason we first prove the following lemma.

Lemma 7 *Let us consider function f defined in (2). Then:*

$$\nabla f(x) \in Q(\Re^n)\ \forall x \in X \quad \Leftrightarrow \quad \exists \overline{x}, \overline{y} \in \Re^n \text{ such that } Q\overline{x} = q \text{ and } Q\overline{y} = b$$

and, in particular, for any given $x \in X$:

$$Qu = \nabla f(x) \quad \Leftrightarrow \quad u = \frac{x + \overline{x} - f(x)\overline{y}}{b^T x + b_0} + k \text{ with } k \in \ker(Q)$$

and $u^T Q u = \frac{p(x)}{(b^T x + b_0)^2}$ with:

$$p(x) = (f(x))^2 b^T \overline{y} + 2f(x)\left[b_0 - b^T \overline{x}\right] + \left(q^T \overline{x} - 2q_0\right)$$

Proof. Suppose that

$$\nabla f(x) = \frac{Qx + q - f(x)b}{b^T x + b_0} \in Q(\Re^n)\ \forall x \in X$$

and let us prove that

$$\exists \overline{x}, \overline{y} \in \Re^n \text{ such that } Q\overline{x} = q \text{ and } Q\overline{y} = b.$$

Since f is not constant, $\exists x_1, x_2 \in X$ such that $f(x_1) \neq f(x_2)$ and hence $\exists u_1, u_2 \in \Re^n$ such that

$$Qu_1 = Qx_1 + q - f(x_1)b \quad \text{and} \quad Qu_2 = Qx_2 + q - f(x_2)b.$$

This implies that

$$Q\left(\frac{u_1 - u_2 - x_1 + x_2}{f(x_2) - f(x_1)}\right) = b$$

and hence $\exists \overline{y} \in \Re^n$ such that $Q\overline{y} = b$. It follows also that $Qu_1 = Qx_1 + q - f(x_1)Q\overline{y}$ which implies $q = Q(u_1 - x_1 + f(x_1)\overline{y})$ and hence $\exists \overline{x} \in \Re^n$ such that $Q\overline{x} = q$.

Suppose now that $\exists \overline{x}, \overline{y} \in \Re^n$ such that $Q\overline{x} = q$ and $Q\overline{y} = b$; then

$$\nabla f(x) = Q\left(\frac{x + \overline{x} - f(x)\overline{y}}{b^T x + b_0}\right),$$

so that $\nabla f(x) \in Q(\Re^n)$ $\forall x \in X$. From the above proved equivalence it follows that $Qu = \nabla f(x)$ if and only if

$$Q\left(u - \frac{x + \overline{x} - f(x)\overline{y}}{b^T x + b_0}\right) = 0$$

and this happens if and only if

$$\left(u - \frac{x + \overline{x} - f(x)\overline{y}}{b^T x + b_0}\right) = k \in \ker(Q).$$

The whole result is finally proved noticing that:

$$\begin{aligned} u^T Q u &= \frac{1}{(b^T x + b_0)^2}\left[(x + \overline{x} - f(x)\overline{y})^T Q (x + \overline{x} - f(x)\overline{y})\right] \\ &= \frac{1}{(b^T x + b_0)^2}\left[(x + \overline{x} - f(x)\overline{y})^T (Qx + q - f(x)b)\right] \\ &= \frac{(f(x))^2 b^T \overline{y} + 2f(x)(b_0 - b^T\overline{x}) + (q^T\overline{x} - 2q_0)}{(b^T x + b_0)^2}. \end{aligned}$$

■

Using Lemma 7 we are able to state the following result, related to Condition *i)* in Corollary 6, which will be a key tool in characterizing the pseudoaffinity of f.

Lemma 8 *Function f defined in (2) can be rewritten in the following form:*

$$f(x) = \alpha b^T x + \beta + \frac{\alpha\gamma}{b^T x + b_0}$$

where $\alpha, \beta, \gamma \in \Re$, $\alpha \neq 0$, if and only if

$$\nu_0(Q) = n - 1, \nabla f(x) \in Q(\Re^n)\ \forall x \in X.$$

Moreover it results $\nabla f(x) \neq 0$ $\forall x \in X$ if and only if $\gamma \leq 0$.

