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Pseudolinearity in scalar and vector optimization

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Abstract

Starting from scalar pseudolinearity revisited, we will present an approach to pseudolinearity in vector optimization with respect to a closed convex and pointed cone; such an approach allow to find new results and to generalize the ones obtained requiring componentwise pseudolinearity.
Key Words Efficient points, generalized scalar and vector convex functions, pseudolinearity.
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1 Introduction

2 Quasilinear and pseudolinear scalar functions revisited

The aim of this chapter is to present the main properties related to optimization of quasilinear and pseudolinear scalar functions. These properties are very important since its allow to suggest sequential methods of simplex-like type for wide classes of optimization problems [22, 27]. For such a reason, revisiting the results appear in the literature, we will suggest an autonomous treatment of this subject.
Let $f : X \to \Re$ be a function defined on an open set X of $\Re^n$ and let S be a convex subset of X. We recall the following classic definitions:

Definition 2.1 *The function f is quasiconvex if*

$$f(x_2) \leq f(x_1) \Rightarrow f((1-\lambda)x_1 + \lambda x_2) \leq f(x_1) \tag{2.1}$$

for every $x_1, x_2 \in S$ and $0 \leq \lambda \leq 1$.

Definition 2.2 *The function f is semistrictly quasiconvex if*

$$f(x_2) < f(x_1) \Rightarrow f((1-\lambda)x_1 + \lambda x_2) < f(x_1) \tag{2.2}$$

for every $x_1, x_2 \in S$ and $0 < \lambda < 1$.

Definition 2.3 *The differentiable function f is pseudoconvex if*

$$x_1, x_2 \in S, f(x_2) < f(x_1) \Rightarrow (x_2 - x_1)^T \nabla f(x_1) < 0. \tag{2.3}$$

We recall that, in the differentiable case, a pseudoconvex function is a semistrictly quasiconvex function, that, in turn, is quasiconvex.
The function f is quasiconcave, semistrictly quasiconcave or pseudoconcave if and only if the function $-f$ is quasiconvex, semistrictly quasiconvex or pseudoconvex, respectively.
The function f is said to be quasilinear (pseudolinear) if it is both quasiconvex and quasiconcave (pseudoconvex and pseudoconcave).
The pseudolinear functions have some properties stated in [12, 17, 24] for which we propose simple proofs.

Theorem 2.1 *Let f be a function defined on an open convex set $S \subset \Re^n$.*
i) If f is pseudolinear and there exists $x_0 \in S$ such that $\nabla f(x_0) = 0$, then f is constant on S.
ii) f is pseudolinear if and only if

$$x, y \in S, \;\; f(x) = f(y) \Longleftrightarrow (y-x)^T \nabla f(x) = 0 \tag{2.4}$$

iii) Assume $\nabla f(x) \neq 0 \; \forall x \in S$. Then f is pseudolinear on S if and only if its normalized gradient mapping $x \to \frac{\nabla f(x)}{\|\nabla f(x)\|}$ is constant on each level set $f(x)$ =constant.

Proof i) It follows taking into account that a stationary point is a global minimum point for a pseudoconvex function and a global maximum point for a pseudoconcave function.
ii) Let f be pseudolinear. Since f is also quasilinear, then $f(x) = f(y)$ implies that f is constant on the line-segment $[x, y]$, so that the directional derivative $(y-x)^T \nabla f(x)$ is equal to zero.

Assume now $(y-x)^T\nabla f(x) = 0$. Since $f(y) < f(x)$ $(f(y) > f(x))$ implies $(y-x)^T\nabla f(x) < 0$ $((y-x)^T\nabla f(x) > 0)$, necessarily we have $f(x) = f(y)$.
Assume that (2.4) holds; we must prove that f is both pseudoconvex and pseudoconcave. If f is not pseudoconvex, there exist $x, y \in S$ with $f(y) < f(x)$ such that $(y-x)^T\nabla f(x) \geq 0$. Since $(y-x)^T\nabla f(x) = 0$ implies $f(x) = f(y)$, we must have $(y-x)^T\nabla f(x) > 0$, so that the direction $d = y - x$ is an increasing direction at x. The continuity of the function f implies the existence of $x^* = x+t^*(y-x), t^* \in]0,1[$ such that $f(x^*) = f(x)$. Consequently $(x^*-x)^T\nabla f(x) = t^*(y-x)^T\nabla f(x) > 0$ and this contradicts (2.4). It follows that f is pseudoconvex. In an analogous way it can be proven that f is pseudoconcave.
iii) Let f be pseudolinear with $\nabla f(x) \neq 0$ $\forall x \in S$. We must prove that

$$f(x) = f(y) \Rightarrow \frac{\nabla f(x)}{\parallel \nabla f(x) \parallel} = \frac{\nabla f(y)}{\parallel \nabla f(y) \parallel} \tag{2.5}$$

