Universit degli Studi di Pisa
Dipartimento di Statistica e Matematica
Applicata all Economia

Report n. 209

A Finite Algorithm for a Class of Non Linear Multiplicative Programs

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Pisa, Luglio 2001

- Stampato in Proprio -

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A Finite Algorithm for a Class of Non Linear Multiplicative Programs

R. Cambini - C. Sodini [(1)]

Abstract. The nonconvex problem of minimizing the product of a strictly convex quadratic function and the p-th power of a linear function over a convex polyhedron is considered. Some theoretical properties of the problem, such as the existence of minimum points and the generalized convexity of the objective function, are deepened on and a finite algorithm solving the problem is proposed.

Key Words. Multiplicative programming, fractional programming, generalized quadratic programming, generalized convexity.
AMS - 2000 Math. Subj. Class. 90C20, 90C26, 90C31.
JEL - 1999 Class. Syst. C61, C63, C62.

1. Introduction

In this paper we consider the nonlinear multiplicative problem

$$(1) \quad \begin{cases} \min f(x) = \left(\frac{1}{2}x^T Qx + q^T x + q_0\right)\left(d^T x + d_0\right)^p \\ x \in X = \{x \in \Re^n : Ax \geq b\} \end{cases}$$

where A is a mxn matrix, q, $d \in R^n$, $b \in R^m$, p, q_0, $d_0 \in R$, Q is a symmetric positive definite nxn matrix and $d^T x + d_0 > 0$, $x \in X$. Note that for p=0 the problem reduces to a strictly convex quadratic one. This class of functions is very used in applicative problems, such as portfolio theory, portfolio selection and risk theory [1-10].

In Section 2 we first study some theoretical properties of the problem, providing conditions guaranteeing the existence of minimum points and proving that the objective function is strictly pseudoconvex in subsets of the feasible region (note that the generalized convexity of these multiplicative functions is generally studied for particular values of p and is limited to the strict quasiconvexity). In Section 3 some local optimality conditions are studied; these conditions allow us to propose a finite algorithm that solves Problem (1) even when X is unbounded.

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2. Theoretical Properties

It is first interesting to point out whether or not Problem (1) admits minimum points. With this aim let us recall the following result given in [5], where X_∞ is the recession cone of X and $f_\infty^X(v)$ is the recession function over X associated to f.

Theorem 2.1 [5] Consider a minimization problem involving a lower semicontinuous function f(x) over an asymptotically-linear closed domain X. If function f is definitely nondecreasing along the directions $v \in ker\left(f_\infty^X\right)$ then:

there exists a minimum point x*∈X $\Leftrightarrow$ $f_\infty^X(v) \geq 0 \;\; \forall v \in X_\infty$

This result allow us to prove conditions guaranteeing the existence of minimum points. These conditions are based on the following preliminary lemma.

Lemma 2.1 Consider Problem (1); the following properties hold:

i) $d^T v \geq 0 \;\; \forall v \in X_\infty$;

ii) $f_\infty^X(v) \geq 0 \;\; \forall v \in X_\infty$;

iii) f is definitely nondecreasing along the directions $v \in X_\infty$ such that $d^T v = 0$;

iv) if p>-2 then f is definitely nondecreasing along the directions $v \in X_\infty$ s.t. $d^T v > 0$.

Proof i) Follows directly being $d^T x + d_0 > 0 \; \forall x \in X$.

ii) Follows being f(x)<0 only in the compact set $\frac{1}{2} x^T Q x + q^T x + q_0 \leq 0$.

iii) Consider an halfline $x(\theta) = x_0 + \theta v$, $\theta \geq 0$ and $x_0 \in X$, with $v \in X_\infty$ such that $d^T v = 0$; the corresponding restriction is

$$z(\theta) = f(x(\theta)) = \left[\frac{1}{2}\theta^2 v^T Q v + \theta v^T (Q x_0 + q) + k_q\right](k_d)^p$$

where $k_q = \frac{1}{2} x_0^T Q x_0 + q^T x_0 + q_0$ and $k_d = d^T x_0 + d_0 > 0$; its derivative results

$$z'(\theta) = \left[\theta v^T Q v + v^T (Q x_0 + q)\right](k_d)^p$$

which is positive for $\theta > 0$ great enough, being $k_d > 0$ and $v^T Q v > 0$.

iv) Consider an halfline $x(\theta) = x_0 + \theta v$, $\theta \geq 0$ and $x_0 \in X$, with $v \in X_\infty$ such that $d^T v > 0$; the corresponding restriction is

$$z(\theta) = f(x(\theta)) = \left[\frac{1}{2}\theta^2 v^T Q v + \theta v^T (Qx_0 + q) + k_q\right]\left(\theta d^T v + k_d\right)^p$$

where $k_q = \frac{1}{2}x_0^T Q x_0 + q^T x_0 + q_0$ and $k_d = d^T x_0 + d_0 > 0$; its derivative results

$$z'(\theta) = \left(\theta d^T v + k_d\right)^{p-1}\left[\frac{1}{2}\theta^2 (p+2) v^T Q v d^T v + \theta c_1 + c_0\right],$$

where $c_1 = \left[(p+1) d^T v v^T (Qx_0 + q) + v^T Q v k_d\right]$ and $c_2 = \left[k_d v^T (Qx_0 + q) + k_q p d^T v\right]$.
It then results that $z'(\theta) > 0$ for $\theta > 0$ great enough, being p+2>0, $\left(\theta d^T v + k_d\right) > 0$, $d^T v > 0$ and $v^T Q v > 0$. ♦

The following result, which follows directly from Theorem 2.1, Lemma 2.1 and the continuity of f, provides some conditions guaranteeing the existence of a minimum point.

Theorem 2.2 Consider Problem (1); the following properties hold:

i) if p>-2 then the minimum exists;

ii) if $\exists x \in X$ such that $f(x) \le 0$ then the minimum exists and is nonpositive.

Note furthermore that Problem (1) always admits a finite infimum, which comes out to be nonnegative if none of the properties of Theorem 2.2 hold.

Another important property, useful from both the theoretical and the algorithmical point of view, is the generalized convexity of the objective function; this property implies for example that all the local minima are also global ones. Problem (1) is not a quasiconvex program in general; neverthless it is possible to state some generalized convexity property of the objective function on subsets of the feasible region.

