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Necessary Optimality Conditions in Vector Optimization

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Abstract

This paper deals with vector optimization problems having a vector valued objective function and three kinds of constraints: inequality constraints, equality constraints, and a set constraint (which covers the constraints which cannot be expressed by means of neither equalities nor inequalities). Necessary optimality conditions and sufficient ones are given in the image space for the nonsmooth case (when the continuity is not required), while necessary conditions in the image and in the decision spaces are given for the nondifferentiable case (when just the Hadamard directional differentiability is assumed). The new concept of U-regularity is introduced in order to study necessary optimality conditions in the decision space. Finally, the results are specialized under differentiability hypothesis, thus obtaining conditions generalizing the so called "maximum principle conditions".

Keywords Vector Optimization, Optimality Conditions, Image Space, Maximum Principle Conditions.
AMS - 2000 Math. Subj. Class. 90C29, 90C46, 90C30
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1 Introduction

The aim of this paper is to study optimality conditions for vector problems having a vector valued objective function and three kinds of constraints: inequality constraints, equality constraints, and a set constraint (which covers the constraints which cannot be expressed by means of neither equalities nor inequalities). The partial ordering in the image of the objective function is given by a closed convex pointed cone C with nonempty interior

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(that is a solid cone, not necessarily the Paretian one), while the inequality constraints are expressed by means of a partial ordering given by a closed convex pointed cone V with nonempty interior.

Problems of this kind have been studied in the literature in finite dimensional spaces with a scalar objective function and under differentiability hypothesis ([1]), obtaining (with some additional hypothesis) necessary optimality conditions of the so called "maximum/minimum principle" type (also called "generalized Lagrange multiplier rule") [3, 23, 21]; these optimality conditions are stated in the decision space, that is to say that they are based on the use of derivatives and multipliers.

The aim of this paper is twofold; first it is to state some optimality conditions by means of the so called image space approach [5, 6, 7, 8, 9, 10, 11], first suggested in [20], then it is to generalize the minimum/maximum principle conditions to multiobjective problems having nondifferentiable functions.

In particular, in Section 3 a characterization of the efficiency of a point is first stated in the image space without any assumptions on the functions of the problem, then some more necessary optimality conditions in the image space are given assuming the Hadamard directional differentiability of the functions.

In Section 4, the existence of necessary optimality conditions in the decision space is studied, still assuming that the functions are Hadamard directionally differentiable; a characterization in the image space of such conditions is provided thus making possible a comparison with the previously stated conditions in the image space. The conditions in the decision space result to be stronger than the image space ones, hence a new regularity concept, called "U-regularity", is introduced in order to commute the conditions in the image space to the ones in the decision space.

Finally, in Section 5 the previously obtained results are specified assuming the differentiability of the functions, it is also pointed out that the given conditions generalize some of the results known in the literature.

2 Statement of the problem

The vector optimization problem studied in this paper has both inequality and equality constraints as well as a set constraint, covering the constraints which cannot be expressed by means of neither equalities nor inequalities:

$$P: \begin{cases} C\_\max \ f(x) & \\ g(x) \in V & \textit{inequality constraints} \\ h(x) = 0 & \textit{equality constraints} \\ x \in X & \textit{set constraint} \end{cases}$$

[1]Minimum/maximum principle optimality conditions are used also in infinite dimensional spaces, for instance in optimal control theory [15, 19, 24, 26].

where $f : A \to \Re^s$, $g : A \to \Re^m$ and $h : A \to \Re^p$ are vector valued functions, with $A \subseteq \Re^n$ open set, $C \subset \Re^s$ and $V \subset \Re^m$ are closed convex cones with nonempty interior (that is to say solid cones), and $X \subseteq A$ is a set verifying no particular topological properties, that is to say that X is not required to be open or convex or with nonempty interior. For the sake of convenience, note that problem P can be rewritten in the following form:

$$P : \begin{cases} C\_\max \; f(x) \\ g(x) \in V \\ x \in (X \cap S) \\ S = \{x \in A : h(x) = 0\} \end{cases}$$