Set $\Gamma_1 = \{d \in \Re^n : d^T\nabla f(x) = 0\}$, $\Gamma_2 = \{d \in \Re^n : d^T\nabla f(y) = 0\}$.
We have $\Gamma_1 = \Gamma_2$. Indeed, if $d \in \Gamma_1$, from ii) it results $f(x+td) = f(x) = f(y)$ for every t such that $x + td \in S$. From ii), it follows $(y-x-td)^T\nabla f(y) = 0$ and $(y-x)^T\nabla f(y) = 0$ so that $d^T\nabla f(y) = 0$ and thus $d \in \Gamma_2$. In an analogous way we can prove that $\Gamma_2 \subset \Gamma_1$.
Since $\Gamma_1 = \Gamma_2$, it results $\frac{\nabla f(x)}{\|\nabla f(x)\|} = \pm\frac{\nabla f(y)}{\|\nabla f(y)\|}$. Set $u = \frac{\nabla f(y)}{\|\nabla f(y)\|}$ and assume that $\frac{\nabla f(x)}{\|\nabla f(x)\|} = -u$; for a suitable $t \in]0, \epsilon[$ the points $z_1 = x + tu$, $z_2 = y + tu$ are such that $f(z_1) < f(x)$, $f(z_2) > f(y)$. The continuity of f implies the existence of $\lambda \in]0,1[$ such that $f(z) = f(x) = f(y)$ with $z = \lambda z_1 + (1-\lambda)z_2$. From ii) we must have $(z-y)^T u = 0$; on the other hand $(z-y)^T u = (\lambda(x-y) + (1-\lambda)tu)^T u = (1-\lambda)t \parallel u \parallel^2 > 0$ so that from ii), $f(y) \neq f(z)$ and this is absurd.
Consequently we have $\frac{\nabla f(x)}{\|\nabla f(x)\|} = \frac{\nabla f(y)}{\|\nabla f(y)\|}$.
Assume now that (2.5) holds. Let $x, y \in S$ and set $\phi(t) = f(x+t(y-x)), t \in [0,1]$. If $\phi'(t)$ is constant in sign, then $\phi(t)$ is quasilinear on the line segment $[0,1]$. Otherwise, from elementary analysis, there exist $t_1, t_2 \in]0,1[$ such that $\phi(t_1) = \phi(t_2)$ with $\phi'(t_1)\phi'(t_2) < 0$. We can assume, without loss of generality, that $t_1 < t_2$, $\phi'(t_1) > 0$, $\phi'(t_2) < 0$. Set $z_1 = x + t_1(y-x), z_2 = x + t_2(y-x)$. Since $f(z_1) = \phi(t_1) = \phi(t_2) = f(z_2)$, we have $\phi'(t_1) = (1-t_1)(y-x)^T\nabla f(z_1) > 0$ and $0 > \phi'(t_2) = (1-t_2)(y-x)^T\nabla f(z_2) = (1-t_2)(y-x)^T\nabla f(z_1)\frac{\|\nabla f(z_2)\|}{\|\nabla f(z_1)\|} > 0$ and this is absurd.
It follows that the restriction of the function over every line-segment contained in S is quasilinear, so that f is quasilinear and also pseudolinear since $\nabla f(x) \neq 0, \forall x \in S$. □

Remark 2.1 *Following the same lines of the proof given in ii) of the previuos theorem, it can be shown that (2.4) is equivalent to the following two statements:*

$$x, y \in S, \ \ f(x) > f(y) \Longleftrightarrow (y-x)^T \nabla f(x) > 0 \tag{2.6}$$

$$x, y \in S, \ \ f(x) < f(y) \Longleftrightarrow (y-x)^T \nabla f(x) < 0 \tag{2.7}$$

which point out that pseudolinearity is equivalent to require that the logical implication in the definition of pseudoconvex (pseudoconcave) function can be reversed.

Remark 2.2 *i) and ii) of Theorem 2.1 do not hold if f is quasilinear, as it is easy to verify considering the function $f(x) = x^3$.*

Let us note that i) and ii) of Theorem 2.1 hold even if S is a relatively open convex set, while in iii) the assumption $intS \neq \emptyset$ cannot be weakened, as it is shown in the following example.

Example 2.1 *Consider the function $f(x,y,z) = xy + xz + \frac{x+y+z}{x-y+z}$ defined on the relatively open convex set $D = \{(x,y,z) : x - y + z > 0\}$*
Consider the convex set $S = \{(x,y,z) : x = 0, y = 0, z > 0\}$; obviously $intS = \emptyset$, while $riS \neq \emptyset$.
By simple calculation, it results $\nabla f(0,0,z) = (z, \frac{2}{z}, 0)$. Let $A = (0,0,1)$, $B = (0,0,2)$, we have $f(A) = f(B) = 1, \nabla f(A) = (1,2,0), \nabla f(B) = (2,1,0)$ so that $\frac{\nabla f(A)}{\|\nabla f(A)\|} \neq \frac{\nabla f(B)}{\|\nabla f(B)\|}$.

Condition iii) of Theorem 2.1 can be streightened when the function f is defined on the whole space $\Re^n$, in the sense stated in the following theorem.

Theorem 2.2 *The non constant function f is pseudolinear on the whole space $\Re^n$ if and only if its normalized gradient mapping $x \longrightarrow \frac{\nabla f(x)}{\|\nabla f(x)\|}$ is constant on $\Re^n$.*

Proof $\Leftarrow$ It follows from iii) of Theorem 2.1.
$\Rightarrow$ Let f be pseudolinear on $\Re^n$ and assume that its normalized gradient mapping is not constant on $\Re^n$. Then there exist $x_1, x_2 \in \Re^n$ such that $\frac{\nabla f(x_1)}{\|\nabla f(x_1)\|} \neq \frac{\nabla f(x_2)}{\|\nabla f(x_2)\|}$. From iii) of Theorem 2.1, we have $f(x_1) \neq f(x_2)$. Set $\Gamma_1 = \{d \in \Re^n : d^T \nabla f(x_1) = 0\}$ and $\Gamma_2 = \{d \in \Re^n : d^T \nabla f(x_2) = 0\}$. Let us note that $x \in x_1 + \Gamma_1$ implies $(x - x_1) \in \Gamma_1$ so that $(x - x_1)^T \nabla f(x_1) = 0$ and, from ii) of Theorem 2.1, $f(x) = f(x_1), \forall x \in x_1 + \Gamma_1$. Analogously we have $f(x) = f(x_2), \forall x \in x_2 + \Gamma_2$. Since $\frac{\nabla f(x_1)}{\|\nabla f(x_1)\|} \neq \frac{\nabla f(x_2)}{\|\nabla f(x_2)\|}$, there exists $\bar{x} \in (x_1 + \Gamma_1) \cap (x_2 + \Gamma_2)$, so that $f(\bar{x}) = f(x_1), f(\bar{x}) = f(x_2)$ and this is