Proof. $\Rightarrow$) By means of simple calculations $Q = [2\alpha b b^T]$, hence $v_0(Q) = n - 1$. Since $\nabla f(x) = \alpha\left[1 - \dfrac{\gamma}{(b^T x + b_0)^2}\right] b$ it is $\nabla f(x) \in Q(\Re^n), \forall x \in X$.
$\Leftarrow$) From Lemma 7 our assumption becomes

$$\nu_0(Q) = n - 1 \text{ and } \exists \overline{x}, \overline{y} \in \Re^n \text{ such that } Q\overline{x} = q \text{ and } Q\overline{y} = b.$$

Since $b \neq 0$ and $\dim(Q(\Re^n)) = 1$ then $Q\overline{x} = q$ if and only if $\exists \mu \in \Re$ such that $q = \mu b$.
Since $b \in Q(\Re^n)$ and $\dim(Q(\Re^n)) = 1$ then b is eigenvector of Q and hence, being Q symmetric, there exists $\alpha \in \Re, \alpha \neq 0$ such that $Q = [2\alpha bb^T]$ and $\overline{y} = \frac{1}{2\alpha\|b\|^2}b$, $2b^T\overline{y} = \frac{1}{\alpha}$. Consequently

$$\begin{aligned} f(x) &= \frac{\alpha(b^Tx)^2 + \mu b^Tx + q_0}{b^Tx + b_0} = \\ &= \frac{\alpha\left[(b^Tx + b_0) - b_0\right]^2 + \mu b^Tx + \mu b_0 - \mu b_0 + q_0}{b^Tx + b_0} \\ &= \alpha b^Tx + (\mu - \alpha b_0) + \frac{\alpha b_0^2 - \mu b_0 + q_0}{b^Tx + b_0}. \end{aligned}$$

The result then follows defining $\beta = (\mu - \alpha b_0)$ and $\gamma = b_0^2 + \frac{1}{\alpha}(q_0 - \mu b_0)$.

To prove the second part of the lemma, note that

$$\nabla f(x) = \alpha \left[1 - \frac{\gamma}{(b^Tx + b_0)^2}\right] b$$

with $\alpha \neq 0$, $b \neq 0$. Hence it results $\nabla f(x) \neq 0 \ \forall x \in X$ if and only if

$$(b^Tx + b_0)^2 \neq \gamma \ \forall x \in X. \tag{3}$$

By definition $\{y \in \Re : y = b^Tx + b_0,\ x \in X\} = \Re_{++}$ so that (3) holds if and only if $\gamma \leq 0$. ∎

We are now ready to provide the following characterization of quadratic fractional pseudoaffine functions.

Theorem 9 *Function f defined in (2) is pseudoaffine on X if and only if f is affine or there exist $\alpha, \beta, \gamma \in \Re$, $\alpha \neq 0$, such that f can be rewritten in the following form:*

$$f(x) = \alpha b^Tx + \beta + \frac{\alpha\gamma}{b^Tx + b_0} \quad \textit{with } \gamma < 0.$$

Proof. $\Rightarrow$) Since f is pseudoaffine, either condition *i)* or condition *ii)* of Corollary 6 holds. If *i)* is verified the results follows from Lemma 8 noticing that f is affine when $\gamma = 0$. Suppose now that condition *ii)* holds; from Lemma 7 it results

$$\nu_-(Q) = \nu_+(Q) = 1,\ \exists \overline{x}, \overline{y} \in \Re^n \text{ s.t. } Q\overline{x} = q \text{ and } Q\overline{y} = b,\ p(x) = 0\ \forall x \in X$$

with $\nabla f(x) \neq 0 \ \forall x \in X$. Being f nonconstant then $p(x) = 0 \ \forall x \in X$ if and only if $b^T\overline{y} = 0$, $b^T\overline{x} = b_0$ and $q^T\overline{x} = 2q_0$.