The aim of this paper is to study optimality conditions for a feasible point $x_0 \in X$ which is assumed, without loss of generality, to bind all the inequality constraints, so that $g(x_0) = 0$. Note that it is not known whether or not x_0 belongs to the boundary of X. The feasible point $x_0 \in X$ is said to be a *local efficient point* if there exists a suitable neighbourhood I_{x_0} of x_0 such that:

$$\nexists y \in I_{x_0} \cap X \text{ such that } f(y) \in f(x_0) + C^0, \;\; g(y) \in V, \;\; h(y) = 0 \qquad (2.1)$$

where $C^0 = C \setminus \{0\}$. For the sake of simplicity the following function is also used:

$$F : A \to \Re^{s+m+p} \text{ such that } F(x) = (f(x), g(x), h(x))$$

By means of function F, $x_0 \in X$ is a local efficient point if and only if there exists a suitable neighbourhood I_{x_0} of x_0 such that:

$$\nexists y \in I_{x_0} \cap X \text{ such that } F(y) \in F(x_0) + (C^0 \times V \times 0) \qquad (2.2)$$

The study of optimality conditions is based on the so called image space approach, originally suggested by Hestenes [20]; with this aim a key tool results to be the *Bouligand Tangent cone to X at x_0*, denoted with $T(X, x_0)$, which is a closed cone defined as follows:

$$\begin{aligned} T(X, x_0) \;\; = \;\; & \{x \in \Re^n : \exists \{x_k\} \subset X, x_k \to x_0, \exists \{\lambda_k\} \subset \Re^{++}, \lambda_k \to +\infty, \\ & x = \lim_{k \to +\infty} \lambda_k (x_k - x_0)\}. \end{aligned}$$

Also subcones of $T(X, x_0)$ are fundamental in this paper; with this aim just recall that particular subcones of $T(X, x_0)$ are the well known *cone of feasible directions to X at x_0* ([2]), denoted with $F(X, x_0)$, and the *cone of interior directions to X at x_0*, denoted with $I(X, x_0)$ (see for example [3, 17, 18]).

[2]Let $X \subseteq \Re^n$ be a nonempty set and let $x_0 \in \mathrm{Cl}(X)$. The *cone of feasible directions to X at x_0* $F(X, x_0)$ and the *cone of interior directions to X at x_0* $I(X, x_0)$ are defined as follows:

$$F(X, x_0) \;\; = \;\; \{x \in \Re^n : \exists \delta > 0 \text{ such that } x_0 + \lambda x \in X \quad \forall \lambda \in (0, \delta]\};$$

Note finally that the results stated in this paper deal also with problems having no equality and/or no inequality constraints. With this regard, it is important to note that the absence of equality constraints is extremely relevant in the optimality conditions expressed in the decision space; for this reason, when necessary, the absence of equality constraints is specified with the condition $p = 0$ (remind that $h : A \to \Re^p$) and in this case $S = A$ is assumed.

3 Optimality conditions in the Image Space

The aim of this section is to state in the image space necessary and/or sufficient optimality conditions for problem P.

By means of an approach similar to the one used in [5, 6, 7, 8, 9, 10, 11], the following subset of the Bouligand tangent cone at $F(x_0)$ in the image space is introduced:

$$\begin{aligned} T_1 &= \{t \in \Re^{s+m+p} : \exists \{x_k\} \subset X, x_k \to x_0, h(x_k) = 0, \exists \{\lambda_k\} \subset \Re, \\ &\quad \lambda_k > 0, \lambda_k \to +\infty, t = \lim_{k \to +\infty} \lambda_k (F(x_k) - F(x_0))\}. \end{aligned} \tag{3.1}$$

The cone T_1 plays a key role in stating optimality conditions in the image space. The forthcoming results extend the ones stated in [6, 7, 8, 10], which can be seen as the particular cases of problem P where X is an open set or where $x_0 \in \mathrm{Int}(X)$.

3.1 The nonsmooth case

The aim of this subsection in to characterize in the image space the efficiency of x_0, this allows also to determine a necessary optimality condition as well as a sufficient one. Note that no hypothesis on the functions f, g and h are assumed, that is to say that they may be not only nondifferentiable but even noncontinuous.