absurd. □

From a geometrical point of wiev, the previous theorem states that the level sets of a non constant pseudolinear function, defined on the whole space $\Re^n$ are parallel hyperplanes; viceversa if the level sets of a differentiable function, with no critical points, are hyperplanes, then the function is pseudolinear.

In any case, if the level sets of a function are hyperplanes, then the function is quasilinear, but the viceversa is not true. In fact the function $\phi(x, y) = f(x)$, where

$$f(x) = \begin{cases} -x^2 & x \in [-1, 0] \\ 0 & x \in]0, 2[\\ (x-2)^2 & x \in [2, 3] \end{cases} \tag{2.8}$$

is quasilinear and the level set $\{(x, y) : \phi(x, y) = 0\} = [0, 2] \times \Re$ is not an hyperplane.

When the non constant pseudolinear function is defined on a convex set $S \subset \Re^n$, from iii) of Theorem 2.1, the level sets are the intersection between S and hyperplanes which are not necessarily parallel (consider for instance the classic case of linear fractional functions).

The above considerations suggest a simple way to construct a pseudolinear function. Consider for instance the family of lines $y = \frac{kx+1}{\sqrt{k+1}}$. It is easy to verify that such lines are the level sets of the function

$$f(x, y) = \frac{-2x + y^2 + y\sqrt{y^2 - 4x + 4x^2}}{2x^2}$$

defined on the set $S = \{(x, y) : x > 1, y > 0\}$; since $\nabla f(x, y) \neq 0 \ \forall (x, y) \in S$, $f(x, y)$ is pseudolinear on S.

Another way to construct a pseudolinear function is to consider a composite function $\phi \circ f$ where f is pseudolinear and $\phi : \Re \rightarrow \Re$ is a differentiable function having a strictly positive (or negative) derivative.

Now we will point out that for a pseudoconvex function, a global maximum or minimum point, if one exists, is attained on the boundary of the feasible region S and, under suitable assumptions on S, it is an extreme point of S. We will begin to prove that if the maximum value of a semistrictly quasiconvex function is reached at a relative interior point of S, then f is constant on S.

We recall that the relative interior of a convex set $C \subset \Re^n$, denoted by riC, is defined as the interior which results when C is regarded as a subset of its affine hull $affC$. In other words,

$$riC = \{x \in affC : \exists \epsilon > 0, (x + \epsilon B) \cap affC \subset C\}$$

where B is the Euclidean unit ball in $\Re^n$.

Lemma 2.1 *Let f be a continuous and semistrictly quasiconvex function on a convex set S. If $x_0 \in riS$ is such that $f(x_0) = max_{x \in S} f(x)$, then f is constant on S.*

Proof Assume that there exists $\bar{x} \in S$ such that $f(\bar{x}) < f(x_0)$. For a known property of convex sets [25], there exists $x^* \in S$ such that $x_0 \in]x^*, \bar{x}[$. Since f is a continuous function, without loss of generality, we can assume that $f(x^*) > f(\bar{x})$. The semistrictly quasiconvexity of f implies $f(x) < f(x^*)$, $\forall x \in]x^*, \bar{x}[$ and this is absurd since $x_0 \in]x^*, \bar{x}[$. □

From Lemma 2.1, we have directly the following result.

Theorem 2.3 *Let f be a continuous and semistrictly quasiconvex function on a convex and closed set S. If f assumes maximum value on S, then it is reached on some boundary point.*

The previous theorem can be strenghtened when the convex set S does not contain lines (such an assumption implies the existence of an extreme point [25]).

Theorem 2.4 *Let f be a continuous and semistrictly quasiconvex function on a convex and closed set S containing no lines. If f assumes maximum value on S, then it is reached on an extreme point.*

Proof If f is constant, then the thesis is trivial. Let x_0 be such that $f(x_0) = max_{x \in S} f(x)$. From Theorem 2.3, x_0 belongs to the boundary of S. Let C be the minimal face of S containing x_0; if x_0 is not an extreme point, then $x_0 \in riC$. It follows from Lemma 2.1 that f is constant on C. On the other hand, C is a convex closed set containing no lines, so that C has at least one extreme point $\bar{x}$ which is also an extreme point of S [25]. Consequently $\bar{x}$ is a global maximum for f on S. □

Taking into account that a pseudolinear function is both semistrictly quasiconvex and semistrictly quasiconcave, we have the following corollary:

Corollary 2.1 *Let f be a pseudoconvex function defined on a convex and closed set S containing no lines. If f assumes maximum or minimum value on S, then it is reached on an extreme point.*

Let us note that a quasiconvex function can have a global maximum point which is not a boundary point. In fact the function

$$f(x) = \begin{cases} -x^2 + 2x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

is nondecreasing, so that it is quasiconvex; on the other hand any point $x > 1$ where f assumes its maximum value is not a boundary point.
If we want to extend Theorem 2.1 to the class of quasiconvex functions, we must require additional assumptions on the convex set S.