Since $\nu_-(Q) = \nu_+(Q) = 1$, from the canonical form of Q we get $Q = [uu^T - vv^T]$ where u and v are eigenvectors of Q with $u^Tv = 0$. From $Q\overline{y} = b$, $b^T\overline{y} = \overline{y}^TQ\overline{y} = 0$ we have

$$\overline{y}^TQ\overline{y} = \left(u^T\overline{y}\right)^2 - (v^T\overline{y})^2 = 0$$

so that $v^T\overline{y} = \pm\, u^T\overline{y}$. Then

$$b = Q\overline{y} = u\left(u^T\overline{y}\right) - \left(v^T\overline{y}\right)v = \left(u^T\overline{y}\right)(u + \delta v),$$

where $\delta = \pm 1$. By defining $a = \frac{1}{2u^T\overline{y}}(u - \delta v)$ and performing simple calculations we get

$$\left(ab^T + ba^T\right) = [uu^T - vv^T] = Q$$

Note that a and b are linearly independent. Let $\overline{x} \in \Re^n$ such that $Q\overline{x} = q$ and define $a_0 = a^T\overline{x}$. It results

$$\begin{aligned} q &= ab^T\overline{x} + ba^T\overline{x} = ab_0 + ba_0 \\ q_0 &= \frac{1}{2}q^T\overline{x} = \frac{1}{2}b_0a^T\overline{x} + a_0b^T\overline{x} = a_0b_0 \\ \frac{1}{2}x^TQx + q^Tx + q_0 &= (b^Tx + b_0)(a^Tx + a_0) \end{aligned}$$

hence $f(x) = a^Tx + a_0$.

$\Leftarrow$) If f is affine it is trivially pseudoaffine. The whole results then follows directly from Lemma 8 and Corollary 6. ■

Remark 10 The proof of Theorem 9 points out that:

i) when $\nu_-(Q) = \nu_+(Q) = 1$ f is pseudoaffine if and only if it is affine,

ii) f may be affine when $\nu_0(Q) = n - 1$ (case $\gamma = 0$),

iii) f is pseudoaffine but not affine only if $\nu_0(Q) = n - 1$ (case $\gamma < 0$).

It is worth noticing that in Theorem 9 it cannot be $\gamma > 0$, as it will be shown in Example 12. However, in this case it is possible to prove that function f is pseudoaffine at least on two disjoint convex sets.

Corollary 11 *Consider function f defined in (2) and suppose that there exist $\alpha, \beta, \gamma \in \Re$, $\alpha \neq 0$, such that f can be rewritten in the following form:*

$$f(x) = \alpha b^T x + \beta + \frac{\alpha\gamma}{b^T x + b_0}$$

i) if $\gamma \leq 0$ then f is pseudoaffine on X.
ii) if $\gamma > 0$ then f is pseudoaffine on $X_1 = \{x \in \Re^n : b^T x + b_0 > \sqrt{\gamma}\}$ and $X_2 = \{x \in \Re^n : 0 < b^T x + b_0 < \sqrt{\gamma}\}$.

Proof. *i)* It has already been proved in Theorem 9.
ii) Observe that $\nabla f(x) = \frac{\alpha}{(b^T x + b_0)^2}[(b^T x + b_0)^2 - \gamma]b$, and consequently $\nabla f(x) \neq 0$ on X_1 and X_2. The result trivially follows from Corollary 6, being $Q = [2\alpha bb^T]$. ■

The following examples use conditions in Theorems 6 and 9 in order to check the pseudoaffinity of three quadratic fractional functions.

Example 12 Consider problem (2) where

$$f(x) = \frac{9x_1^2 + 24x_1x_2 + 16x_2^2 + 6x_1 - 8x_2 + 1}{3x_1 + 4x_2}.$$

Observe that f is not pseudoaffine since it is nonconstant and $\nabla f(x)$ vanishes at $3x_1 + 4x_2 = 1$. In this case we get:

$$Q = \begin{bmatrix} 18 & 24 \\ 24 & 32 \end{bmatrix}, q = \begin{bmatrix} 6 \\ 8 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, b_0 = 0,\ q_0 = 1;$$

by simple calculations we obtain $\nu_0(Q) = 1 = \nu_+(Q)$, $f(x) = 3x_1 + 4x_2 + 2 + \frac{1}{3x_1 + 4x_2}$ hence $\gamma = 1 > 0$.