Theorem 2.3 Consider problem (1) and let:

$$X_{pos}=\{x \in X: f(x) \ge 0\} \text{ and } X_{neg}=\{x \in X: f(x) \le 0\}.$$

The following properties hold:

i) if p=0 then f is strictly convex over X ;

ii) if p=–1 then f is strictly pseudoconvex over X ;

iii) if p<–1 or p>0 then f is strictly pseudoconvex over every convex subset of X_{neg} ;

iv) if –1<p<0 then f is strictly pseudoconvex over every convex subset of X_{pos} .

Proof If $p=0$ the result is trivial. Let us now denote the quadratic factor of f with $h(x)=(\frac{1}{2}x^TQx + q^Tx + q_0)$ and let us define the function $g(x)=(d^Tx + d_0)^{-p}$ over X, so that $f(x)=h(x)[g(x)]^{-1}$. First notice that g is convex for $p\leq-1$ or $p\geq 0$ and it is concave for $-1\leq p\leq 0$ since its hessian matrix is $H_g(x)=\frac{p(p+1)}{(d^Tx+d_0)^{p+2}}dd^T$.

Function f is strictly pseudoconvex over X if for any $x,y\in X$, $x\neq y$, it is:

$$f(y)\leq f(x) \Rightarrow \nabla f(x)^T(y-x)<0$$

Assume $f(y)\leq f(x)$; note that this condition can be rewritten as $h(y)[g(y)]^{-1}\leq h(x)[g(x)]^{-1}$ that is, being $g(x)>0$ $\forall x\in X$, $h(y)\leq h(x)\frac{g(y)}{g(x)}$.

Being Q positive definite function $h(x)$ is strictly convex, hence:

$$\nabla h(x)^T(y-x)<h(y)-h(x)\leq h(x)\left(\frac{g(y)}{g(x)}-1\right)=f(x)[g(y)-g(x)].$$

It results also:

$$\nabla f(x)=[g(x)]^{-1}\nabla h(x)-h(x)[g(x)]^{-2}\nabla g(x)$$

so that, being $g(x)>0$ $\forall x\in X$, we obtain

$$\begin{aligned}\nabla f(x)^T(y-x) &= [g(x)]^{-1}\nabla h(x)^T(y-x)-h(x)[g(x)]^{-2}\nabla g(x)^T(y-x)\\ &< [g(x)]^{-1}f(x)[g(y)-g(x)]-f(x)[g(x)]^{-1}\nabla g(x)^T(y-x)\\ &= f(x)\,[g(x)]^{-1}\,[g(y)-g(x)-\nabla g(x)^T(y-x)]\end{aligned}$$

If $p=-1$ g is affine, hence $g(y)-g(x)-\nabla g(x)^T(y-x)=0$ and $\nabla f(x)^T(y-x)<0$, so that f is strictly pseudoconvex over X.

If $p<-1$ or $p>0$ g is convex, hence $g(y)-g(x)-\nabla g(x)^T(y-x)\geq 0$ and consequently when $f(x)\leq 0$ it results $\nabla f(x)^T(y-x)<0$, that is to say that function f is strictly pseudoconvex over X_{neg}.

If $-1<p<0$ g is concave, hence $g(y)-g(x)-\nabla g(x)^T(y-x)\leq 0$ and for $f(x)\geq 0$ it results $\nabla f(x)^T(y-x)<0$, in other words function f is strictly pseudoconvex over X_{pos}. ♦

Note that Theorem 2.3 extends the results given in [1,2,3,8] which prove only the strict quasiconvexity of the function and just for the particular cases p=-1 and p=-2.

Theorem 2.3 points out also that it is useful to distinguish between the cases $X_{neg}=\emptyset$ and $X_{neg}\neq\emptyset$, with this aim in the forthcoming results some conditions are stated in order to guarantee the absence of feasible points with nonpositive image.

First it is useful to define the minimum feasible level $\xi_{min}>0$, given by the solution of the linear problem

$$\begin{cases} min\ d^Tx+d_0 \\ x\in X \end{cases};$$

note that the minimum ξ_{min} exists since X is closed, $d^Tx + d_0>0\ \forall x\in X$ and the objective function $d^Tx + d_0$ is linear.

Definition 2.1 Consider Problem (1) and its quadratic factor:

$$\left(\frac{1}{2}x^TQx + q^Tx + q_0\right)$$

From now on we denote with:

i) $x_u = -Q^{-1}q$ (unconstrained minimum of the quadratic factor)

ii) $\xi_u = d_0 - d^TQ^{-1}q = d^Tx_u + d_0$ (level corresponding to x_u)

iii) $q_u = q_0 - \frac{1}{2}q^TQ^{-1}q = \frac{1}{2}x_u^TQx_u + q^Tx_u + q_0$ (value of the quadratic factor in x_u)

iv) $\delta = 2d^TQ^{-1}d$ ($\delta>0$ since Q positive definite implies Q^{-1} positive definite)

Lemma 2.2 Consider Problem (1). For any given level $d^Tx + d_0 = \xi$ the quadratic factor

$$\left(\frac{1}{2}x^TQx + q^Tx + q_0\right)$$

attains the unconstrained minimum in

$$x(\xi) = -Q^{-1}\left(q - 2\frac{\xi-\xi_u}{\delta}d\right)$$

with minimum value $\overline{Q}(\xi) = q_u + \frac{(\xi-\xi_u)^2}{\delta}$.

Proof Consider the minimum constrained problem

$$\begin{cases} min\ \frac{1}{2}x^TQx + q^Tx + q_0 \\ d^Tx + d_0 = \xi \end{cases},$$

the minimum point verifies the necessary and sufficient optimality condition

$$\begin{cases} Qx + q = \lambda d \\ d^Tx + d_0 = \xi \end{cases};$$

since Q is positive definite it is also non singular, hence $x = -Q^{-1}(q - \lambda d)$ and, by means of simple calculations, we obtain

$$\begin{cases} \lambda = 2\frac{\xi-\xi_u}{\delta} \\ x = x(\xi) = -Q^{-1}\left(q - 2\frac{\xi-\xi_u}{\delta}d\right) \end{cases}$$

and $\overline{Q}(\xi) = \frac{1}{2}x(\xi)^TQx(\xi) + q^Tx(\xi) + q_0 = \frac{1}{2}\lambda^2d^TQ^{-1}d + q_u = q_u + \frac{(\xi-\xi_u)^2}{\delta}$. ♦

By means of Lemma 2.2 it possible to state the following conditions related to the positivity of the objective function f.