Theorem 2.5 *Let f be a continuous and quasiconvex function on a convex and compact set S. Then there exists some extreme point on which f assumes its maximum value.*

Proof From Weierstrass Theorem, there exists $\bar{x} \in S$ with $f(\bar{x}) = max_{x \in S} f(x)$. Since S is convex and compact, it is also the convex hull of its extreme points, so that there exists a finite number $x^1, ..., x^h$ of extreme points such that $\bar{x} = \sum_{i=1}^{h} \lambda_i x_i$, $\sum_{i=1}^{h} \lambda_i = 1$, $\lambda_i \geq 0$. From the quasiconvexity of f we have $f(\bar{x}) \leq max\{f(x^1), ..., f(x^h)\}$ and the thesis follows. □

Corollary 2.2 *Let f be a pseudoconvex function defined on a convex and compact set S. Then there exists some extreme point on which f assumes its maximum and minimum value.*

Remark 2.3 *From a computational point of view, Theorem 2.2 is very important since it establish that we must investigate the boundary of the feasible set (in particular the extreme points if one exists) in order to find a global maximum or minimum of a pseudoconvex function. In particular, for a pseudolinear objective function defined on a polyhedral set S, we have the property that when the maximum and the minimum value exist, they are reached on a vertex of S. This nice property have suggested some simplex-like procedure for pseudolinear problems.. These programs include linear programs and linear fractional programs which arise in many practical applications [13, 27]. Algorithms for a linear fractional problem have been suggested by several authors [2, 11, 22]. Computational comparisons between algorithms for linear fractional programming are given in [14].*

3 First order optimality conditions in vector optimization

In order to suggest an approach which extends to multiobjective functions the results given in Section 2, we need of some preliminary results regarding separation theorems and first order optimality conditions.
With this aim let $U_1 \subset \Re^t$, $U_2 \subset \Re^k$ closed convex pointed cones with nonempty interiors and let $W_1 \subset \Re^t$, $W_2 \subset \Re^k$ linear subspaces. We recall,

first of all, the concept of base of a cone.
A base of a cone K is a set B satisfying the following properties a), b), c):
a) $0 \notin B$;
b) for each $x \in K$, $x \neq 0$, there are unique $b \in B$, $\mu > 0$ such that $x = \mu b$;
c) $K = \{x = \mu b : b \in B, \mu \geq 0\}$.
In finite dimensional space, a closed, convex and pointed cone has a closed convex bounded base [19] . A simple proof of this property is given in the following Lemma, where K^* denotes the positive polar of K.

Lemma 3.1 *Let K be a closed, convex and pointed cone of some finite dimensional space. Let $\beta \in intK^*$ and let Γ be a hyperplane of equation $\beta^T x = \beta_0$, $\beta_0 > 0$.*
Then $B = \Gamma \cap K$ is a compact convex base of K.

Proof Obviously $0 \notin B$ and since Γ and K are closed and convex sets, B is closed and convex too.
For each $x \in K$, the halfline of equation $z = \mu x, \mu > 0$ meet B in the unique point $b = \frac{\beta_0}{\beta^T x} x$ with $\mu = \frac{\beta_0}{\beta^T x}$ (let us note that $intK^* \neq \emptyset$ since K is closed and pointed and $\beta \in intK^*$ implies $\beta^T x > 0$). It remains to prove that B is bounded. If not, there exists a sequence $\{b_n\} \subset B$ with $\| b_n \| \to +\infty$. It results $\beta^T b_n = \beta_0$ and thus $\beta^T \frac{b_n}{\|b_n\|} = \frac{\beta_0}{\|b_n\|}$ so that there exists a subsequence of $\{\frac{b_n}{\|b_n\|}\}$, which we can redenominate in the same way, converging to an element $b^* \in B = K \cap \Gamma$. Consequently, we have $\beta^T b^* = 0$ and this is absurd since $b^* \in K$, $\beta \in intK^*$ implies $\beta^T b^* > 0$. □

Now we are able to state the following separations theorems, where int() denotes the interior of ().

Theorem 3.1 *The following properties hold:*
i) $W_1 \cap intU_1 = \emptyset$ if and only if

$$\exists \lambda \in U_1^* \setminus \{0\} : \lambda^T w = 0, \ \ \forall w \in W. \tag{3.9}$$

ii) $W_1 \cap U_1 = \{0\}$ if and only if

$$\exists \lambda \in intU_1^* : \lambda^T w = 0, \ \ \forall w \in W. \tag{3.10}$$

Proof i) It is a well-known result.
ii) Let B be a base of U_1 existing from Lemma3.1. If $W_1 \cap U_1 = \{0\}$ then W_1, B are non-empty disjoint closed convex sets having no common directions of recessions so that there exists a hyperplane separating W_1 and B strongly [25], that is there exists $\lambda \in \Re^t$, $\lambda_0 > 0$ such that $\lambda^T w < \lambda_0$ $\forall w \in W_1$,

$\lambda^T b > \lambda_0\ \forall b \in B$. This last inequality implies $\lambda^T u > \lambda_0 > 0\ \forall u \in U_1$, so that $\lambda \in intU_1^*$. It remains to prove that $\lambda^T w = 0,\ \forall w \in W_1$. Since W_1 is a linear subspace, the relation $\lambda^T w < \lambda_0\ \forall w \in W_1$, implies $\lambda^T kw = k\lambda^T w < \lambda_0\ \forall k \in \Re$, so that necessarily we have $\lambda^T w = 0\ \forall w \in W_1$.
Assume now that 3.10 holds. The existence of $u \in W_1 \cap U_1,\ u \neq 0$, implies $\lambda^T u > 0$ and this is absurd.

