Università degli Studi di Pisa
Dipartimento di Statistica e Matematica
Applicata all'Economia

Report n.210

A method for calculating subdifferential
Convex vector functions

Alberto Cambini – Dinh The Luc – Laura Martein

Pisa, Luglio 2001

- Stampato in Proprio -

Via Cosimo Ridolfi, 10 – 56124 PISA – Tel. Segr. Amm. 050 945231 Segr. Stud. 050 945317 Fax 050 945375
Cod. Fisc. 80003670504 - P. IVA 00286820501 - Web http://statmat.ec.unipi.it/ - E-mail: dipstat@ec.unipi.it

A method for calculating subdifferential of convex vector functions

Alberto Cambini[1] Dinh The Luc [2]
Laura Martein[1]
[1]Department of Statistics and Applied Mathematics, University of Pisa
[2]Department of Mathematics, University of Avignon, France

July 2001

Abstract

Generalized convex vector functions are characterized by using order preserving transformations and some calculus rules for subdifferential of convex vector functions are obtained.

Key Words Partial order, efficient point, generalized convex functions, subdifferential.
AMS subject classifications 90C29

1 Introduction

Let C be a closed and convex cone in $\Re^n$. The partial order induced by C is denoted by $\geq_C$ and defined by

$$x \geq_C y \ if \ x - y \in C.$$

Let f be a vector function from a convex subset $X \subseteq \Re^\ell$ to $\Re^n$. We say that f is C-convex if for each $x, y \in X$ and for each $\lambda \in (0,1)$ one has

$$\lambda f(x) + (1-\lambda) f(y) \geq_C f(\lambda x + (1-\lambda) y).$$

In the case where C is the positive orthant cone

$$\Re^n_+ := \{x = (x_1, ..., x_n) : x_i \geq 0, \ \ i = 1, ..., n\},$$

it follows directly that f is $\Re^n_+$-convex if and only if the n components $f_1, ..., f_n$ of f are convex in the sense of scalar functions. This fact allows us to expect a rich calculus for convex vector functions.

The class of convex vector functions plays an important role in the theory of vector optimization and related areas. It has been investigated by several authors (see for instance [7, 8, 11]and references given therein).

The concept of convex subdifferential developed mainly by Rockafellar [16], can succesfully be extended to the vector case. Despites of numerous litterature on theoretical aspects of subdifferential of convex vector functions, very little is known about numerical results which are evidently indispensable in applications.

The purpose of this note is to present a simple method to calculate the subdifferential of a convex vector function when the ordering cone is polyhedral. The main tool we are going to use is a special linear transformation, introduced in [3],which replaces the ordering cone C by the positive orthant cone with respect to which the subdifferential of a convex function is componently-wise computed. By an inverse transformation one obtain the subdifferential of the function with respect to the original order.

The paper is organized as follows. Section 2 is devoted to the chain rule of subdifferential under linear transformations. In Section 3 (4) the calculus of the subdifferential of a nondifferentiable (differentiable) function is solved by means of a matrix equation. Several examples to illustrate our method are given.

2 Linear transformations and subdifferential

Throughout this paper C is a closed and convex cone in $\Re^n$ and L is a linear transformation from $\Re^n$ to $\Re^m$. The image $L(C)$ of the cone C is a closed and convex cone in $\Re^m$. A relation between the partial order $(\geq_C)$ and $(\geq_{L(C)})$ is given in the following lemma:

Lemma 2.1 *For every $x, y \in \Re^n$ one has : $x \geq_C y \Rightarrow L(x) \geq_{L(C)} L(y)$. The converse is also true provided $KerL \subseteq C$.*

Proof The implication $(\Rightarrow)$ is straightforward. For the converse, $L(x) \geq=_{L(C)} L(y)$ means that $L(x-y) \in L(C)$. Hence $x - y \in C + KerL \subseteq C$ because $KerL \subseteq C$, which implies $x \geq=_C y$. □

Now let f be a vector function from $X \subseteq \Re^\ell$ to $\Re^n$. The composite function $L \circ f$ is then from X to $\Re^m$. As a direct consequence of Lemma 2.1, we obtain

Lemma 2.2 *Let X be a nonempty convex set in $\Re^\ell$. If f is C-convex, then $L \circ f$ is $L(C)$-convex. The converse is also true provided $Ker L \subseteq C$.*

We recall ([11, 13, 18]) that the subdifferential of f at $x \in X$ is the set

$$\partial_C f(x) := \{A \in L(\Re^\ell, \Re^n) : f(y) - f(x) \geq_C A(y - x), \;\; for \;\; all \;\; y \in X\}$$

where $L(\Re^\ell, \Re^n)$ denotes the space of $n \times \ell$-matrices.
When $C = \Re^n_+$, it is known that C-convexity is equivalent to componentwise convexity, so that

$$\partial_C f(x) = \partial f_1(x) \times ... \times \partial f_n(x)$$

where $f_1, ..., f_n$ are components of f and $\partial f_1, ..., \partial f_n$ are their classical convex subdifferential.