Theorem 3.1 *Consider problem P. If $x_0 \in X$ is a local efficient point then:*

$$T_1 \cap (\mathrm{Int}(C) \times \mathrm{Int}(V) \times 0) = \emptyset \tag{3.2}$$

$$I(X, x_0) = \{x \in \Re^n : \exists \epsilon > 0, \exists \delta > 0 \text{ such that } \lambda \in (0, \delta), \|y - x\| < \epsilon \text{ imply } x_0 + \lambda y \in X\}.$$

It is very well known that for any set X:

$$I(X, x_0) \subseteq \mathrm{Int}(F(X, x_0)) \subseteq F(X, x_0) \subseteq F(\mathrm{Cl}(X), x_0) \subseteq \mathrm{Cl}(F(X, x_0)) \subseteq T(X, x_0).$$

Proof The result is proved by contradiction. Suppose that $\exists t^* \in T_1 \cap (\text{Int}(C) \times \text{Int}(V) \times 0)$; then $\exists\{x_k\} \subset X$, $x_k \to x_0$, $h(x_k) = 0$, $\exists\{\lambda_k\} \subset \Re$, $\lambda_k > 0$, $\lambda_k \to +\infty$, such that $t^* = \lim_{k\to+\infty} \lambda_k(F(x_k) - F(x_0))$.
Being $t^* \in (\text{Int}(C) \times \text{Int}(V) \times 0)$ and being $h(x_k) = 0\ \forall k$ then for a known limit theorem:

$$\exists \bar{k} > 0 \text{ such that } \lambda_k(F(x_k) - F(x_0)) \in (\text{Int}(C) \times \text{Int}(V) \times 0) \quad \forall k > \bar{k}$$

so that, being $\lambda_k > 0$, $F(x_k) \in F(x_0) + (\text{Int}(C) \times \text{Int}(V) \times 0)\ \forall k > \bar{k}$ and this contradicts the local efficiency of x_0. □

The next theorem shows that it is possible to characterize in the image space the optimality of x_0.

Theorem 3.2 *Consider problem P. The point $x_0 \in X$ is a local efficient point if and only if the following condition holds:*

> *$\forall t \in T_1 \cap (C \times V \times 0)$, $t \neq 0$, and $\forall\{x_k\} \subset X$, $x_k \to x_0$, $h(x_k) = 0$, such that $\exists\{\lambda_k\} \subset \Re$, $\lambda_k > 0$, $\lambda_k \to +\infty$, with $t = \lim_{k\to+\infty} \lambda_k(F(x_k) - F(x_0))$, there exists an integer $\bar{k} > 0$ such that:*
>
> $$F(x_k) \notin F(x_0) + (C^0 \times V \times 0)\ \forall k > \bar{k}.$$

Proof $\Rightarrow$) If x_0 is a local efficient point then, for (2.2), $\forall\{x_k\} \subset X$, $x_k \to x_0$, $h(x_k) = 0$, there exists an integer $\bar{k} > 0$ such that $F(x_k) \notin F(x_0) + (C^0 \times V \times 0)\ \forall k > \bar{k}$, and this is true also for particular sequences such that $t = \lim_{k\to+\infty} \lambda_k(F(x_k) - F(x_0))$ with $t \in T_1 \cap (C \times V \times 0)$.
$\Leftarrow$) The result is proved by contradiction. Suppose that $x_0 \in X$ is not a local efficient point, then by means of (2.2) $\exists\{x_k\} \subset X$, $x_k \to x_0$, such that $F(x_k) \in F(x_0) + (C^0 \times V \times 0)\ \forall k$, so that in particular $h(x_k) = 0\ \forall k$. Let us consider now the sequence $\{d_k\} \subset \Re^{s+m+p}$ with $d_k = \frac{F(x_k)-F(x_0)}{\|F(x_k)-F(x_0)\|}$; since the unit ball is a compact set, we can suppose (substituting $\{d_k\}$ with a suitable subsequence, if necessary) that $\lim_{k\to+\infty} d_k = t^* \neq 0$, $t^* \in T_1$. On the other hand, $d_k = \frac{F(x_k)-F(x_0)}{\|F(x_k)-F(x_0)\|} \in (C^0 \times V \times 0)$ so that its limit $t^* \in (C \times V \times 0)$. It then results that $t^* \in T_1 \cap (C \times V \times 0)$, $t \neq 0$, and this contradicts the assumptions since $t^* = \lim_{k\to+\infty} \frac{F(x_k)-F(x_0)}{\|F(x_k)-F(x_0)\|}$ and $F(x_k) \in F(x_0) + (C^0 \times V \times 0)\ \forall k$. □

Directly from Theorem 3.2 we can state the following sufficient optimality condition.