□

Theorem 3.2 *Set $W = W_1 \times W_2$. The following properties hold:*
i) $(W_1 \times W_2) \cap (intU_1 \times intU_2) = \emptyset$ if and only if

$$\exists \lambda_1 \in U_1^*, \exists \lambda_2 \in U_2^*, (\lambda_1, \lambda_2) \neq 0 : \lambda_1^T w_1 + \lambda_2^T w_2 = 0,\quad \forall (w_1, w_2) \in W. \quad (3.11)$$

ii) $(W_1 \times W_2) \cap (intU_1 \times U_2) = \emptyset$ if and only if

$$\exists \lambda_1 \in U_1^* \setminus \{0\}, \exists \lambda_2 \in U_2^* : \lambda_1^T w_1 + \lambda_2^T w_2 = 0\ \forall (w_1, w_2) \in W. \quad (3.12)$$

iii) $(W_1 \times W_2) \cap ((U_1 \setminus \{0\}) \times U_2) = \emptyset$ if and only if

$$\exists \lambda_1 \in intU_1^*, \exists \lambda_2 \in U_2^* : \lambda_1^T w_1 + \lambda_2^T w_2 = 0\ \forall (w_1, w_2) \in W. \quad (3.13)$$

Proof i) It is a well-known result.
ii), iii). Obviously 3.12, 3.13 imply the thesis.
If $(W_1 \times W_2) \cap (intU_1 \times U_2) = \emptyset\quad ((W_1 \times W_2) \cap ((U_1 \setminus \{0\}) \times U_2) = \emptyset)$, necessarily we have $W_1 \cap intU_1 = \emptyset\quad (W_1 \cap (U_1 \setminus \{0\}) = \emptyset$ otherwise $u \in W_1 \cap intU_1\quad (u \in W_1 \cap (U_1 \setminus \{0\}))$ implies $(u, 0) \in (W_1 \times W_2) \cap (intU_1 \times U_2)$ $((u, 0) \in (W_1 \times W_2) \cap ((U_1 \setminus \{0\}) \times U_2)$ and this is absurd.
From 3.9 $\exists \lambda_1 \in U_1^* \setminus \{0\} : \lambda_1^T w = 0,\ \forall w \in W_1$ (from 3.10 $\exists \lambda_1 \in intU_1^* : \lambda_1^T w = 0,\ \forall w \in W_1$).
If $U_2 \subset W_2$, then any hyperplane of equation $\lambda_2^T w = 0$ containing W_2 is such that $\lambda_2^T u = 0\ \forall u \in U_2$, so that $\lambda_2 \in U_2^*$. If $U_2 \not\subset W_2$, then $intU_2 \cap W_2 = \emptyset$, so that from 3.9 $\exists \lambda_2 \in U_2^* \setminus \{0\} : \lambda_2^T w = 0,\ \forall w \in W_2$. In any case it follows that (λ_1, λ_2) satisfies 3.12 (3.13).

□

Consider now a closed, convex and pointed cone with a nonempty interior $C \subset \Re^s$ and set $C^0 = C \setminus \{0\}$.
The cone C induces a partial order relation in $\Re^s$ which allow to extends the definitions of a maximum or minimum point for a vector function.
More exactly, let $F : X \to \Re^s$ be a function defined on the open set X of $\Re^n$ and consider the following vector optimization problems:

$$P_{min} : minF(x), x \in S \subset X$$

$$P_{max} : maxF(x), x \in S \subset X$$

A point $x_0 \in S$ is said to be :
- weakly efficient for P_{min} if $F(x) \notin F(x_0) - intC, \ \forall x \in S$
- efficient for P_{min} if $F(x) \notin F(x_0) - C^0, \ \forall x \in S$.
- weakly efficient for P_{max} if $F(x) \notin F(x_0) + intC, \ \forall x \in S$
- efficient for P_{max} if $F(x) \notin F(x_0) + C^0, \ \forall x \in S$.

If the previous conditions are verified in $I \cap S$, where I is a suitable neighbourhood of x_0, then x_0 is said to be a local weak efficient point or a local efficient point, respectively.
In the scalar case (s=1) a (local) weak efficient point and an (local) efficient point reduce to the ordinary definition of a (local) minimum or maximum point.
When C is the Paretian cone

$$C = \Re^s_+ = \{z = (z_1, ..., z_s) \in \Re^s : z_i \geq 0, i = 1, ..., s\}$$

an efficient point is usually referred to a Pareto point. Obviously (local) efficiency implies (local) weak efficiency.
From now on, we will assume that F is differentiable and that S is described by constraint functions, that is
$S = \{x \in X : G(x) \in -V\}$ in problem P_{min}
$S = \{x \in X : G(x) \in V\}$ in problem P_{max}.
where $G : X \to \Re^m$ is a differentiable function and V is a closed convex pointed cone of $\Re^m$ with nonempty interior.
We will denote with $F'(u)$ and $G'(u)$ the Jacobian matrices of F and G, respectively, evaluated at $u \in X$.
The following theorems hold:

Theorem 3.3 *If x_0 is an interior local weak efficient point for P_{min} or P_{max}, then*

$$\exists \alpha \in C^* \setminus \{0\} \ \textit{such that} \ \alpha^T F'(x_0) = 0 \tag{3.14}$$