The following fundamental theorem states that the subdifferential of a C-convex funtion is nonempty.

Theorem 2.1 *Let X be an open convex subset of $\Re^l$ and let $f : X \to \Re^m$ be a C-convex function. Then for every $x \in X$ it results $\partial_C f(x) \neq \emptyset$.*

Proof The proof is trivial if $C = \Re^n$.
Let $l(C) = C \cap (-C)$ be the lineality space of C and decompose $\Re^n$, C as $\Re^n = l(C) \times \Re^k$, $C = l(C) \times K$. Obviously, K is a pointed closed and convex cone. Setting $f = (f_1, f_2)$, we have that f is C-convex iff f_2 is K-convex.
A matrix A belongs to $\partial_C f(x)$ iff $f(y) - f(x) \geq_C A(y - x)$, that is, setting $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$,

$$f_1(y) - f_1(x) \geq_{l(C)} A_1(y - x) \tag{2.1}$$

$$f_2(y) - f_2(x) \geq_K A_2(y - x) \tag{2.2}$$

Since (2.1) holds for every matrix A_1, we have that $A \in \partial_C f(x))$ if and only if $A_2 \in \partial_K f_2(x)$. Taking into account that f_2 is K-convex and K is pointed closed and convex, it results that f_2 is locally Lipschitz so that $\partial_K f_2(x) \neq \emptyset$ (see for instance [13, 18]). Consequently $\partial_C f(x) \neq \emptyset$ and it is characterized as the set

$$\partial_C f(x) = \{\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} : A_2 \in \partial_K f_2(x)\}$$

□

Below there is a chain rule for convex subdifferential.

Lemma 2.3 *The following chain rule is true:*

$$L \circ \partial_C f(x) \subseteq \partial_{L(C)}(L \circ f)(x)$$

Equality holds provided L is an isomorphism.

Proof Let $A \in \partial_C f(x)$. By definition, one has

$$f(y) - f(x) \geq_C A(y - x), \forall y \in X \tag{2.3}$$

Hence

$$L \circ f(y) - L \circ f(x) \geq_{L(C)} L \circ A(y - x), \forall y \in X \tag{2.4}$$

which means that $L \circ A \in \partial_{L(C)}(L \circ f)(x)$.
Now, if L is an isomorphism, then by using the above inclusion for L^{-1}, we obtain

$$L^{-1} \circ \partial_{L(C)}(L \circ f)(x) \subseteq \partial_{L^{-1}(L(C))} L^{-1} \circ L \circ f(x) = \partial_C f(x)$$

Consequently,

$$\partial_{L(C)} L \circ f(x) \subseteq L \circ \partial_C f(x)$$

and equality follows. □

We are now interested in the case where C is a polyhedral cone given by the following system of linear inequalities :

$$< \xi_i, x > \geq 0, \quad i = 1, ..., k, \tag{2.5}$$

where $\xi_1, ..., \xi_k \in \Re^n$.
Let us denote by T the following transformation from $\Re^n$ to $\Re^k$:

$$T(x) = (< \xi_1, x >, ..., < \xi_k, x >), \quad for \ every \ x \in \Re^n \tag{2.6}$$

This transformation possesses several properties [3]. Some of them are given next.

Lemma 2.4 *Let T defined by (2.4). Then one has*
i) T is linear and $KerT = l(C)$, where $l(C)$ denotes the linear part of C, that is $l(C) = C \cap -C$;
ii) T is injective if and only if C is pointed, that is $l(C) = \{0\}$;
iii) T is an isomorphism if and only if C is pointed and $k = n$.

The conclusions of Lemma 2.1, 2.2 and 2.3 applied to T yeld the next result.

Corollary 2.1 *Let C be defined by (2.3) and T defined by (2.4). Let f be a vector function from a convex subset $X \subseteq \Re^\ell$ to $\Re^n$. Then one has:*
i) For $y_1, y_2 \in \Re^n$, $y_1 \geq_C y_2$ if and only if $T(y_1) \geq_{\Re^k_+} T(y_2)$;
ii) f is C-convex if and only if $T \circ f$ is $\Re^k_+$-convex;
iii) $T \circ \partial_C f(x) \subseteq \partial_{\Re^k_+}(T \circ f)(x) = \partial(\xi_1 \circ f)(x) \times ... \times \partial(\xi_k \circ f)(x)$.