Corollary 3.1 *Consider problem P. If the following condition holds then $x_0 \in X$ is a local efficient point:*

$$T_1 \cap (C \times V \times 0) = \{0\} \tag{3.3}$$

3.2 The nondifferentiable case

The previously stated optimality conditions are extremely general since no properties are assumed regarding to functions f, g and h. On the other hand, those conditions are not easy to be verified, since the cone T_1 is not trivial to be determined.

Some more "easy to use" necessary optimality conditions, still based on the image space approach, can be proved with the following assumption.

(H_N) *Nondifferentiability Assumptions*

- Functions f, g and h are Hadamard directionally differentiable at the point $x_0 \in X$ ([3]).

A complete study of Hadamard directionally differentiable functions can be found for example in [16] (see also [1, 2, 25, 28]). The nondifferentiability hypothesis (H_N) allows to define the following cones, which play a key role in stating further necessary optimality conditions in the image space.

Definition 3.1 Consider problem P, suppose (H_N) holds and let $U \subseteq \Re^n$ be a cone. The following sets are defined:

$$\begin{aligned}
Ker_{\partial h} &= \{0\} \cup \{v \in \Re^n \setminus \{0\} : \frac{\partial h}{\partial v}(x_0) = 0\} \\
Ker^C_{\partial h} &= \Re^n \setminus Ker_{\partial h} = \{v \in \Re^n \setminus \{0\} : \frac{\partial h}{\partial v}(x_0) \neq 0\} \\
Im_{\partial h}(U) &= \{0\} \cup \{t \in \Re^p : t = \frac{\partial h}{\partial v}(x_0), v \neq 0, v \in U\} \\
L(X, S, x_0) &= T(X \cap S, x_0) \cup Ker^C_{\partial h} = \Re^n \setminus (Ker_{\partial h} \setminus T(X \cap S, x_0)) = L \\
K_L &= \{t \in \Re^{m+s+p} : t = (\frac{\partial f}{\partial v}(x_0), \frac{\partial g}{\partial v}(x_0), \frac{\partial h}{\partial v}(x_0)), v \neq 0, v \in L\}
\end{aligned}$$

[3]Let $f : A \to \Re$, with $A \subseteq \Re^n$ open set. The limit:

$$\lim_{\lambda \to 0^+, h \to v} \frac{f(x_0 + \lambda h) - f(x_0)}{\lambda}$$

is called the *Hadamard directional derivative of $f(x)$ at $x_0 \in A$ in the direction v*; if this derivative exists and is finite for all v then $f(x)$ is *Hadamard directionally differentiable at* $x_0 \in A$. In order to verify the Hadamard directional derivability, remind that a function $f(x)$ is Hadamard directionally differentiable at x_0 (see [16]) if and only if its derivative $\frac{\partial f}{\partial v}(x_0) \stackrel{\text{def}}{=} \lim_{\lambda \to 0^+} \frac{f(x_0 + \lambda v) - f(x_0)}{\lambda}$ is continuous as a function of direction and the function itself is *Dini uniformly directionally differentiable at* x_0 (hence directionally differentiable at x_0), that is to say that:

$$\lim_{\|v\| \to 0} \left| f(x_0 + v) - f(x_0) - \frac{\partial f}{\partial v}(x_0) \right| = 0$$

Recall also that if a function $f(x)$ is Hadamard directionally differentiable at x_0 then it is also continuous at x_0. A vector valued function $F : A \to \Re^m$ is Hadamard directionally differentiable at x_0 if all its components verify this property.

$$K_U = \{t \in \Re^{m+s+p} : t = (\frac{\partial f}{\partial v}(x_0), \frac{\partial g}{\partial v}(x_0), \frac{\partial h}{\partial v}(x_0)), v \neq 0, v \in U\}$$

Note that $Ker_{\partial h}$, $Ker_{\partial h}^C$, $Im_{\partial h}(U)$, K_L and K_U are cones, since $\frac{\partial f}{\partial v}(x_0)$, $\frac{\partial g}{\partial v}(x_0)$ and $\frac{\partial h}{\partial v}(x_0)$ are positively homogeneous (of the first degree) as functions of direction v, due to the Hadamard directional differentiability of f, g and h ([4]). In the rest of the paper, cones $U \subseteq L(X, S, x_0)$ will be very used, with this aim note that:

$$U \subseteq L(X, S, x_0) \quad \Leftrightarrow \quad U \cap Ker_{\partial h} \subseteq T(X \cap S, x_0)$$