Proof Consider the line-segment $x = x_0 + td, t \in [0, \epsilon[, d \in \Re^n$. The local weak efficiency of x_0 with respect to P_{min} implies $\frac{F(x_0+td)-F(x_0)}{t} \notin -intC$, so that $F'(x_0)d = \lim_{t \to 0^+} \frac{F(x_0+td)-F(x_0)}{t} \notin -intC$. Setting $W = \{F'(x_0)d, d \in \Re^n\}$, it results $W \cap (-intC) = \emptyset$; the thesis follows from i) of Theorem 3.1. The proof is similar when x_0 is a weak efficient point for P_{max}. □

Theorem 3.4 *If $x_0 \in S$ with $G(x_0) = 0$ is a weak efficient point for for P_{min} or P_{max}, then*

$$\exists \ \alpha \in C^*, \exists \ \beta \in V^*, (\alpha, \beta) \neq 0 : \alpha^T F'(x_0) + \beta^T G'(x_0) = 0 \tag{3.15}$$

Proof Let us note that $G'(x_0)d \in -intV$ implies that d is a feasible direction of S at $x_0 \in S$. If x_0 is a weak efficient point with respect to P_{min}, $F'(x_0)d \notin -intC\ \forall d:\ G'(x_0)d \in -intV$ or, equivalently, $(F'(x_0)d, G'(x_0)d) \notin (-intC)\times(-intV)\ \forall d \in \Re^n$. Setting $W_1 = \{F'(x_0)d,\ d \in \Re^n\}$, $W_2 = \{G'(x_0)d,\ d \in \Re^n\}$, we have $W_1 \times W_2 \cap (-intC) \times (-intV) = \emptyset$. The thesis follows from i) of Theorem 3.2.The proof is similar when x_0 is a weak efficient point for P_{max}. □

4 Pseudolinearity in vector optimization

In order to suggest an approach which generalizes scalar pseudolinearity, we will introduce the concepts of quasiconvexity (quasiconcavity) and pseudoconvexity (pseudoconcavity) for a multiobjective function.
As is known, there are different ways in extending the definitions of generalized convex functions to the vector case; we will address to the following two classes of vector pseudoconvex functions which reduce, when $s = 1$, to the classical definition given by Mangasarian [21].

Definition 4.1 *The function F is said to be (C^0, C^0)-pseudoconvex (on S) if*

$$x_1, x_2 \in S, F(x_2) \in F(x_1) - C^0 \Rightarrow F'(x_1)(x_2 - x_1) \in -C^0$$

Definition 4.2 *The function F is said to be $(intC, intC)$-pseudoconvex (on S) if*

$$x_1, x_2 \in S, F(x_2) \in F(x_1) - intC \Rightarrow F'(x_1)(x_2 - x_1) \in -intC$$

Definition 4.3 *The function F is said to be (C^0, C^0)-pseudoconcave (on S) if*

$$x_1, x_2 \in S, F(x_2) \in F(x_1) + C^0 \Rightarrow F'(x_1)(x_2 - x_1) \in C^0$$

Definition 4.4 *The function F is said to be $(intC, intC)$-pseudoconcave (on S) if*

$$x_1, x_2 \in S, F(x_2) \in F(x_1) + intC \Rightarrow F'(x_1)(x_2 - x_1) \in intC$$

Obviously, F is pseudoconcave iff $-F$ is pseudoconvex, so that any results related to pseudoconvexity can easily translated in a result related to pseudoconcavity. In what follows we will refer mainly to pseudoconvexity.
When C is the Paretian cone, we have:
- if any component of F is pseudoconvex, then F is (intC, intC)- pseudoconvex and also (C^0, C^0)-pseudoconvex;

- if any component of F is quasiconvex and at least one is strictly pseudoconvex, then F is (C^0, C^0)-pseudoconvex.
The converse of these statements are not true in general, as it is shown in the following example.

Example 4.1 *The function $F(x,y) = (x, -x, -x^2, y)$ is (intC,intC)-pseudoconvex and (C^0, C^0)-pseudoconvex on $S = \Re^2$, with $C = \Re^4_+$, but it is not componentwise quasiconvex or pseudoconvex.*
The function $F(x) = (x, -x, -x^2)$ is $(C^0, intC)$-pseudoconvex on $S = \Re$, with $C = \Re^3_+$, but its components are not strictly pseudoconvex.

The following example point out that there are not inclusion relationships between (C^0, C^0)-pseudoconvexity and (intC, intC)-pseudoconvexity.

Example 4.2 *Consider the function $F(x) = (-x^2, x^2(x-1)^3)$, $S = \{x \in \Re, x \geq 0\}$, $C = \Re^2_+$.*
It is easy to prove that there do not exist $x_1, x_2 \in S$ such that $F(x_2) \in F(x_1) - intC$, so that F is (intC,intC)-pseudoconvex. On the other hand, setting $x_2 = 1, x_1 = 0$, we have $F(1) \in F(0) - C^0$, while $F'(0) \cdot 1 \notin -C^0$ and thus F is not (C^0, C^0)-pseudoconvex.
Consider now the function $F(x) = (-x, -x^2)$, $S = \{x \in \Re, x \geq 0\}$, $C = \Re^2_+$. Setting $x_2 = 1, x_1 = 0$, we have $F(1) \in F(0) - intC$ and $F'(0) \cdot 1 \notin -intC$, so that F is not (intC,intC)-pseudoconvex, while simple calculations show that F is (C^0, C^0)-pseudoconvex.

Conditions (2.6) and (2.7) suggest to define pseudolinearity with respect to a cone requiring that the logical implication in the definitions of (C^0, C^0)-pseudoconvexity ($(intC, intC)$-pseudoconvexity) and (C^0, C^0)-pseudoconcavity ($(intC, intC)$-pseudoconcavity) can be reversed.