Theorem 2.4 Consider Problem (1). The following properties hold:

i) if $q_u > 0$ then $f(x)>0\ \forall x \in X$,

ii) if $q_u \le 0$ and $\xi_u + \sqrt{-\delta q_u} > 0$ then:

$$f(x) \le 0 \;\Rightarrow\; \xi_u - \sqrt{-\delta q_u} \le d^T x + d_0 \le \xi_u + \sqrt{-\delta q_u} \;,$$

iii) if $q_u \le 0$ and $\xi_u + \sqrt{-\delta q_u} < \xi_{min}$ then $f(x)>0\ \forall x \in X$.

Proof *i)* The result follows trivially being q_u the unconstrained minimum value of the quadratic factor and being $d^T x + d_0 > 0\ \forall x \in X$.

ii) Let $x \in X$ and $\xi > 0$ such that $d^T x + d_0 = \xi$, then $f(x) \le 0$ implies $\overline{Q}(\xi) \le 0$ and hence $(\xi - \xi_u)^2 \le -\delta q_u$, that is $\xi_u - \sqrt{-\delta q_u} \le \xi \le \xi_u + \sqrt{-\delta q_u}$.

iii) Follows directly from *ii)* being $d^T x + d_0 > 0\ \forall x \in X$. ♦

Remark 2.1 Theorem 2.4 suggests a smart procedure to study problem (1) in the case

$$q_u \le 0 \text{ and } \xi_u + \sqrt{-\delta q_u} \ge \xi_{min}$$

Split the feasible region X in the following subsets, so that $X = X_1 \cup X_2 \cup X_3$ where:

$X_1 = X \cap \{x \in R^n : \xi_u - \sqrt{-\delta q_u} \le d^T x + d_0 \le \xi_u + \sqrt{-\delta q_u}\}$

$X_2 = X \cap \{x \in R^n : \xi_u - \sqrt{-\delta q_u} > d^T x + d_0\}$

$X_3 = X \cap \{x \in R^n : d^T x + d_0 > \xi_u + \sqrt{-\delta q_u}\}$

First solve the problem $\{\min f(x),\ x \in X_1\}$; if the infimum/minimum value computed is nonpositive, then it is also the infimum/minimum value of Problem (1), otherwise solve the two other problems $\{\min f(x),\ x \in X_2\}$ and $\{\min f(x),\ x \in X_3\}$ and compare the obtained results, taking into account that for $-1 \le p \le 0$ the function is strictly pseudo-convex on X_2 and X_3 and hence every local minimum is a global one.

We conclude this section studying conditions which could be used as stop criterion in algorithms solving Problem (1). With this aim, let us consider the following program associated to (1):

$$\begin{cases} \min\ f(x) \\ d^T x + d_0 = \xi > 0 \end{cases};$$

the minimum is attained again at $x(\xi)$ and the minimum values, associated to the levels $\xi > 0$, are given by:

$$\varphi(\xi) = \frac{\xi^p}{\delta}\left[(\xi - \xi_u)^2 + \delta q_u\right] = \frac{\xi^{p+2}}{\delta}\left[\left(1 - \frac{\xi_u}{\xi}\right)^2 + \frac{\delta q_u}{\xi^2}\right]$$

By means of simple calculations, we obtain the corresponding first derivative:

$$\varphi'(\xi)=\frac{\xi^{p-1}}{\delta}\left[(p+2)\xi^2-2\xi_u(p+1)\xi+p(\xi_u^2+\delta q_u)\right]$$

which allow us to study the behaviour of the unconstrained minimum level values.

Note also that it results:

$$\lim_{\xi\to+\infty}\varphi(\xi)=\begin{cases}+\infty & if\ \ p>-2\\ 1/\delta & if\ \ p=-2\\ 0 & if\ \ p<-2\end{cases}$$

Some optimality conditions, which could be used as stop criterions in solving algorithms, can be provided when $\varphi(\xi)$ is definitely increasing.

Theorem 2.5 Consider Problem (1), let ξ_{min}>0 the minimum feasible level and let x*∈X and ξ*≥ξ_{min} be such that $f(x^*)\le\varphi(\xi^*)$. If one of the following conditions hold:

i) p>-2 and $\xi_u^2-p(p+2)\delta q_u\le 0$,

ii) p>-2, $\xi_u^2-p(p+2)\delta q_u>0$ and $\xi^*\ge\xi_u+\dfrac{-\xi_u+\sqrt{\xi_u^2-p(p+2)\delta q_u}}{p+2}$,

iii) p=-2, $\xi_u>0$ and $\xi^*\ge\dfrac{\xi_u^2+\delta q_u}{\xi_u}$,

iv) p=-2, $\xi_u=0$ and $q_u<0$,

then $f(x^*)\le f(x)\ \ \forall x\in X,\ d^Tx+d_0\ge\xi^*$.

Proof We prove the result showing that these conditions imply the increasness of $\varphi(\xi)$, function of the uncostrained minimum values associated to a feasible level ξ, so that

$$f(x^*)\le\varphi(\xi^*)\le\varphi(\xi)\le f(x)\ \ \forall x\in X,\ d^Tx+d_0=\xi\ge\xi^*.$$

i),ii) Let p>-2, being ξ>0 the derivative $\varphi'(\xi)$ is positive when

$$(p+2)\xi^2-2\xi_u(p+1)\xi+p(\xi_u^2+\delta q_u)\ge 0;$$

solving the second order inequality we obtain that for $\frac{\Delta}{4}=\xi_u^2-p(p+2)\delta q_u\le 0$ function $\varphi(\xi)$ is increasing $\forall\xi>0$, while for $\frac{\Delta}{4}=\xi_u^2-p(p+2)\delta q_u>0$ it is definitely increasing for $\xi\ge\xi_u+\dfrac{-\xi_u+\sqrt{\xi_u^2-p(p+2)\delta q_u}}{p+2}$.

iii),iv) Let p=-2, then $\varphi(\xi)=\dfrac{1}{\delta}\left[\left(1-\dfrac{\xi_u}{\xi}\right)^2+\dfrac{\delta q_u}{\xi^2}\right]$ and $\varphi'(\xi)=2\dfrac{\xi^{-3}}{\delta}\left[\xi_u\xi-(\xi_u^2+\delta q_u)\right]$;

hence if $\xi_u>0$ $\varphi(\xi)$ is increasing for $\xi\ge\dfrac{\xi_u^2+\delta q_u}{\xi_u}$, while if $\xi_u=0$ it is $\varphi'(\xi)>0$ just when $q_u<0$. ♦