Remark 3.1 Since $L(X, S, x_0) = T(X \cap S, x_0) \cup Ker_{\partial h}^C$ it is worth noticing that if h is Hadamard directionally differentiable at $x_0 \in X$ then ([5]):

$$T(X \cap S, x_0) \subseteq T(S, x_0) \subseteq Ker_{\partial h}$$

In order to verify this property, firstly note that $T(X \cap S, x_0) \subseteq T(S, x_0)$ being $X \cap S \subseteq S$. Being $t = 0 \in T(S, x_0) \cap Ker_{\partial h}$ just the case $t \in T(S, x_0)$, $t \neq 0$, has to be considered. By means of the definition of tangent cone, $\exists\{x_k\} \subset S$, $x_k \to x_0$, $\exists\{\lambda_k\} \subset \Re$, $\lambda_k > 0$, $\lambda_k \to +\infty$, such that $t = \lim_{k\to+\infty} \lambda_k(x_k - x_0)$; it can be supposed also (eventually substituting $\{x_k\}$ with a proper subsequence) that $v = \lim_{k\to+\infty} \frac{x_k - x_0}{\|x_k - x_0\|}$. Since $\{x_k\} \subset S$ it yields $h(x_0) = h(x_k) = 0$ $\forall k > 0$ so that, by means of the Hadamard directional differentiability of $h(x)$, it is:

$$0 = \lim_{k\to+\infty} \frac{h(x_k) - h(x_0)}{\|x_k - x_0\|} = \lim_{\gamma_k\to 0^+, d_k \to v} \frac{h(x_0 + \gamma_k d_k) - h(x_0)}{\gamma_k} = \frac{\partial h}{\partial v}(x_0)$$

where $\gamma_k = \|x_k - x_0\|$ and $d_k = \frac{x_k - x_0}{\|x_k - x_0\|}$, so that $v \in Ker_{\partial h}$.
By means of the definition it results:

$$t = \lim_{k\to+\infty} \lambda_k(x_k - x_0) = \lim_{k\to+\infty} \lambda_k \|x_k - x_0\| \lim_{k\to+\infty} \frac{x_k - x_0}{\|x_k - x_0\|} = \mu v$$

where $\mu = \lim_{k\to+\infty} \lambda_k \|x_k - x_0\| \geq 0$ and $\|v\| = 1$. Being $Ker_{\partial h}$ a cone and being $v \in Ker_{\partial h}$ it follows that $t \in Ker_{\partial h}$. □

By means of these cones the following necessary optimality conditions in the image space can be stated.

[4]Note also that the given definition of K_L generalizes the one given in [5, 6, 7, 8, 9, 10, 11] for differentiable problems having no set constraints; in particular these papers consider $K_L = \{t \in \Re^{s+m} : t = [J_f(x_0), J_g(x_0)]v, v \in \Re^n\}$, which is nothing but the image of $[J_f(x_0), J_g(x_0)]$.

[5]It is also known, see for instance [3], that if h is differentiable at $x_0 \in X$ it is:

$$T(S, x_0) \subseteq \text{Cl}(\text{Co}(T(S, x_0))) \subseteq Ker_{\partial h}$$

where $\text{Co}(X)$ denotes the convex hull of the set X.

Theorem 3.3 *Consider Problem P and suppose (H_N) holds; if the feasible point $x_0 \in X$ is a local efficient point then the two following equivalent conditions hold:*

$$K_L \cap (\text{Int}(C) \times \text{Int}(V) \times 0) = \emptyset \tag{3.4}$$
$$(K_L - (C \times V \times 0)) \cap (\text{Int}(C) \times \text{Int}(V) \times 0) = \emptyset \tag{3.5}$$

In addiction, for any cone $U \subseteq \Re^n$ such that $U \cap Ker_{\partial h} \subseteq T(X \cap S, x_0)$ the two following further equivalent conditions hold:

$$K_U \cap (\text{Int}(C) \times \text{Int}(V) \times 0) = \emptyset \tag{3.6}$$
$$(K_U - (C \times V \times 0)) \cap (\text{Int}(C) \times \text{Int}(V) \times 0) = \emptyset \tag{3.7}$$