Definition 4.5 *The function F is (C^0, C^0)-pseudolinear (on S) if the following two statements hold:*

$$x_1, x_2 \in S, \quad F(x_2) \in F(x_1) - C^0 \Leftrightarrow F'(x_1)(x_2 - x_1) \in -C^0 \tag{4.16}$$

$$x_1, x_2 \in S, \quad F(x_2) \in F(x_1) + C^0 \Leftrightarrow F'(x_1)(x_2 - x_1) \in C^0 \tag{4.17}$$

Definition 4.6 *The function F is $(intC, intC)$-pseudolinear (on S) if the following two statements hold:*

$$x_1, x_2 \in S, \quad F(x_2) \in F(x_1) - intC \Leftrightarrow F'(x_1)(x_2 - x_1) \in -intC \tag{4.18}$$

$$x_1, x_2 \in S, \quad F(x_2) \in F(x_1) + intC \Leftrightarrow F'(x_1)(x_2 - x_1) \in intC \tag{4.19}$$

Remark 4.1 *If C is the Paretian cone and F is componentwise pseudolinear, then F is (C^0, C^0)-pseudolinear; the converse is not true, as it can be easily verified considering the class of functions $F : \Re \to \Re^2$, $F(x) = (F_1(x), F_2(x))$, where F_1, F_2 are such that $F_1'(x) > 0 \ \forall x \in \Re, F_2'(x) \geq 0 \ \forall x \in \Re$ and there exist $\bar{x}, x^* \in \Re$ such that $F_2'(\bar{x}) > 0, F_2'(x^*) = 0$.*

For a $(intC, intC)$-pseudoconvex function, a local weak efficient point for P_{min} is also weakly efficient so that this property is maintained for a $(intC, intC)$-pseudolinear function. Now, we will point out that (C^0, C^0)-pseudolinearity implies that a local efficient point is efficient too, even if such a property does not hold for (C^0, C^0)-pseudoconvex functions.

Theorem 4.1 *Let F be (C^0, C^0)-pseudolinear (on S). If $x_0 \in S$ is a local efficient point for P, then x_0 is efficient for P.*

Proof Assume that there exists $\bar{x} \in S$ such that $F(\bar{x}) \in F(x_0) - C^0$. Then $F'(x_0)(\bar{x} - x_0) \in -C^0$. Consider the line-segment $]x_0, \bar{x}] = \{x = x_0 + t(\bar{x} - x_0), t \in]0, 1]\}$. It results $F'(x_0)(x - x_0) = tF'(x_0)(\bar{x} - x_0) \in -C^0$, so that for the pseudolinearity of F, $F(x) \in F(x_0) - C^0 \ \ \forall x \in]x_0, \bar{x}]$ and this contradicts the local efficiency of x_0. □

The efficiency of an interior point can be completed characterized for the classes of $(intC, intC)$-pseudolinear and (C^0, C^0)-pseudolinearfunctions by means of the following theorem:

Theorem 4.2 *The following properties hold:*
i) Let F be (C^0, C^0)-pseudolinear (on S) and let $x_0 \in S$ be an interior point of S. Then x_0 is an efficient point either for P_{min} or for P_{max} if and only if

$$\exists \alpha \in intC^* \ \ such \ that \ \ \alpha^T F'(x_0) = 0 \tag{4.20}$$

ii) Let F be $(intC, intC)$-pseudolinear (on S) and let $x_0 \in S$ be an interior point of S. Then x_0 is a weak efficient point either for P_{min} or for P_{max} if and only if

$$\exists \alpha \in C^* \setminus \{0\} \ \ such \ that \ \ \alpha^T F'(x_0) = 0 \tag{4.21}$$

Proof i) The pseudolinearity of F and the efficiency of x_0 with respect to P_{min}, imply $F'(x_0)(x - x_0) \notin -C^0, \ \ \forall x \neq x_0$. (4.20) follows from ii) of Theorem 3.1.

Assume now that (4.20 holds; if x_0 is not an efficient point for P_{min}, then there exists $\bar{x} \in S : F(\bar{x}) \in C^0$, so that we have $F'(x_0)(\bar{x} - x_0) \in C^0$ and this implies $\alpha^T F'(x_0)(\bar{x} - x_0) > 0$ and this is absurd.

The proof is similar when x_0 is a weak efficient point for P_{max}.

ii) The proof is similar to the one given in i). □

Corollary 4.1 *Let C the paretian cone and let F be a componentwise pseudolinear or an affine function (on S). Then $x_0 \in S$ is an interior efficient point if and only if*

$$\exists \alpha \in intC^* \text{ such that } \alpha^T F'(x_0) = 0.$$

When all the components F_i of F are pseudolinear on the whole space $\Re^n$, the existence of a stationary point implies that any point of $\Re^n$ is efficient as it is shown in the following theorem.