3. Some local optimality conditions

In this section we give some local optimality conditions for problem (1). If we add the constraint $d^Tx+d_0=\xi$, $\xi\in R^+$, to problem (1), the following strictly convex quadratic problem is obtained:

$$(P(\xi))\qquad \begin{aligned} z(\xi) &= \min\ \left(\tfrac{1}{2}x^TQx + q^Tx + q_0\right)\xi^p \\ x &\in X(\xi) \end{aligned}$$

where $X(\xi)=X\cap\{x\in R^n : d^Tx + d_0 = \xi\}$. The parameter ξ is said to be a <u>feasible level</u> if the set $X(\xi)$ is nonempty. An optimal solution of problem $P(\xi)$ is called an <u>optimal level solution</u>.

Clearly, problem (1) is equivalent to problem $P(\xi)$, when ξ is the level corresponding to an optimal solution of problem (1).

In this section we give some optimality conditions which allow us to detect if an optimal level solution is a local minimum of problem (1).

Let x' be the optimal solution of problem $P(\xi')$ and let Nx=k be the equations of the constraints binding at x'. We can always choose a subset of these constraints, making a submatrix M of N and correspondingly a subvector h of k, such that the rows of M and the vector d are linearly independent. Being problem $P(\xi)$ convex, then x' is an optimal solution if and only if the Kuhn-Tucker conditions are verified.

Since Q is positive definite and the rows of M and d are linearly independent, the matrix of the following Kuhn-Tucker linear system is non singular:

$$(3.1)\qquad \begin{aligned} Qx - M^T\mu - d\lambda &= -q \\ Mx &= h \\ d^Tx &= \xi'-d_0 \end{aligned}$$

where μ is the vector of the Lagrange multipliers associated to the constraints Mx=h and λ is the Lagrange multiplier of the parametric constraint $d^Tx=\xi'-d_0$. The solution x', μ', λ' of (3.1) is then unique, note also that being x' an optimal solution then $\mu'\geq 0$.

Let us consider the parametric program:

$$(P(\xi'+\theta))\qquad \begin{aligned} z(\xi'+\theta) &= \min\ \left(\tfrac{1}{2}x^TQx + q^Tx + q_0\right)(\xi'+\theta)^p \\ x &\in X(\xi'+\theta) \end{aligned}$$

where $X(\xi'+\theta)=X\cap\{x\in R^n : d^Tx + d_0 = \xi'+\theta\}$.

Let

$$(3.2)\qquad \begin{aligned} x'(\theta) &= x' + \theta\,\alpha \\ \mu'(\theta) &= \mu' + \theta\,\gamma \\ \lambda'(\theta) &= \lambda' + \theta\,\beta \end{aligned}$$

be the solutions of the Kuhn-Tucker system:

$$(3.3)\qquad \begin{aligned} Qx - M^T\mu - d\lambda &= -q \\ Mx &= h \\ d^Tx &= \xi'-d_0+\theta \end{aligned}$$

Note that (α,γ,β) is the unique solution of the linear system

$$(3.4)\qquad \begin{aligned} Qx - M^T\mu - d\lambda &= 0 \\ Mx &= 0 \\ d^Tx &= 1 \end{aligned}$$

so that it results $Q\alpha = M^T\gamma + d\beta$, $M\alpha = 0$, $d^T\alpha = 1$ and $\beta = \alpha^TQ\alpha$. Note also that, being Q positive definite, it is $\beta > 0$ if and only if $\alpha \neq 0$.

Note that the solutions of the Kuhn-Tucker system (3.3) can be computed also by means of the explicit inverse of the matrix

$$D = \begin{bmatrix} Q & -M^T & -d \\ M & 0 & 0 \\ d^T & 0 & 0 \end{bmatrix}$$

as it has been described in [6].

Set $F(\theta)=\{\theta: x'(\theta)\in X\}$, $O(\theta)=\{\theta: \mu'(\theta)\geq 0\}$, $H(\theta)=F(\theta)\cap O(\theta)$. Clearly, $x'(\theta)$ is an optimal level solution for $\theta\in H(\theta)$. Set $z(\theta)=z(\xi'+\theta)$, $z'=\frac{1}{2}x'^TQx'+q^Tx'+q_0$. The following lemma gives an explicit form for the function $z(\theta)$, $\theta\in H(\theta)$.

Lemma 3.1

If $H(\theta)\neq\{0\}$, then $z(\theta) =(\xi'+\theta)^p\,(\frac{1}{2}\beta\,\theta^2 + \lambda'\,\theta + z')$.

Proof. We have $z(\theta) = (\xi'+\theta)^p\,(\frac{1}{2}(x'+\theta\,\alpha)^TQ(x'+\theta\,\alpha) + q^T(x' + \theta\,\alpha) + q_0) = (\xi'+\theta)^p$ $(\frac{1}{2}x'^TQx'+ \alpha^TQx'\,\theta+1/2\,\alpha^TQ\alpha\,\theta^2+q^Tx'+\theta\,q^T\alpha+q_0)$; note also that from (3.3) it results

$\alpha^T Qx' = \lambda' - \alpha^T q$. From direct substitution we obtain $z(\theta)=(\xi'+\theta)^p(\frac{1}{2}\beta\,\theta^2+\lambda'\theta+z')$ ♦

Now, the following lemma can be derived.

Lemma 3.2

If $pz'+\lambda'\xi'>0$ ($pz'+\lambda'\xi'<0$), then $z(\theta)$ is increasing (decreasing) at $\theta=0$.

Proof. We have $z'(\theta) = (\xi'+\theta)^{p-1}\,[\beta(1+\frac{1}{2}p)\theta^2+(p\lambda'+\beta\xi'+\lambda')\,\theta + pz'+\lambda'\xi']$. Hence $z'(0) = \xi'^{p-1}\,(pz'+\lambda'\xi)$. ♦

If $(p\lambda'+\beta\xi'+\lambda')^2-4\beta(1+\frac{1}{2}p)(pz'+\lambda'\xi')>0$, set $\theta^1<\theta^2$ the two roots of $z'(\theta) = 0$ and, if $(p\lambda'+\beta\xi'+\lambda')^2-4\beta(1+\frac{1}{2}p)(pz'+\lambda'\xi')=0$ set θ', the unique root of $z'(\theta) = 0$. Furthermore let $U(\theta)$ the connected set, containing $\theta=0$, such that $U(\theta)=\{\theta\in H(\theta)\colon \theta\geq 0$ and $z(\theta)$ is decreasing$\}$ or $U(\theta)=\{\theta\in H(\theta)\colon \theta\leq 0$ and $z(\theta)$ is increasing$\}$.