Proof Condition (3.4) is proved by contradiction. Suppose that there exists $t = (t_f, t_g, t_h) \in K_L \cap (\text{Int}(C) \times \text{Int}(V) \times 0)$, so that $\exists \mu > 0$, $\exists v \in L(X, S, x_0)$, $\|v\| = 1$, such that

$$t = \mu(\frac{\partial f}{\partial v}(x_0), \frac{\partial g}{\partial v}(x_0), \frac{\partial h}{\partial v}(x_0)) \in (\text{Int}(C) \times \text{Int}(V) \times 0).$$

Being $\frac{\partial h}{\partial v}(x_0) = 0$ then $v \in Ker_{\partial h}$ which implies that $v \notin Ker^C_{\partial h}$ and $v \in T(X \cap S, x_0)$. By means of the definition of $T(X \cap S, x_0)$ it yields that $\exists \{x_k\} \subset (X \cap S)$, $x_k \to x_0$, $\exists \{\lambda_k\} \subset \Re$, $\lambda_k > 0$, $\lambda_k \to +\infty$, such that $v = \lim_{k \to +\infty} v_k$ where $v_k = \lambda_k(x_k - x_0)$. Being functions f and g Hadamard directionally differentiable it results:

$$\lim_{k \to +\infty} \frac{f(x_k) - f(x_0)}{\frac{1}{\lambda_k}} = \lim_{k \to +\infty} \frac{f(x_0 + \frac{1}{\lambda_k} v_k) - f(x_0)}{\frac{1}{\lambda_k}} = \frac{\partial f}{\partial v}(x_0) \in \text{Int}(C)$$

and, in the same way:

$$\lim_{k \to +\infty} \frac{g(x_k) - g(x_0)}{\frac{1}{\lambda_k}} = \frac{\partial g}{\partial v}(x_0) \in \text{Int}(V)$$

By means of a well known limit theorem it then exists $\bar{k} > 0$ such that $f(x_k) - f(x_0) \in \text{Int}(C)$ and $g(x_k) - g(x_0) \in \text{Int}(V)$ for any $k > \bar{k}$; this means that the sequence $\{x_k\} \subset (X \cap S)$, $x_k \to x_0$, is feasible for $k > \bar{k}$ and that x_0 is not a local efficient point, which is a contradiction.

The equivalence of (3.4) and (3.5) can be easily verified; the whole result then follows noticing that $U \subseteq L(X, S, x_0)$ implies $K_U \subseteq K_L$. □

Remark 3.2 For the sake of completeness, note that (3.4) can be stated as a corollary of Theorem 3.1.
Denoting with $B = \{t = (t_f, t_g, t_h) \in \Re^{s+m+p} : t_h \neq 0\}$, directly from Theorem 3.1 it follows that the efficiency of x_0 implies:

$$(T_1 \cup B) \cap (\text{Int}(C) \times \text{Int}(V) \times 0) = \emptyset.$$

It is now just needed to verify that $K_L \subseteq (T_1 \cup B)$. Let $t = \mu\frac{\partial F}{\partial v}(x_0) \in K_L$, $v \in L(X, S, x_0)$, $\|v\| = 1$, $\mu \geq 0$; if $\mu = 0$ then $t = \mu\frac{\partial F}{\partial v}(x_0) = 0 \in T_1$ while if $\mu \neq 0$ and $v \in Ker^C_{\partial h}$ then $\frac{\partial h}{\partial v}(x_0) \neq 0$ and $t \in B$. Suppose now $\mu \neq 0$ and $v \in T(X \cap S, x_0)$, then $\exists\{x_k\} \subset X$, $x_k \to x_0$, $h(x_k) = 0$, such that $v = \lim_{k\to+\infty} \frac{x_k - x_0}{\|x_k - x_0\|}$; let also $\lambda_k = \|x_k - x_0\|^{-1}$. By means of the Hadamard directional differentiability of $F(x)$ at x_0 it is:

$$\frac{\partial F}{\partial v}(x_0) = \lim_{k\to+\infty} \frac{F(x_k) - F(x_0)}{\|x_k - x_0\|} = \lim_{k\to+\infty} \lambda_k(F(x_k) - F(x_0)) \in T_1;$$

being T_1 a cone it then follows that $t = \mu\frac{\partial F}{\partial v}(x_0) \in T_1$ too. □

4 Optimality conditions in the Decision Space: the nondifferentiable case

In the literature some necessary optimality conditions expressed in the decision space are stated for particular problems P having a scalar objective function and assuming the differentiability of functions f, g and h [3, 21, 23]. These conditions are useful in the applications (consider for all the optimal control theory) and are known as "maximum/minimum principle" conditions.