Example 13 Consider problem (2) where

$$f(x) = \frac{8x_1^2 + 2x_2^2 + 18x_3^2 - 8x_1x_2 - 24x_1x_3 + 12x_2x_3 + 10x_1 - 5x_2 - 15x_3 - 4}{-2x_1 + x_2 + 3x_3 - 3}.$$

In this case we get:

$$Q = \begin{bmatrix} 16 & -8 & -24 \\ -8 & 4 & 12 \\ -24 & 12 & 36 \end{bmatrix}, q = \begin{bmatrix} 10 \\ -5 \\ -15 \end{bmatrix}, b = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, b_0 = -3,\ q_0 = -4$$

Since Q is semidefinite positive with $\nu_0(Q) = n-1$, we have to verify condition i) of Corollary 6. Note that $\nabla f(x) = \left(2 + \frac{1}{(-2x_1 + x_2 + 3x_3 - 3)^2}\right) b \neq 0$ $\forall x \in X$, $\nabla f(x) \in Q(\Re^2)$, hence f is pseudoaffine. The same result can be obtained by means Theorem 9. In fact simple calculations give

$$f(x) = -4x_1 + 2x_2 + 6x_3 + 1 - \frac{1}{-2x_1 + x_2 + 3x_3 - 3},$$

so that $\alpha = 2$, $\gamma = -\frac{1}{2} < 0$ and hence f is pseudoaffine.

Example 14 Consider problem (2) where

$$f(x) = \frac{-8x_1^2 - 24x_2^2 + 32x_3^2 + 16x_1x_2 + 64x_2x_3 + 16x_2 + 16x_3 + 2}{-8x_1 - 8x_2 - 16x_3 - 4}.$$

In this case we get:

$$Q = \begin{bmatrix} -16 & 16 & 0 \\ 16 & 48 & 64 \\ 0 & 64 & 64 \end{bmatrix}, q = \begin{bmatrix} 0 \\ 16 \\ 16 \end{bmatrix}, b = \begin{bmatrix} -8 \\ -8 \\ -16 \end{bmatrix}, b_0 = -4, \; q_0 = 2$$

Q is indefinite with $\nu_+(Q) = \nu_-(Q) = 1$, hence f is pseudoaffine if and only if it is affine (see also Remark 10). In fact, by means of simple calculations we obtain

$$f(x) = x_1 - 3x_2 - 2x_3 - \frac{1}{2},$$

4 A wider class of quadratic functions

The aim of this section is to study a class of functions wider than the one considered so far. Specifically speaking, we aim to characterize the pseudoaffinity of the following type of functions:

$$g(x) = \frac{\frac{1}{2}x^TQx + q^Tx + q_0}{b^Tx + b_0} + c^Tx = f(x) + c^Tx \tag{4}$$

where as usual $X = \{x \in \Re^n : b^Tx + b_0 > 0\}$, Q is a $n \times n$ symmetric matrix, $q, x, b, c \in \Re^n$, $b \neq 0$, and $q_0, b_0 \in \Re$; note that g is of the kind (2) when $c = 0$ and $Q \neq 0$. First of all observe that if f is pseudoaffine then g is trivially pseudoaffine; on the other hand it may happen that g is pseudoaffine even if f is not. This is pointed out in the following example.

Example 15 Consider problem (4) where

$$g(x) = \frac{x_1^2 + x_1x_2 - x_1 + 2x_2 + 1}{x_1 + x_2} - x_1.$$

Observe that $f(x) = \dfrac{x_1^2 + x_1x_2 - x_1 + 2x_2 + 1}{x_1 + x_2} = x_1 - 1 + \dfrac{3x_2 + 1}{x_1 + x_2}$ is not pseudoaffine while

$$g(x) = \frac{-x_1 + 2x_2 + 1}{x_1 + x_2}$$

is pseudoaffine being a linear fractional function.

The characterization of the pseudoaffinity of g follows from Theorem 9.