Proof For the first statement, by Lemma 2.4, $KerT = l(C) \subseteq C$. Hence Lemma 2.1 implies that $y_1 \geq_C y_2$ if and only if $T(y_1) \geq_{T(C)} T(y_2)$. It suffices to observe that $T(C) \subseteq \Re^k_+$, hence $T(y_1) \geq_{T(C)} T(y_2)$ if and only if $T(y_1) \geq_{\Re^k_+} T(y_2)$. The second statement is deduced from the first one. For the last statement, the inclusion $T(C) \subseteq \Re^k_+$ implies $\partial_{T(C)}(T \circ f)(x) \subseteq \partial_{\Re^k_+}(T \circ f)(x)$. An application of Lemma 2.3 achieves the proof. □

3 A matrix equation in nonsmooth case

From now on, f is defined on $\Re^\ell$. The cone C and the transformation T are defined respectively by (2.3) and (2.4). In this section we study the case where f is non-differentiable, or more generally $T \circ f$ is non-differentiable. We have seen in iii) of Corollary 2.1 that

$$T \circ \partial_C f(x) \subseteq \partial_{\Re^k_+}(T \circ f)(x) = \partial(\xi_1 \circ f)(x) \times ... \times \partial(\xi_k \circ f)(x). \qquad (3.1)$$

We will see later that the inclusion is strict. The following theorem characterizes the subdifferential $\partial_C f(x)$.

Theorem 3.1 *Assume that $T \circ f$ is a $\Re^k_+$-convex function. Then f is C-convex and a matrix $A \in \partial_C f(x)$ if and only if there exixts a matrix $B \in \partial_{\Re^k_+}(T \circ f)(x)$ such that $TA = B$.*

Proof From Corollary 2.1 it results that f is C-convex; taking into account Lemma 2.3 we have to show that if A verifies the equation $TA = B$, then $A \in \partial_C f(x)$. In fact, since $T \circ f$ is $\Re^k_+$-convex, one derives

$$T \circ f(y) - T \circ f(x) \geq_{\Re^k_+} T \circ A(y - x), \quad \textit{for every } y \in \Re^\ell$$

By Corollary 2.1, this in turn implies

$$f(y) - f(x) \geq_C A(y - x), \ \ \forall y \in \Re^\ell$$

which shows that $A \in \partial_C f(x)$. □

In order to calculate $\partial_C f(x)$, according to Theorem 3.1 we must solve the matrix equation $TA = B$.
With this aim, in what follows with respect to a matrix X of order $s \times t$, the notation

$$X = \begin{pmatrix} X_{u \times v} & X_{u \times (t-v)} \\ X_{(s-u) \times v} & X_{(s-u) \times (t-v)} \end{pmatrix}$$

means that X has been decomposed in submatrices of order $u \times v$, $u \times (t-v)$, $(s-u) \times v$, $(s-u) \times (t-v)$,respectively, with the convention that :

- $X = \begin{pmatrix} X_{u\times t} \\ X_{(s-u)\times t} \end{pmatrix}$, when $v \geq t$ and $s > u$;
- $X = \begin{pmatrix} X_{s\times v} & X_{s\times(t-v)} \end{pmatrix}$, when $u \geq s$ and $t > v$;
- $X = X_{s\times t}$ when $v \geq t$ and $u \geq s$.

Let $p = rankT$ and let Γ be the nonsingular matrix which reduces T to its Jordan canonical form $\overline{T}$. Without any loss of generality, we will assume that $\overline{T}$ and Γ are of the form:

$$\overline{T} = \begin{pmatrix} I_{p\times p} & H_{p\times(n-p)} \\ O_{(k-p)\times p} & O_{(k-p)\times(n-p)} \end{pmatrix}, \quad \Gamma = \begin{pmatrix} \Gamma_{p\times p} & O_{p\times(k-p)} \\ \Gamma_{(k-p)\times p} & \Gamma_{(k-p)\times(k-p)} \end{pmatrix} \tag{3.2}$$

where $I_{p\times p}$ denotes the identity matrix of order p and O denotes the null matrix.

According to the form of $\overline{T}$, we partition the matrices $A \in \partial_C f(x)$ and $B \in \partial_{\Re^k_+}(T \circ f)(x)$ in the following way:

$$A = \begin{pmatrix} A_{p\times p} & A_{p\times(\ell-p)} \\ A_{(n-p)\times p} & A_{(n-p)\times(\ell-p)} \end{pmatrix}, \quad B = \begin{pmatrix} B_{p\times p} & B_{p\times(\ell-p)} \\ B_{(k-p)\times p} & B_{(k-p)\times(\ell-p)} \end{pmatrix} \tag{3.3}$$

Now we are able to characterize the subdifferential $\partial_C f(x)$.