The following theorem holds:

Theorem 3.1

a) If $pz'+\lambda'\xi'= 0$ and $p\lambda'+\beta\xi'+\lambda'>0$, then x' is a local minimum for problem (1).

b) Case $\beta(1+\frac{1}{2}p)>0$.
- If $pz'+\lambda'\xi'<0$ and $\theta^2\in H(\theta)$, then $x'(\theta^2)$ is a local minimum for problem (1);
- If $pz'+\lambda'\xi'>0$, $\theta^2<0$ and $\theta^2\in H(\theta)$, then $x'(\theta^2)$ is a local minimum for problem (1);

c) Case $\beta(1+\frac{1}{2}p)<0$.
- If $pz'+\lambda'\xi'<0$, $\theta^1>0$ and $\theta^1\in H(\theta)$, then $x'(\theta^1)$ is a local minimum for problem (1);
- If $pz'+\lambda'\xi'>0$, $\theta^1<0$ and $\theta^1\in H(\theta)$, then $x'(\theta^1)$ is a local minimum for problem (1).

Proof. a) $pz'+\lambda'\xi'= 0$ and $p\lambda'+\beta\xi'+\lambda'>0$ imply $z'(0) = 0$, $z'(\theta)$ negative on the left of $\theta=0$ and $z'(\theta)$ positive on the right of $\theta=0$; hence $x'(0)=x'$ is a local minimum. b) We have $z'(\theta^2)=0$, $z'(\theta)$ negative on the left of $\theta=\theta^2$ and $z'(\theta)$ positive on the right of $\theta=\theta^2$; this implies that $x'(\theta^2)$ is a local minimum for problem (1). c) We have $z'(\theta^1)=0$, $z'(\theta)$ negative on the left of $\theta=\theta^1$ and $z'(\theta)$ positive on the right of $\theta=\theta^1$; this implies that $x'(\theta^1)$ is a local minimum for problem (1). ♦

Let x' be a vertex of X; in x' at least n constraints of X are binding as well as the parametric constraint and thus x' is a degenerate basic solution. Clearly, the different bases containing the parametric constraint are n if x' is a non degenerate vertex of X; more than n if x' is a degenerate vertex of X. A basis B is said to be feasible if $\mu'_B \geq 0$. To point out the dependence of $z(\theta)$, $H(\theta)$, etc. on the basis B, we write $z_B(\theta)$, $H_B(\theta)$, etc..

Theorem 3.2

a) If there are two different feasible bases B_1 and B_2 such that either $pz'+\lambda'_{B_1}\xi'>0$, $\sup H_{B_1}(\theta) > 0$, $pz'+\lambda'_{B_2}\xi'<0$, $\inf H_{B_2}(\theta) < 0$ or $pz'+\lambda'_{B_1}\xi'<0$, $\inf H_{B_1}(\theta) < 0$, $pz'+\lambda'_{B_2}\xi'>0$, $\sup H_{B_2}(\theta) > 0$, then x' is a local minimum for problem (1).

b) If we have $U_B(\theta)=\{0\}$ for any feasible basis B, then x' is a local minimum for problem (1).

Proof. a) In wiew of Lemma 3.2 condition

$$pz'+\lambda'_{B_1}\xi'>0,\ pz'+\lambda'_{B_2}\xi'<0\ (pz'+\lambda'_{B_1}\xi'<0,\ pz'+\lambda'_{B_2}\xi'>0)$$

implies $z(\theta) \geq z(0)$ in a neighborhood of 0. Hence x' is a local minimum for problem (1).

b) This follows directly from the definition of $U_B(\theta)$. ♦

4. A finite algorithm for problem (1)

Since problem (1) is nonconvex, in general, it is necessary to solve problem $P(\xi)$ for all feasible levels in order to find a global minimum, assuming one exists. In this section we will show that this can be done by means of a finite number of iterations, using the results of the previous section.

Let ξ' be a feasible level and suppose that x* is the incumbent global minimum for $\xi\leq\xi'$, i.e. is the best optimal level solution for $\xi\leq\xi'$. Clearly UB=f(x*) is an upper bound for the value of $z(\xi)$ for $\xi>\xi'$.

Let $\xi_{max} = \sup\{d^Tx, x\in X\}$ (of course ξ_{max} may be equal to $+\infty$).

Let us consider the parametric problem $P(\xi'+\theta)$ for $\theta\geq 0$ and determine $x'(\theta)$, $\mu'(\theta)$, $\lambda'(\theta)$, $z(\theta)$, $\theta^1<\theta^2$ [if $\Delta=(p\lambda'+\beta\xi'+\lambda')^2-4\beta(1+1/2p)(pz'+\lambda'\xi')>0$], $F(\theta)$, $O(\theta)$, $H(\theta)$ as well $\sup F(\theta)$, $\sup O(\theta)$, $\sup H(\theta)$. For each $\theta\in O(\theta)$, $z(\theta)$ is a lower bound for $P(\xi'+\theta)$; in fact if $\theta\in F(\theta)$, then $x'(\theta)$ is an optimal level solution; otherwise, if $\theta\notin F(\theta)$, $x'(\theta)$ is unfeasible for $P(\xi'+\theta)$ but is an optimal solution of a problem with the same objective function of $P(\xi'+\theta)$ and a feasible region containing $X(\xi'+\theta)$.