Theorem 16 *Let g be of the kind (4); the following statements hold:*

i) g is affine if and only if f is affine;

ii) g is pseudoaffine but not affine if and only if either $Q + bc^T + cb^T = 0$ or there exist $\alpha, \xi \in \Re$, $\alpha \neq 0$, such that:

$$Q + bc^T + cb^T = 2\alpha bb^T, \quad q + b_0c = \xi b, \quad b_0^2 < \frac{\xi b_0 - q_0}{\alpha}$$

Proof. *i)* The result is trivial provided that g is the sum of f and an affine function.

ii) By means of simple calculations g can be rewritten as follows

$$g(x) = \frac{\frac{1}{2}x^T[Q + bc^T + cb^T]x + [q + b_0c]^Tx + q_0}{b^Tx + b_0}.$$

If $Q + bc^T + cb^T = [0]$ then g is a linear fractional function which is known to be pseudoaffine. In the other case, from Theorem 9, g is pseudoaffine but not affine if and only if there exist $\alpha, \beta, \gamma \in \Re$, $\alpha \neq 0$, such that it can be rewritten in the following form:

$$g(x) = \alpha b^Tx + \beta + \frac{\alpha\gamma}{b^Tx + b_0} \text{ with } \gamma < 0$$

and so

$$g(x) = \frac{\frac{1}{2}x^T[2\alpha bb^T]x + (\beta + \alpha b_0)b^Tx + (\beta b_0 + \alpha\gamma)}{b^Tx + b_0}.$$

This means that:

$$Q + bc^T + cb^T = 2\alpha bb^T, \quad q + b_0 c = (\beta + \alpha b_0)b, \quad q_0 = \beta b_0 + \alpha\gamma.$$

Defining $\xi = \beta + \alpha b_0$, so that $\beta = \xi - \alpha b_0$ and $\gamma = \dfrac{q_0 - \beta b_0}{\alpha} = b_0^2 - \dfrac{\xi b_0 - q_0}{\alpha}$, the result then follows from $\gamma < 0$. ■

Note that Theorem 16 can be applied also to functions of the kind (2) just assuming $c = 0$.

The next examples clarify the use of the conditions given in Theorem 16.

Example 17 Consider again function g in Example 15. Observe that

$$Q = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \; q = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \; q_0 = 1, \; b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, b_0 = 1, \; c = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

$Q + bc^T + cb^T = 0$ and hence g is pseudoaffine.

Example 18 Consider problem (4) where

$$g(x) = \frac{x_2^2 - 2x_1x_2 + 3x_2x_3 - 2x_1 - 2x_2 + 3x_3 - 4}{-2x_1 + x_2 + 3x_3 - 3} - 4x_1 + x_2 + 6x_3.$$

Observe that since

$$Q = \begin{bmatrix} 0 & -2 & 0 \\ -2 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}, \; q = \begin{bmatrix} -2 \\ -2 \\ 3 \end{bmatrix}, \; b = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \; c = \begin{bmatrix} -4 \\ 1 \\ 6 \end{bmatrix},$$

$q_0 = -4$, $b_0 = -3$, it results

$$Q + bc^T + cb^T = \begin{bmatrix} 16 & -8 & -24 \\ -8 & 2 & 12 \\ -24 & 12 & 36 \end{bmatrix} = 2\alpha bb^T \text{ with } \alpha = 2,$$

$$q + b_0 c = \begin{bmatrix} 10 \\ -5 \\ -15 \end{bmatrix} = \xi b \text{ with } \xi = -5,$$

$$b_0^2 = 9 < \frac{19}{2} = \frac{\xi b_0 - q_0}{\alpha},$$

hence g is pseudoaffine.

5 Final remarks

Theorem 9 shows that every pseudoaffine quadratic fractional function f, is the sum of a linear and a linear fractional one. This properties can be efficiently used in order to study problems of the kind

$$\min_{x\in S} / \max \quad f(x) = \frac{\frac{1}{2}x^TQx + q^Tx + q_0}{b^Tx + b_0} \tag{5}$$

where $S \subseteq X$. It has been already stated that function f, when it is pseudoaffine and the corresponding Q is indefinite, can be written as

$$f(x) = \frac{(a^Tx + a_0)(b^Tx + b_0)}{b^Tx + b_0} = a^Tx + a_0 \qquad x \in X$$

so that

$$\arg\min_{x\in S}\{f(x)\} = \arg\min_{x\in S}\{a^Tx\} \quad \text{and} \quad \arg\max_{x\in S}\{f(x)\} = \arg\max_{x\in S}\{a^Tx\}.$$

Define now $\varphi(t) = \alpha t + \beta + \frac{\alpha\gamma}{t+b_0}$; we have proved that when f verifies condition

$$\nu_0(Q) = n - 1, \nabla f(x) \in Q(\Re^n) \ \forall x \in X,$$

it can be rewritten as

$$f(x) = \alpha b^Tx + \beta + \frac{\alpha\gamma}{b^Tx + b_0} = \varphi(b^Tx).$$