The following theorem holds:

Theorem 3.2 *i) If $rankT = p = k$, then $A \in \partial_C f(x)$ if and only if*

$$A = \bar{A} + \begin{pmatrix} \Gamma_{p\times p} B_{p\times p} & \Gamma_{p\times p} B_{p\times(\ell-p)} \\ O_{(n-p)\times p} & O_{(n-p)\times(\ell-p)} \end{pmatrix}, \quad B \in \partial_{\Re^k_+}(T \circ f)(x) \tag{3.4}$$

where every column of the $n \times \ell$ matrix $\bar{A}$ is an arbitrary element of $KerT$, that is a linear combination of the columns of the matrix $\begin{pmatrix} -H \\ I_{n-p} \end{pmatrix}$.

ii) If $rankT = p < k$, then $A \in \partial_C f(x)$ if and only if there exists $B \in \partial_{\Re^k_+}(T \circ f)(x)$ such that (3.5), (3.6) , (3.7) hold.

$$\Gamma_{(k-p)\times p} B_{p\times p} + \Gamma_{(k-p)\times(k-p)} B_{(k-p)\times p} = O_{(k-p)\times p} \tag{3.5}$$

$$\Gamma_{(k-p)\times p} B_{p\times(\ell-p)} + \Gamma_{(k-p)\times(k-p)} B_{(k-p)\times(\ell-p)} = O_{(k-p)\times(\ell-p)} \tag{3.6}$$

$$A = \bar{A} + \begin{pmatrix} \Gamma_{p\times p} B_{p\times p} & \Gamma_{p\times p} B_{p\times(\ell-p)} \\ O_{(n-p)\times p} & O_{(n-p)\times(\ell-p)} \end{pmatrix}, \quad B \in \partial_{\Re^k_+}(T \circ f)(x) \tag{3.7}$$

Proof The matrix equation $TA = B$ is equivalent to $\Gamma TA = \overline{T}A = \Gamma B$. By means of simple calculations, we achieve (3.4) - (3.7). □

The following examples show the way to calculate $\partial_C f(x)$ when f is nondifferentiable and C is a polyhedral cone.

Example 3.1 *Consider the nondifferentiable function* $f : \Re^2 \longrightarrow \Re^3$, *defined as* $f(x,y) = (|x|, |x| + 3|y|, |y|)$ *and the polyhedral cone* $C = \{(\alpha, \beta, \gamma) : \alpha + \beta + \gamma \geq 0,\ \ \beta - \gamma \geq 0\}$.

It results $(T \circ f)(x,y) = (2|x| + 4|y|, |x| + 2|y|)$. *Since any component of* $T \circ f$ *is a convex function, the function* f *is* C*-convex. For sake of simplicity we limit ourselves to calculate* $\partial_C f(0,0)$. *Taking into account of (3.1), we have*

$$\partial_{\Re^2_+}(T \circ f)(0,0) = \{B \in L(\Re^2, \Re^2) : B = \begin{pmatrix} 2a & 4b \\ c & 2d \end{pmatrix};\ \ a, b, c, d \in [-1, 1]\}$$

Since $p = k = 2$ *and* $\Gamma = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, *according to (5.4), we have* $A \in \partial_C f(0,0)$ *if and only if*

$$A = \begin{pmatrix} -2\xi_1 & -2\xi_2 \\ \xi_1 & \xi_2 \\ \xi_1 & \xi_2 \end{pmatrix} + \begin{pmatrix} 2a - c & 4b - 2d \\ c & 2d \\ 0 & 0 \end{pmatrix},\ \ \xi_1, \xi_2 \in \Re,\ \ a, b, c, d \in [-1, 1].$$

Example 3.2 *Consider the nondifferentiable function as in Example 3.1 and the polyhedral cone*

$$C = \{(\alpha, \beta, \gamma) : \alpha + \beta + \gamma \geq 0,\ \ \beta - \gamma \geq 0,\ \ \alpha + \gamma \geq 0,\ \ \gamma \geq 0\}.$$

It results $(T \circ f)(x,y) = (2|x| + 4|y|, |x| + 2|y|, |x| + |y|, |y|)$, *so that for (3.1) we have*

$$\partial_{\Re^4_+}(T \circ f)(0,0) = \{B \in L(\Re^2, \Re^4) : B = \begin{pmatrix} 2a & 4b \\ c & 2d \\ e & f \\ 0 & g \end{pmatrix}, a, b, c, d, e, f, g \in [-1, 1]\}$$

The linear transformation associated to the cone C *is* $T = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$,

so that $p = 3 < k = 4$.