If p>-2, then the following two cases can occur:

A1) if $\Delta \le 0$ or $\Delta > 0$ and $\theta^2 \le 0$, then two subcases need to be considered:

A1a) if $\sup O(\theta) = +\infty$, then problem (1) is solved and x^* is a global minimum;

A1b) if $\sup O(\theta) = \theta'' < +\infty$, then: i) if $\xi'' = \xi' + \theta'' \ge \xi_{max}$ then problem (1) is solved and x^* is a global minimum; ii) if $\xi'' < \xi_{max}$ then we consider the new feasible level ξ'' and the corresponding parametric problem $P(\xi''+\theta)$;

A2) if $\Delta > 0$ and $\theta^2 > 0$, then three subcases need to be considered:

A2a) if $\sup H(\theta) = +\infty$, then problem (1) is solved; in fact if $UB \le z(\theta^2)$, then x^* is a global minimum, otherwise $x'(\theta^2)$ is a global minimum;

A2b) if $\theta^2 \le \sup H(\theta) < +\infty$, then two subcases need to be considered:

A2b1) if $\sup O(\theta) = +\infty$, then problem (1) is solved; in fact if $UB \le z(\theta^2)$, then x^* is a global minimum, otherwise $x'(\theta^2)$ is a global minimum;

A2b2) if $\sup O(\theta) = \theta'' < +\infty$, then: i) if $\xi'' = \xi' + \theta'' \ge \xi_{max}$ then problem (1) is solved; in fact if $UB \le z(\theta^2)$, then x^* is a global minimum, otherwise $x'(\theta^2)$ is a global minimum; ii) if $\xi'' < \xi_{max}$ then we consider the new feasible level ξ'' and the corresponding parametric problem $P(\xi''+\theta)$ with $x^* = x'(\theta^2)$, $UB = z(\theta^2)$ if $UB > z(\theta^2)$;

A2c) if $\sup H(\theta) = \theta^* < \theta^2$, then two subcases need to be considered:

A2c1) if $\sup O(\theta) = +\infty$ and $UB \le z(\theta^2)$, then problem (1) is solved and x^* is a global minimum;

A2c2) otherwise; i) if $\xi'' = \xi' + \theta^* = \xi_{max}$ then problem (1) is solved, in fact if $UB \le z(\theta^*)$, then x^* is a global minimum, otherwise $x'(\theta^*)$ is a global minimum; ii) if $\xi'' < \xi_{max}$ then we consider the new feasible level ξ'' and the corresponding parametric problem $P(\xi''+\theta)$ with $x^* = x'(\theta^*)$, $UB = z(\theta^*)$ if $UB > z(\theta^*)$.

If $p < -2$, then the following two cases can occur:

B1) if $\Delta \le 0$ or $\Delta > 0$ and $\theta^1 \le 0$, then three subcases need to be considered:

B1a) if $\sup H(\theta) = +\infty$, then problem (1) is solved; in fact if $UB \le 0$ then x^* is a global minimum, otherwise the minimum is not attained and $\inf_{x \in X} f(x) = 0$;

B1b) if $\sup O(\theta) = +\infty$ and $UB \le 0$ or $\sup H(\theta) = \theta^* < \sup O(\theta) = \theta'' < +\infty$, $\xi'' = \xi' + \theta'' \ge \xi_{max}$ and $UB \le z(\theta'')$, then problem (1) is solved and x^* is a global minimum;

B1c) otherwise; i) if $\xi'' = \xi' + \theta^* = \xi_{max}$ then problem (1) is solved; in fact if $UB \le z(\theta^*)$, then x^* is a global minimum, otherwise $x'(\theta^*)$ is a global minimum; ii) if $\xi'' < \xi_{max}$, then we consider the new feasible level ξ'' and the corresponding parametric problem $P(\xi''+\theta)$ with $x^*=x'(\theta^*)$, $UB=z(\theta^*)$ if $UB>z(\theta^*)$;

B2) if $\Delta>0$ and $\theta^1>0$, then three subcases need to be considered:

B2a) if $\sup H(\theta)=+\infty$, then problem (1) is solved; in fact if $\min\{UB, z(\theta^1)\}\leq 0$, then x^*, or $x'(\theta^1)$ if $z(\theta^1)<UB$, is a global minimum, otherwise the minimum is not attained and $\inf_{x\in X} f(x)=0$;

B2b) if $\theta^1\leq\sup H(\theta)=\theta^*<+\infty$, then two subcases need to be considered:

B2b1) if $\sup O(\theta)=+\infty$ and $\min\{UB, z(\theta^1)\}\leq 0$ or $\theta^*<\sup O(\theta)=\theta''<+\infty$, $\xi''=\xi'+\theta''\geq\xi_{max}$ and $\min\{UB, z(\theta^1)\}\leq z(\theta'')$; then problem (1) is solved and x^*, or $x'(\theta^1)$ if $z(\theta^1)<UB$, is a global minimum;

B2b2) otherwise; i) if $\xi''=\xi'+\theta^*=\xi_{max}$, then problem (1) is solved and the global minimum is x^* if $UB\leq\min\{z(\theta^1), z(\theta^*)\}$, $x'(\theta^1)$ if $z(\theta^1)<\min\{UB, z(\theta^*)\}$, $x'(\theta^*)$ if $z(\theta^*)<\min\{UB, z(\theta^1)\}$; ii) if $\xi''<\xi_{max}$ then we consider the new feasible level ξ'' and the corresponding parametric problem $P(\xi''+\theta)$ with $x^*=x'(\theta^1)$ ($x^*=x'(\theta^*)$), $UB=z(\theta^1)$ ($UB=z(\theta^*)$) if $\min\{UB, z(\theta^*)\}>z(\theta^1)$ ($\min\{UB, z(\theta^1)\}>z(\theta^*)$);

B2c) if $\sup H(\theta)=\theta^*<\theta^1$, then two subcases need to be considered:

B2c1) if $\sup O(\theta)=+\infty$ and $UB\leq\min\{0, z(\theta^1)\}$; then problem (1) is solved and x^* is a global minimum;

B2c2) otherwise; i) if $\xi''=\xi'+\theta^*=\xi_{max}$ then problem (1) is solved, in fact if $UB\leq z(\theta^*)$, then x^* is a global minimum, otherwise $x'(\theta^*)$ is a global minimum; ii) if $\xi''<\xi_{max}$ then we consider the new feasible level ξ'' and the corresponding parametric problem $P(\xi''+\theta)$ with $x^*=x'(\theta^*)$, $UB=z(\theta^*)$ if $UB>z(\theta^*)$;

If p=-2, then the derivative of $z(\theta)$ is $z'(\theta) = (\xi'+\theta)^{-3}[(\beta\xi'-\lambda')\theta+\lambda'\xi'-2z']$ and the following three cases can occur:

C1) if $\lambda'\xi'-2z'>0$ and $\beta\xi'-\lambda'\geq 0$, then two subcases need to be considered:

C1a) if $\sup O(\theta)=+\infty$, then problem (1) is solved and x^* is a global minimum;

C1b) if $\sup O(\theta)=\theta''<+\infty$, then: i) if $\xi''=\xi'+\theta''\geq\xi_{max}$ then problem (1) is solved and x^* is a global minimum; ii) if $\xi''<\xi_{max}$ then we consider the new feasible level ξ'' and the corresponding parametric problem $P(\xi''+\theta)$;

C2) if $\lambda'\xi'-2z'\geq 0$ and $\beta\xi'-\lambda'<0$ or $\lambda'\xi'-2z'<0$ and $\beta\xi'-\lambda'\leq 0$, then three subcases need to be considered:

C2a) if $\sup H(\theta)=+\infty$, then problem (1) is solved; in fact if $UB\leq\frac{1}{2}\beta$ then x^* is a global minimum, otherwise the minimum is not attained and $\inf_{x\in X} f(x)=\frac{1}{2}\beta$;

C2b) if $\sup H(\theta)=\theta^*<\sup O(\theta)=+\infty$ and $UB\leq\frac{1}{2}\beta$ or $\sup H(\theta)=\theta^*<\sup O(\theta)=\theta''<+\infty$, $\xi''=\xi'+\theta''\geq\xi_{max}$ and $UB\leq z(\theta'')$, then problem (1) is solved and x* is a global minimum;

C2c) otherwise; i) if $\xi''=\xi'+\theta^*=\xi_{max}$ then problem (1) is solved; in fact if $UB\leq z(\theta^*)$, then x* is a global minimum, otherwise $x'(\theta^*)$ is a global minimum; ii) if $\xi''<\xi_{max}$, then we consider the new feasible level ξ'' and the corresponding parametric problem $P(\xi''+\theta)$ with $x^*=x'(\theta^*)$, $UB=z(\theta^*)$ if $UB>z(\theta^*)$;

C3) if $\lambda'\xi'-2z'\leq 0$ and $\beta\xi'-\lambda'>0$, then three subcases need to be considered:

C3a) if $\sup H(\theta)=+\infty$, then problem (1) is solved; in fact if $UB\leq z(\hat{\theta})$, where $\hat{\theta}=-\frac{\lambda'\xi'-2z'}{\beta\xi'-\lambda'}$, then x* is a global minimum, otherwise the minimum is $x'(\hat{\theta})$;

C3b) if $\hat{\theta}\leq\sup H(\theta)=\theta^*<\sup O(\theta)=+\infty$, then problem (1) is solved; in fact if $UB\leq z(\hat{\theta})$, then x* is a global minimum, otherwise the minimum is $x'(\hat{\theta})$;

C3c) otherwise; i) if $\xi''=\xi'+\theta^*=\xi_{max}$ then problem (1) is solved; in fact if $UB\leq z(\theta^*)$, then x* is a global minimum, otherwise $x'(\theta^*)$ is a global minimum; ii) if $\xi''<\xi_{max}$, then we consider the new feasible level ξ'' and the corresponding parametric problem $P(\xi''+\theta)$ with $x^*=x'(\theta^*)$, $UB=z(\theta^*)$ if $UB>z(\theta^*)$;

Starting from the solution x' corresponding to the level ξ', we arrive at one of the following situations:

i) x* is an optimal solution;
ii) the problem is unbounded;
iii) a level greater than ξ' has been found together with the best incumbent solution.

In order to propose a finite algorithm to solve problem (1), it remains to consider an appropriate initialization and to show how it is possible to obtain the optimal level solution corresponding to the new level ξ'' in a finite number of iterations.

Let us solve the following linear programs:

(P_1) $\quad \min\ d^Tx+d_0,\ x\in X;$

If x' is the unique optimal solution of (P_1) and $\xi'=d^Tx'+d_0$ is the corresponding level then $X(\xi')=\{x'\}$ and clearly x' is an optimal level solution; in this case we can start with $x^*=x'$ and only increasing value of ξ need to be considered. Otherwise if $X(\xi')\neq\{x'\}$ the optimal level solution x' corresponding to the feasible level $\xi'=d^Tx'+d_0$ must be found and then we can start with $x^*=x'$ considering only increasing value of ξ.

It remains to consider the problem of obtaining the optimal level solution x' corresponding to the new level $\xi''=\xi'+\tilde{\theta}$ in a finite number of iterations. If $\tilde{\theta}=\sup H(\theta)=\sup F(\theta)$, then $x'=x'(\tilde{\theta})$ and at least one new constraint is binding at x', while if $\tilde{\theta}=\sup H(\theta)=\sup O(\theta)$ at least one of the Lagrange multipliers $\mu'(\tilde{\theta})$ are zero and the corresponding constraint can be deleted. If $\tilde{\theta}>\sup H(\theta)$, then $x'(\tilde{\theta})$ is unfeasible and the optimal level solution x' must be determined. Starting from the level ξ' the level $\xi'+\sup H(\theta)$ is obtained together with the optimal level solution $x'(\sup H(\theta))$, then starting from the level $\xi'=\xi'+\sup H(\theta)$ the new level $\xi'+\sup H(\theta)$ is obtained and so on until a level $\xi'\geq\xi''$ is reached. The proposed procedure is finite since for each new level either at least one new constraint is added or at least one old constraint is deleted.

Let us consider the following three numerical examples: the first for the case $p>-2$, the second for the case $p<-2$ and the third for the case $p=-2$.

Example n. 1

$$\min f(x_1, x_2) = \left(\frac{1}{2}(x_1, x_2)\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + (-2, 1)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} -4\right)(x_1+2x_2+1)^3$$

$$(1)\ x_1 \geq 0,\quad (2)\ x_2 \geq 0,\quad (3)\ 5x_1 - 10x_2 \geq -2,\quad (4)\ -x_1 + 3x_2 \geq -4.$$

Starting from the optimal solution $x'=(0, 0)$ of the linear program

$\min \{x_1 + 2x_2:\ x_1 \geq 0, x_2 \geq 0, 5x_1 - 10x_2 \geq -2, -x_1 + 3x_2 \geq -4\}$

we obtain the following steps:

- base {(2), P} (P is the parametric constraint $x_1+2x_2=1+\theta$), $\xi'=1$, $x'=(0, 0)$, $x'(\theta)=(\theta, 0)$, $\mu'_2(\theta)=5-5\theta$, $\lambda'(\theta)=-2+3\theta$, $z'=-4$, $x^*=(0, 0)$, UB=-4, $z(\theta)=(1+\theta)^3(3/2\theta^2-2\theta-4)$, $z'(\theta)=(1+\theta)^2(15/2\theta^2-5\theta-14)$, $\sup O(\theta)=1$, $\sup F(\theta)=4$, $\sup H(\theta)=1$, $\Delta=445$, $\theta^1=-1.073$, $\theta^2=1.73966$; case A2c2) holds;
- $x'(1)=(1, 0)$, $z(1)=-36<$UB$=-4$, $x^*=(1, 0)$, UB=-36, base {P}, $\xi'=1+1=2$, $x'=(1, 0)$ $x'(\theta)=(1, 1/2\theta)$, $\lambda'(\theta)=1+1/2\theta$, $z'=-9/2$, $z(\theta)=(2+\theta)^3(1/4\theta^2+\theta-9/2)$, $z'(\theta)=(2+\theta)^2(5/4\theta^2+5\theta-23/2)$, $\sup O(\theta)=+\infty$, $\sup F(\theta)=7/5$, $\sup H(\theta)=7/5$, $\Delta=82.5$, $\theta^1=-5.633$, $\theta^2=1.633$; case A2c2) holds;
- $x'(7/5)=(1, 7/10)$, $z(7/5)=-83.3245<$UB$=-36$, $x^*=(1, 7/10)$, UB=-83.3245, base{(3), P}, $\xi'=2+7/5=12/5$, $x'=(1, 7/10)$, $x'(\theta)=(1+1/2\theta, 7/10+1/4\theta)$, $\lambda'(\theta)=17/10+9/8\theta$, $z'=4/50$,

$z(\theta)= (17/5+\theta)^3\,(9/16\theta^2+17/10\theta+4/50)$, $z'(\theta) = (2+\theta)^2\,(45/16\theta^2+437/40\theta+301/50)$, sup $O(\theta)=+\infty$, sup $F(\theta)=+\infty$, sup $H(\theta)=+\infty$, $\Delta=51.6306$, $\theta^1=-3.2196$, $\theta^2=-0.6648$; case A1a) holds and $x^*=(1, 7/10)$ with $f(x^*)=-83.3245$ is the optimal solution.

Example n. 2

$$\min f(x_1, x_2) = \left(\frac{1}{2}(x_1, x_2)\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + (-2, 1)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 4\right)(x_1+2x_2+1)^{-3}$$

(1) $x_1 \geq 0$, (2) $x_2 \geq 0$, (3) $x_1 - x_2 \geq -2$, (4) $-x_1 + 2x_2 \geq -4$.

Starting from the optimal solution $x'=(0, 0)$ of the linear program

$\min \{x_1 + 2x_2:\ x_1 \geq 0, x_2 \geq 0, x_1 - x_2 \geq -2, -x_1 + 2x_2 \geq -4\}$

we obtain the following steps:

- base {(2), P}, $\xi'=1$, $x'=(0, 0)$, $x'(\theta)=(\theta, 0)$, $\mu_2'(\theta)=5-5\theta$, $\lambda'(\theta)=-2+3\theta$, $z'=-4$, $x^*=(0, 0)$, UB=-4, $z(\theta)=(1+\theta)^{-3}(3/2\theta^2-2\theta-4)$, $z'(\theta)=(1+\theta)^{-4}(-3/2\theta^2+7\theta+10)$, sup $O(\theta)=1$, sup $F(\theta)=4$, sup $H(\theta)=1$, $\Delta=109$, $\theta^1=-1.146768$, $\theta^2=5.813435$; case B1c) holds;
- $x'(1)=(1, 0)$, $z(1)=-9/16>UB=-4$, $x^*=(0, 0)$, UB=-4, base {P}, $\xi'=1+1=2$, $x'=(1, 0)$ $x'(\theta)=(1, 1/2\theta)$, $\lambda'(\theta)=1+1/2\theta$, $z'=-9/2$, $z(\theta)=(2+\theta)^{-3}(1/4\theta^2+\theta-9/2)$, $z'(\theta)=(2+\theta)^{-4}(-1/4\,\theta^2-\theta+31/2)$, sup $O(\theta)=+\infty$, sup $F(\theta)=6$, sup $H(\theta)=6$, $\Delta=16.5$, $\theta^1=-10.124038$, $\theta^2=6.124038$; case B1b) holds; since UB=-4<0, $x^*=(0, 0)$ is the optimal solution.

Example n. 3

$$\min f(x_1, x_2) = \left(\frac{1}{2}(x_1, x_2)\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + (-2, 1)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 4\right)(x_1+2x_2+1)^{-2}$$

(1) $x_1 \geq 0$, (2) $x_2 \geq 0$, (3) $x_1 - x_2 \geq -2$, (4) $-x_1 + 2x_2 \geq -4$.

Starting from the optimal solution $x'=(0, 0)$ of the linear program

$\min \{x_1 + 2x_2:\ x_1 \geq 0, x_2 \geq 0, x_1 - x_2 \geq -2, -x_1 + 2x_2 \geq -4\}$

we obtain the following steps:

- base {(2), P}, $\xi'=1$, $x'=(0, 0)$, $x'(\theta)=(\theta, 0)$, $\mu_2'(\theta)=5-5\theta$, $\lambda'(\theta)=-2+3\theta$, $z'=-4$, $x^*=(0, 0)$, UB=-4, $z(\theta)=(1+\theta)^{-2}(3/2\theta^2-2\theta-4)$, $z'(\theta)=(1+\theta)^{-3}(5\theta+6)$, sup $O(\theta)=1$, sup $F(\theta)=4$, sup $H(\theta)=1$; case C1b) holds;
- $x'(1)=(1, 0)$, $z(1)=-9/8>UB=-4$, $x^*=(0, 0)$, UB=-4, base {P}, $\xi'=1+1=2$, $x'=(1, 0)$ $x'(\theta)=(1, 1/2\theta)$, $\lambda'(\theta)=1+1/2\theta$, $z'=-9/2$, $z(\theta)=(2+\theta)^{-2}(1/4\theta^2+\theta-9/2)$, $z'(\theta)=11(2+\theta)^{-3}$,

$\sup O(\theta)=+\infty$, $\sup F(\theta)=6$, $\sup H(\theta)=6$; case C1a) holds; $x^*=(0, 0)$ is the optimal solution.

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