Since $\varphi'(t) = \alpha\left(1 - \frac{\gamma}{(t+b_0)^2}\right)$ when $\gamma \leq 0$ (i.e. f is pseudoaffine) we have that $\varphi'(t) > 0$ $[< 0]$ if and only if $\alpha > 0$ $[< 0]$ and hence:

$$\begin{aligned}
\alpha &> 0 \Rightarrow \arg\min_{x\in S}\{f(x)\} = \arg\min_{x\in S}\{b^Tx\},\ \arg\max_{x\in S}\{f(x)\} = \arg\max_{x\in S}\{b^Tx\}, \\
\alpha &< 0 \Rightarrow \arg\min_{x\in S}\{f(x)\} = \arg\max_{x\in S}\{b^Tx\},\ \arg\max_{x\in S}\{f(x)\} = \arg\min_{x\in S}\{b^Tx\}.
\end{aligned}$$

Note that, being $Q = [2\alpha bb^T]$, it is $\alpha > 0$ $[< 0]$ if and only if Q is positive [negative] semidefinite. We can then conclude that, whenever f is pseudoaffine, its optimal points can be studied by means of a linear problem.

On the other hand, in the case $\gamma > 0$, even if f is not pseudoaffine on X, Problem (5) can be still studied by means of linear problems, splitting the set X as $X = X_1 \cup X_2 \cup X_3$, where

$$\begin{aligned}
X_1 &= \{x \in \Re^n : b^Tx + b_0 > \sqrt{\gamma}\}, \\
X_2 &= \{x \in \Re^n : 0 < b^Tx + b_0 < \sqrt{\gamma}\}, \\
X_3 &= \{x \in \Re^n : b^Tx + b_0 = \sqrt{\gamma}\},
\end{aligned}$$

and defining the sets

$$
\begin{array}{ll}
S_1^m = \arg\min\limits_{x\in S\cap X_1}\{b^T x\}, & S_1^M = \arg\max\limits_{x\in S\cap X_1}\{b^T x\}, \\
S_2^m = \arg\min\limits_{x\in S\cap X_2}\{b^T x\}, & S_2^M = \arg\max\limits_{x\in S\cap X_2}\{b^T x\}, \\
S_3^m = \arg\min\limits_{x\in S\cap X_3}\{\frac{1}{2}x^T Q x + q^T x\}, & S_3^M = \arg\max\limits_{x\in S\cap X_3}\{\frac{1}{2}x^T Q x + q^T x\}.
\end{array}
$$

In fact, if $\alpha > 0$ then $\varphi'(t) > 0$ when $t + b_0 > \sqrt{\gamma}$ while it is $\varphi'(t) < 0$ when $0 < t + b_0 < \sqrt{\gamma}$, hence we get

$$
\arg\min_{x\in S}\{f(x)\} = \arg\min_{x\in S_1^m\cup S_2^M\cup S_3^m}\{f(x)\}, \quad \arg\max_{x\in S}\{f(x)\} = \arg\max_{x\in S_1^M\cup S_2^m\cup S_3^M}\{f(x)\};
$$

analogously for $\alpha < 0$ it is

$$
\arg\min_{x\in S}\{f(x)\} = \arg\min_{x\in S_1^M\cup S_2^m\cup S_3^m}\{f(x)\}, \quad \arg\max_{x\in S}\{f(x)\} = \arg\max_{x\in S_1^m\cup S_2^M\cup S_3^M}\{f(x)\}.
$$

Appendix - Pseudoaffinity Test

Characterization in Theorem 9 suggests the following procedure, written with MapleV, which check the pseudoaffinity of a quadratic fractional function.

This procedure divides the numerator of function f by its denominator and computes the quotient and the reminder. If the reminder is zero, then the function is affine and the procedure stops; otherwise the procedures checks the degree of the quotient and the reminder; if they are 1 and 0, respectively, the procedure recalls a subroutine which establishes the pseudoaffinity of f.