The matrix $\Gamma = \begin{pmatrix} -1 & 1 & 2 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & -1 & 0 \\ -1 & 1 & 1 & 1 \end{pmatrix}$ *reduces* T *to the following canonical form* $\overline{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. *According to (3.5),(3.6), and taking into account that* $\ell = 2 < p = 3$, *we have*

$\Gamma_{1\times 3} = \begin{pmatrix} -1 & 1 & 1 \end{pmatrix}$, $B_{3\times 2} = \begin{pmatrix} 2a & 4b \\ c & 2d \\ e & f \end{pmatrix}$, $\Gamma_{1\times 1} = 1$, $B_{1\times 2} = \begin{pmatrix} 0 & g \end{pmatrix}$

so that $\Gamma_{1\times 3}B_{3\times 2} + \Gamma_{1\times 1}B_{1\times 2} = \begin{pmatrix} -2a + c + e & -4b + 2d + f + g \end{pmatrix}$.
It follows that the matrix equation $TA = B$ *has solution if and only if* $-2a + c + e = 0, \ -4b + 2d + f + g = 0$
According to (3.7), $A \in \partial_C f(0,0)$ *if and only if*

$$A = \begin{pmatrix} e & f - g \\ c & 2d + g \\ 0 & g \end{pmatrix}, \quad c, d, e, f, g \in [-1, 1].$$

Remark 3.1 *Conditions (3.5) and (3.6) point out that the inclusion in (3.1) is, in general, strict. Referring to Example 3.2, the matrix* $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \in \partial_{\Re^4_+}(T \circ f)(0,0)$, *but simple calculations show that (3.5) is not verified, so that the matrix equation* $TA = B$ *does not have solution.*

4 A matrix equation in smooth case

In this section we will assume that f is differentiable or, more generally, that the composite function $T \circ f$ is differentiable and its Jacobian matrix is written as $J(T \circ f)(x)$.

Lemma 4.1 *Assume that* f *is* C*-convex. Then the subdifferential* $\partial_C f(x)$ *consists of all solutions to the matrix equation :*

$$TA = J(T \circ f)(x). \tag{4.1}$$

Proof If $A \in \partial_C f(x)$ then, by iii) of Corollary 2.1, we have $TA \in J(T \circ f)(x)$, and taking into account that $J(T \circ f)(x)$ is a singleton set, we achived (3.1).

Conversely, let A be a solution of (4.1), such a solution exists since $\partial_C f(x)$ is nonempty and $J(T \circ f)(x)$ is a singleton set. From the convexity of the function $T \circ f$, we have $(T \circ f)(x+v) - (T \circ f)(x) \geq_{\Re^k_+} TAv, \ \forall v \in \Re^\ell$. From ii) of Corollary 2.1 we have $f(x+v) - f(x) \geq_C Av, \ \forall v \in \Re^\ell$, so that $A \in \partial_C f(x)$. □

Now we will establish a simple rule for calculating the set of all matrices A satisfying (3.1), when the function f is differentiable.
With this aim let $\overline{T}$ be the Jordan canonical form of T. Without any loss of generality, we will assume that $\overline{T}$ is of the form:

$$\overline{T} = \begin{pmatrix} I_p & H \\ O_1 & O_2 \end{pmatrix} \tag{4.2}$$

where $p = rankT$, I_p denotes the identity matrix of order p, H is a $p \times (n-p)$, O_1, O_2 are the null matrices of order $(k-p) \times p$ and $(k-p) \times (n-p)$, respectively , with the convention that $\overline{T} = \begin{pmatrix} I_p & H \end{pmatrix}$ when $p = k$.
Consider the Jacobian matrix $Jf(x)$ of the function f evaluated at $x \in \Re^\ell$; according to the form of $\overline{T}$, we partition the matrices $Jf(x)$ and A in the following way:
case $p < \ell$

$$Jf(x) = \begin{pmatrix} (Jf(x))_p & (Jf(x))^* \\ (Jf(x))_1 & (Jf(x))_2 \end{pmatrix}, \quad A = \begin{pmatrix} A_p & A^* \\ A_1 & A_2 \end{pmatrix} \tag{4.3}$$

where $(Jf(x))^*$, A^* are $p \times (\ell - p)$ matrices, $(Jf(x))_1$, A_1 are $(n-p) \times p$ matrices and $(Jf(x))_2$, A_2 are $(n-p) \times (\ell - p)$ matrices.
case $p \geq \ell$

$$Jf(x) = \begin{pmatrix} (Jf(x))_{p \times \ell} \\ (Jf(x))_1 \end{pmatrix}; \quad A = \begin{pmatrix} A_{p \times \ell} \\ A_1 \end{pmatrix} \tag{4.4}$$

where $(Jf(x))_1$, A_1 are now $(n-p) \times \ell$ matrices .