Theorem 4.3 *Let C the Paretian cone and let F be componentwise pseudolinear on $\Re^n$. If there exists an efficient point for F, then any point of $\Re^n$ is efficient.*

Proof Without loss of generality, we can suppose that $\nabla F_i(x) \neq 0, \ \forall i = 1, ..., s, \ \forall x \in \Re^n$; indeed if there exists z such that $\nabla F_j(z) = 0$, then F_j is a constant function and the efficiency of a point x_0 with respect to F is equivalent to the efficiency of x_0 with respect to the function $(F_1, ..., F_{j-1}, F_{j+1}, ..., F_s)$. Let x_0 be an efficient point for F; then from Corollary 4.1 $\exists \ \alpha^T = (\alpha_1, ..., \alpha_s) \in int\Re^s_+$ such that $\sum_{i=1}^s \alpha_i \nabla F_i(x_0) = 0$. From Theorem 2.2, we have $\nabla F_i(x_0) = \beta_i \nabla F_i(x)$ with $\beta_i = \frac{\|\nabla F_i(x_0)\|}{\|\nabla F_i(x)\|} > 0 \ \forall i$, so that $\sum_{i=1}^s \alpha_i \beta_i \nabla F_i(x) = 0 \ \forall x \in \Re^n$ and thus, taking into account Corollary 4.1, $x \in \Re^n$ is an efficient point. □

Let us note that if F is componentwise pseudolinear on a convex subset of $\Re^n$, the property stated in Theorem 4.3 does not hold even if there exist stationary points, as is shown in the following example.

Example 4.3 *Consider the function*

$$F(x_1, x_2) = (\frac{x_1 + x_2 - 1}{x_2 + 1}, \frac{-x_1 + x_2 - 3}{-x_2 + 3}),$$

$(x_1, x_2) \in S = \{(x_1, x_2) : 0 \leq x_1 \leq 2, 0 \leq x_2 \leq 2\}$.
F is (C^0, C^0)-pseudolinear on S since its components are pseudolinear on S. Consider the interior point $x_0 = (\frac{5}{4}, \frac{1}{2})$. It results $F'(x_0) = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{5} & -\frac{1}{5} \end{bmatrix}$, so that, setting $\alpha^T = (3, 5)$, we have $\alpha^T F'(x_0) = 0$, that is x_0 is a stationary point for F and, consequently, x_0 is an efficient point either with respect to C or with respect to -C, as it can be verified applying the definitions. On the other hand $x_0 = (0, 0) \in S$ is not an efficient point, so that Theorem 4.3 does not hold if we substitute $\Re^n$ with a subset S. Let us note that the line-segment $[A, B]$ with $A = (\frac{3}{2}, 0), B = (1, 1)$ is the set of all efficient points of F.

The following example points out that the classes of (C^0, C^0) and (intC,intC)-pseudolinear functions are not comparable as it happens for vector pseudoconvexity.

Example 4.4 *Consider the function*

$$F(x_1, x_2) = (x_1, -x_1, x_1 + x_2^3, -x_2), (x_1, x_2) \in \Re^2,$$

$C = \Re^4_+$. *It can be verified that F is (intC,intC)-pseudolinear. Obviously* $x_0 = (0,0)$ *is an efficient point for F, but* $\alpha^T F'(x_0) = 0$, *with* $\alpha^T = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, *holds if and only if* $\alpha_4 = 0, \alpha_1 - \alpha_2 + \alpha_3 = 0$, *so that it is not possible to have the positivity of all multipliers and this imply that F is not* (C^0, C^0)*-pseudolinear.*
Consider now the function $F(x) = (x^3 + x, x^3), x \in \Re, C = \Re^2_+$. *Such a function is* (C^0, C^0)*-pseudolinear but it is not (intC,intC)-pseudolinear.*

The following theorem states a necessary and sufficient optimality condition of the Kuhn-Tucker type.

Theorem 4.4 *Consider problem* P^*, *where F is* (C^0, C^0)*-pseudolinear, G is* (V^0, V^0)*-pseudolinear and such that*

$$G'(y)(x - y) = 0 \Leftrightarrow G(x) = G(y) \tag{4.22}$$

Then a feasible point x_0 *is efficient either for* P_{min} *or for* P_{max} *if and only if*

$$\exists\ \alpha \in int\Re^s_+, \exists\ \beta \in \Re^m_+ : \alpha^T F'(x_0) + \beta^T G'(x_0) = 0 \tag{4.23}$$

Proof **Necessity**. Consider the linear subspace

$$W = \{(F'(x_0)(x - x_0), G'(x_0)(x - x_0)), x \in \Re^n\}.$$

Now we will prove that $W \cap (-\Re^s_+ \setminus \{0\}) \times (-\Re^m_+)) = \emptyset$.
Assume that there exists $\bar{x} \in \Re^n$ such that

$$F'(x_0)(\bar{x} - x_0) \in -C^0 \tag{4.24}$$

$$G'(x_0)(\bar{x} - x_0) \in -V \tag{4.25}$$

Let us note that (4.24) and (4.25) hold for any point of the intersection between the convex set X and the line-segment $[x_0, \bar{x}]$, so that we can assume without loss of generality that $\bar{x} \in X$.
The pseudolinearity of G and property (4.22) imply the feasibility of $\bar{x}$. On the other hand, (4.24) contradicts the efficiency of x_0, so that $W \cap (-C^0) \times (-V)) = \emptyset$. From iii) of Theorem 3.2, we have (4.23).
Sufficiency. The proof is trivial.. □

Remark 4.2 *In the previous theorem, we have assumed that the constraint vector function G belongs to the subclass of (V^0, V^0)-pseudolinear functions verifying the property (4.22); this subclass contains the componentwise pseudolinear functions and the inclusion is proper since the function introduced in Remark 4.1 belongs to such a subclass.*

As a direct consequence of the previous theorem, we obtain the following result given by Chew-Choo [12].

Corollary 4.2 *Consider problem P^*, where the objective function and the constraints are componentwise pseudolinear. A point x_0 is an efficient solution of problem P^* if and only if*

$$\exists\ \alpha \in int\Re^s_+, \exists\ \beta \in \Re^m_+ : \alpha^T F'(x_0) + \beta^T G'(x_0) = 0$$

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