Procedure 19

```
>
> ##  Subroutine  ##
>
> case_pseudo := proc(reminder,index)
> local g,i,alpha,gamma;
> g:=vector(vectdim(b));
> for i from 1 to vectdim(b) do g[i]:=coeff(quotient,x[i]) od;
> alpha:=g[index]/b[index];
> if equal(vector(vectdim(b),0),evalm(g-alpha*b))
>    then gamma:=reminder/alpha;
>         if gamma<0
>            then writeline(default,"f is pseudoaffine on ");
```

```
>                   print(X={evalm(transpose(b)&*x+b0)>0})
>              else writeline(default,"f is not pseudoaffine on ");
>                   print(X={evalm(transpose(b)&*x+b0)>0});
>                   writeline(default,"but it is pseudoaffine on ");
>                   print(X1={evalm(transpose(b)&*x+b0)>sqrt(gamma)},
>                         X2={0<evalm(transpose(b)&*x+b0) and
>                         evalm(transpose(b)&*x+b0)<sqrt(gamma)})
>         fi;
>         writeline(default,"and f(x)=");
>         quotient+(reminder/evalm(transpose(b)&*x+b0))
>    else writeline(default,"f is not pseudoaffine");
>         writeline(default,"quotient and denominator not proportional");
>         print()
> fi;
> end;
>
> ##  Main Procedure  ##
>
> isaffine := proc(Q,q,q0,b,b0)
> local num,den,reminder,i,index,vars;
> num:=evalm((1/2)*transpose(x)&*Q&*x+transpose(q)&*x+q0);
> den:=evalm(transpose(b)&*x+b0);
> index:=0:i:=1:
> while (index=0 and i<= vectdim(b)) do
>   if  b[i]<>0 then index:=i fi:
>   i:=i+1:
> od:
> reminder:=rem(num, den, x[index], 'quotient');
> if reminder=0
>    then writeline(default,"f is affine and");
>         writeline(default,"f(x)=");
>         quotient
>    else vars:=seq(x[i],i=1..vectdim(x));
>         if degree(reminder,{vars})=0 and degree(quotient,{vars})=1
>            then case_pseudo(reminder,index);
>            else writeline(default,"f is not pseudoaffine");
>                 writeline(default,"wrong degree of quotient
>                           and/or reminder");
>                 print()
>         fi;
> fi;
> end;
```

The following examples show the use of Procedure 19 in order to check the pseudoaffinity of the functions in Examples 12, Examples 13 and 14.

Example 20

```
> n:=2:
> x:=vector(n):
> Q:=matrix(n,n,[18,24,24,32]):
> q:=vector(n,[6,8]):
> q0:=1:
> b:=vector(n,[3,4]):
> b0:=0:
> isaffine(Q,q,q0,b,b0);
```

$$\left[\begin{array}{l} f \text{ is not pseudoaffine on} \\ X = \{0 < 3x_1 + 4x_2\} \\ \text{but it is pseuodaffine on} \\ X_1 = \{1 < 3x_1 + 4x_2\},\ X_2 = \{-3x_1 - 4x_2 < 0 \text{ and } 3x_1 + 4x_2 - 1 < 0\} \\ \text{and } f(x) = 3x_1 + 4x_2 + 2 + \frac{1}{3x_1+4x_2} \end{array}\right.$$

Example 21

```
> n:=3:
> x:=vector(n):
> Q:=matrix(n,n,[16,-8,-24,-8,4,12,-24,12,36]):
> q:=vector(n,[10,-5,-15]):
> q0:=-4:
> b:=vector(n,[-2,1,3]):
> b0:=-3:
> isaffine(Q,q,q0,b,b0);
```

$$\left[\begin{array}{l} f \text{ is pseudoaffine and} \\ f(x) = -4x_1 + 2x_2 + 6x_3 + 1 - \frac{1}{-2x_1+x_2+3x_3-3} \end{array}\right.$$

Example 22

```
> n:=3:
> x:=vector(n):
> Q:=matrix(n,n,[-16,16,0,16,48,64,0,64,64]):
> q:=vector(n,[0,16,16]):
> q0:=2:
> b:=vector(n,[-8,-8,-16]):
> b0:=-4:
> isaffine(Q,q,q0,b,b0);
```

$$\left[\begin{array}{l} f \text{ is affine and} \\ f(x) = x_1 - 3x_2 - 2x_3 - \frac{1}{2} \end{array}\right.$$

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