The following theorem holds:

Theorem 4.1 *Let C and T defined as (2.5) and (2.6) respectively. If f is C-convex and differentiable at $x \in \Re^\ell$ then $\partial_C f(x)$ consists of all matrices of the kind :*
i) ***case*** $p < \ell$

$$A = \bar{A} + \begin{pmatrix} (Jf(x))_p + H(Jf(x))_1 & (Jf(x))^* + H(Jf(x))_2 \\ O_3 & O_4 \end{pmatrix} \tag{4.5}$$

where O_3, O_4 are the null matrices of order $(n-p) \times p$ and $(n-p) \times (\ell - p)$ respectively, and where every column of the $n \times \ell$ matrix $\bar{A}$ is an arbitrary *element of $KerT$, that is a linear combination of the columns of the matrix* $\begin{pmatrix} -H \\ I_{n-p} \end{pmatrix}$.

ii) ***case*** $p \geq \ell$

$$A = \bar{A} + \begin{pmatrix} (Jf(x))_{p\times\ell} + H(Jf(x))_1 \\ O_5 \end{pmatrix} \tag{4.6}$$

where O_5 is the null matrix of order $(n-p)\times\ell$.

Proof Denote by P the non singular matrix which reduces T to its Jordan canonical form, that is such that $PT = \overline{T}$.

From (4.1), taking into account that f is a differentiable function, we have $TA = T(Jf(x))$, so that $PTA = PT(Jf(x))$, that is $\overline{T}A = \overline{T}(Jf(x))$.

i) **case** $p < \ell$

According to (4.3), $\overline{T}A = \overline{T}(Jf(x))$ becomes

$$\begin{pmatrix} I_p & H \\ O_1 & O_2 \end{pmatrix}\begin{pmatrix} A_p & A^* \\ A_1 & A_2 \end{pmatrix} = \begin{pmatrix} I_p & H \\ O_1 & O_2 \end{pmatrix}\begin{pmatrix} (Jf(x))_p & (Jf(x))^* \\ (Jf(x))_1 & (Jf(x))_2 \end{pmatrix}$$

Consequently

$$A_p = -HA_1 + (Jf(x))_p + H(Jf(x))_1$$
$$A^* = -HA_2 + (Jf(x))^* + H(Jf(x))_2$$

so that, setting $A_1 = I_{n-p}A_1$, $A_2 = I_{n-p}A_2$, we have

$$A = \begin{pmatrix} -H \\ I_{n-p} \end{pmatrix}\begin{pmatrix} A_1 & A_2 \end{pmatrix} + \begin{pmatrix} (Jf(x))_p + H(Jf(x))_1 & (Jf(x))^* + H(Jf(x))_2 \\ O_3 & O_4 \end{pmatrix}$$

where the elements of A_1 and A_2 are free parameters. It remains to prove that every column of $\bar{A} = \begin{pmatrix} -H \\ I_{n-p} \end{pmatrix}\begin{pmatrix} A_1 & A_2 \end{pmatrix}$ is an arbitrary element of $KerT$.

Let a^j be a column of the matrix A_1 or A_2. The set

$$W^j = \{z = \begin{pmatrix} -H \\ I_{n-p} \end{pmatrix} a^j, \ a^j \in \Re^{n-p}\}$$

is a linear subspace of $\Re^n$ with $dimW^j = n - p$. Furthermore

$$Tz = T\begin{pmatrix} -H \\ I_{n-p} \end{pmatrix} a^j = P^{-1}PT\begin{pmatrix} -H \\ I_{n-p} \end{pmatrix} a^j = P^{-1}\overline{T}\begin{pmatrix} -Ha^j \\ a^j \end{pmatrix} = P^{-1}(0) = 0$$

so that $W^j \subseteq KerT$. On the other hand $dimKerT = n - p = dimW^j$ and thus $W^j = KerT$.

ii) **case** $p \geq \ell$
According to (4.4), $\overline{T}A = \overline{T}(Jf(x))$ becomes

$$\begin{pmatrix} I_p & H \\ O_1 & O_2 \end{pmatrix} \begin{pmatrix} A_{p\times\ell} \\ A_1 \end{pmatrix} = \begin{pmatrix} I_p & H \\ O_1 & O_2 \end{pmatrix} \begin{pmatrix} (Jf(x))_{p\times\ell} \\ (Jf(x))_1 \end{pmatrix}$$

Consequently

$$A_{p\times\ell} = -HA_1 + (Jf(x))_{p\times\ell} + H(Jf(x))_1$$

Following the same line of the proof given in i) we get (4.6). □

Corollary 4.1 *Let C and T defined as (2.5) and (2.6) respectively. If f is C-convex and differentiable at $x \in \Re^\ell$, and if $rankT = n \leq k$ then $\partial_C f(x)$ is a singleton set and we have $\partial_C f(x) = \{Jf(x)\}$.*

Now we present some examples to illustrate the method for calculating any matrix $A \in \partial_C f(x)$.

Example 4.1 *Consider the differentiable function $f : \Re^2 \to \Re^3$, defined as $f(x, y) = (-x^2 + y^2, x^2 - y^2 + 4x - y, x + 2y)$ and the polyhedral cone $C = \{(\alpha, \beta, \gamma) : \alpha + \beta + \gamma \geq 0, \ -\alpha - \beta - \gamma \geq 0\}$.*
The linear transformation associated to the cone C is :

$$T = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}$$

it results $(T\circ f)(x, y) = (5x+y, -5x-y)$. Since any component of $(T\circ f)(x, y)$ is a convex function, the function $T \circ f$ is $\Re^2_+$-convex, so that, for Corollary 2.1, the function f is C-convex and its Jacobian matrix is $Jf(x, y) = \begin{pmatrix} -2x & 2y \\ 2x+4 & -2y-1 \\ 1 & 2 \end{pmatrix}$. *In order to calculate $\partial_C f(x, y)$, we consider the following Jordan canonical form of T :*

$$\overline{T} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Since $p = 1 < \ell = 2$, $A \in \partial_C f(x, y)$ if and only if A is of the form (4.5).
According to (4.2) and (4.3), it results $H = \begin{pmatrix} 1 & 1 \end{pmatrix}$, $(Jf(x, y))_p = (-2x)$, $(Jf(x, y))^* = (2y)$, $(Jf(x, y))_1 = \begin{pmatrix} 2x+4 \\ 1 \end{pmatrix}$, $(Jf(x, y))_2 = \begin{pmatrix} -2y-1 \\ 2 \end{pmatrix}$.
So that

$$Jf(x,y))_p + H(Jf(x))_1 = (-2x) + \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 2x+4 \\ 1 \end{pmatrix} = 5\ ,$$

$$Jf(x,y))^* + H(Jf(x,y))_2 = (2y) + \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} -2y-1 \\ 2 \end{pmatrix} = 1.$$

Taking into account that $KerT = \{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \lambda_1 + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \lambda_2,\ \lambda_1, \lambda_2 \in \Re \}$,

(4.5) gives

$$A = \begin{pmatrix} -\xi_1 - \xi_2 & -\mu_1 - \mu_2 \\ \xi_1 & \mu_1 \\ \xi_2 & \mu_2 \end{pmatrix} + \begin{pmatrix} 5 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and $\partial_C f(x,y) = \{A : \ \xi_1, \xi_2, \mu_1, \mu_2 \in \Re\}$.

Example 4.2 *Consider the differentiable function* $f : \Re^2 \longrightarrow \Re^3$ *, defined as* $f(x,y) = (-x^2 + y^2, 2x^2 + y, 2y)$ *and the polyhedral cone* $C = \{(\alpha, \beta, \gamma) : \alpha + \beta + \gamma \geq 0,\ \ \beta - \gamma \geq 0\}$.
The linear transformation associated to the cone C is :

$$T = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

it results $(T \circ f)(x,y) = (x^2 + y^2 + 3y, 2x^2 - y)$. *Since any component of* $(T \circ f)(x,y)$ *is a convex function, the function* $T \circ f$ *is* $\Re^2_+$*-convex, so that, for Corollary 2.1, the function f is C-convex and its Jacobian matrix is*
$Jf(x,y) = \begin{pmatrix} -2x & 2y \\ 4x & 1 \\ 0 & 2 \end{pmatrix}$. *In order to calculate the subdifferential* $\partial_C f(x)$, *we consider the following Jordan canonical form of T :*

$$\overline{T} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}$$

Since $p = 2 = \ell$, $A \in \partial_C f(x,y)$ *if and only if A is of the form (4.6).*
According to (4.2) and (4.4), it results $H = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $(Jf(x,y))_{p\times\ell} = \begin{pmatrix} -2x & 2y \\ 4x & 1 \end{pmatrix}$,
$(Jf(x,y))_1 = \begin{pmatrix} 0 & 2 \end{pmatrix}$, *so that* $Jf(x))_{p\times\ell} + H(Jf(x))_1 = \begin{pmatrix} -2x & 2y \\ 4x & 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} \begin{pmatrix} 0 & 2 \end{pmatrix} = \begin{pmatrix} -2x & 2y+4 \\ 4x & -1 \end{pmatrix}$.

Taking into account that $KerT = \{ \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \lambda,\ \ \lambda \in \Re \}$, *(4.6) gives*

$$A = \begin{pmatrix} -2\xi_1 & -2\xi_2 \\ \xi_1 & \xi_2 \\ \xi_1 & \xi_2 \end{pmatrix} + \begin{pmatrix} -2x & 2y+4 \\ 4x & -1 \\ 0 & 0 \end{pmatrix}$$

and $\partial_C f(x,y) = \{A : \ \xi_1, \xi_2 \in \Re\}$.

Example 4.3 *Consider the differentiable function* $f : \Re^2 \to \Re^3$ *, defined as* $f(x,y) = (-x^2+y^2, 2x^2+y^2-x, 2y)$ *and the polyhedral cone* $C = \{(\alpha,\beta,\gamma) : \alpha+\beta \geq 0, \ \beta+\gamma \geq 0, \ \alpha+2\beta \geq 0, \ -\alpha+\beta+\gamma \geq 0\}$.
The linear transformation associated to the cone C *is :*

$$T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

it results $(T \circ f)(x,y) = (x^2+2y^2-x, 2x^2+y^2-x+2y, 3x^2+3y^2-2x, 3x^2-x+2y)$. *Since any component of* $(T \circ f)(x,y)$ *is a convex function, the function* f *is* C*-convex. In order to calculate* $\partial_C f(x,y)$*, we consider the following Jordan canonical form of* T *:*

$$\overline{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Since $p = n = 3$*, from Corollary 4.1, we have*

$$\partial_C f(x,y) = \{Jf(x,y) = \begin{pmatrix} -2x & 2y \\ 4x-1 & 2y \\ 0 & 2 \end{pmatrix}\} \ .$$

References

[1] A. Cambini and L. Martein, Generalized concavity and optimality conditions in vector and scalar optimization, in "Generalized convexity", eds. Komlosi S. et als., Lecture Notes in Economics and Mathematical Systems, 405, Springer-Verlag, Berlin, 1994.

[2] A. Cambini and L. Martein, Generalized concavity in multiobjective programming, in "Generalized convexity, Generalized motonicity", eds. Crouzex et als., Nonconvex Optimization and its applications, 27, Kluwer Academic Publishers, 1998.

[3] A. Cambini, D.T. Luc and L. Martein, Order Preserving Transformations and Applications, Technical report n. 182, Department of Statistics and Applied Mathematics, University of Pisa.

[4] R. Cambini, Some new classes of generalized concave vector valued functions, Optimization 36 (1996), 11-24.

[5] F. Clarke, Optimization and Nonsmooth Analysis, John Wiley and Sons, New York, 1983.

[6] A. Guerraggio,E. Molho and A. Zaffaroni, On the notion of proper efficiency in vector optimization, J. of Optimization Theory and Applications, 82 (1994), 1-21.

[7] J.Jahn, Mathematical Vector Optimization in Partially ordered Linear Space, Verlag Peter Lang, 1986.

[8] J. Jahn and E. Sach, Generalized quasiconvex mappings and vector optimization, SIAM journal on Control and Otimization, 24 (1986), 306-322.

[9] M. I. Henig, Proper efficiency with respect to cones, J. of Optimization Thery and Applications, 36 (1982), 387-407.

[10] P. Q. Khaul, Proper solutions in vector optimization problems, J. of Optimization Thery and Applications, 74 (1992), 105-130.

[11] D. T. Luc, Theory of Vector Optimization, LEMS 319, Springer-Verlag, 1989.

[12] D. T. Luc, On the properly efficient points of nonconvex sets, European Journal of Operational Research, 86 (1995), 332-336.

[13] D. T. Luc, N. X. Tan and P. N. Truh, Convex vector functions and their subdifferential, Acta Mathematica Vietnamica, 23 (1998), 107-127.

[14] D. T. Luc and J. P. Penot, Convergence of asymptotic directions, Preprint, 1999.

[15] A. L. Peressini, Ordered Topological Vector Spaces, Harper and Row Pubblication, New York, 1967.

[16] R. T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, New Jersey, 1970.

[17] Y. Sawaragi, H. Nakayama and T. Tanino, Theory of Multiobjective Optimization, Academic Press, 1985.

[18] P. N. Tinh, D. T. Luc and N. X. Tan, Subdifferential characterization of quasiconvex and convex vector functions, Vietnam Journal of Mathematics, 26 (1998), 53-69.