The aim of this section is to generalize those conditions for Hadamard directionally differentiable functions and for multiobjective problems. In other words, the necessary optimality conditions in the decision space (hence involving the directional derivatives and some multipliers) which are going to be studied in this section are the followings:

(C_N) $\exists\alpha_f \in C^+$, $\exists\alpha_g \in V^+$, $\exists\alpha_h \in \Re^p$, $(\alpha_f, \alpha_g, \alpha_h) \neq 0$, such that:

$$\alpha_f^T \frac{\partial f}{\partial v}(x_0) + \alpha_g^T \frac{\partial g}{\partial v}(x_0) + \alpha_h^T \frac{\partial h}{\partial v}(x_0) \leq 0 \ \forall v \in \mathrm{Cl}(U) \setminus \{0\}$$

where $U \subseteq \Re^n$ is a cone and (H_N) is assumed.

It is important to note that conditions (C_N), depending on the particular chose cone U, do not hold in general even if x_0 is an efficient point. This is shown in the following example, which implicitly points out that condition (3.4) is more general than (C_N) ones.

Example 4.1 Consider the following problem:

$$P : \{\max f(x_1, x_2) = x_1, g(x_1, x_2) = x_2 \geq 0, x \in X\}$$

where $X = X_1 \cup X_2 \cup X_3$ with:

$$\begin{aligned} X_1 &= \{(x_1, x_2) \in \Re^2 : x_1 + x_2 \geq 0, 2x_1 + x_2 \leq 0\}, \\ X_2 &= \{(x_1, x_2) \in \Re^2 : x_1 \leq 0, x_2 \leq 0\}, \\ X_3 &= \{(x_1, x_2) \in \Re^2 : x_1 + x_2 \geq 0, x_1 + 2x_2 \leq 0\} \end{aligned}$$

and $x_0 = (0, 0)$; since the problem has no equality constraints it is $p = 0$ and $S = \Re^2$. Note that $(\mathrm{Int}(C) \times \mathrm{Int}(V)) = \Re^2_{++}$ and $X = T(X \cap S, x_0) = K_L$ since $[J_f(x_0), J_g(x_0)]$ is equal to the identity matrix. The point x_0 is the global efficient point of the problem and the necessary optimality condition (3.4) is verified being $X \cap \Re^2_{++} = \emptyset$; on the other hand the sets X, $I(X, x_0)$, $T(X \cap S, x_0)$ and K_L are not convex.

Assume now $U = T(X \cap S, x_0)$; even if $x_0 \in X$ is a global efficient point it can be easily verified that (C_N) does not hold; this points out that condition (3.4) is more general than (C_N) one.

In this section it is going to be proved that the additional assumption, needed in order to state the necessary optimality conditions in the decision space, is the existence of a separation hyperplane between the cone $(\mathrm{Int}(C) \times \mathrm{Int}(V) \times 0)$ and K_U or K_L. This result is stated by means of separating theorems and the use of multipliers, hence a key tool of this approach is the positive polar of a cone K, denoted with K^+.

4.1 Characterization in the image space

The aim of this subsection is to characterize conditions (C_N) in the image space, thus making possible a complete comparison with condition (3.4). With this aim, the following preliminary results are needed.

Lemma 4.1 *Consider problem P with $p \geq 1$, suppose (H_N) holds and let $U \subseteq \Re^n$ be a cone such that $\mathrm{Co}(Im_{\partial h}(U)) \neq \Re^p$. Then $\exists \alpha_h \in \Re^p$, $\alpha_h \neq 0$, such that:*

$$\alpha_h^T \frac{\partial h}{\partial v}(x_0) \leq 0 \;\; \forall v \in \mathrm{Cl}(U) \setminus \{0\}$$

and hence (C_N) is verified.

Proof Since $\mathrm{Co}(Im_{\partial h}(U)) \neq \Re^p$ there exists a support hyperplane for the convex cone $\mathrm{Co}(Im_{\partial h}(U))$, so that $\exists \alpha_h \in \Re^p$, $\alpha_h \neq 0$, such that $\alpha_h^T t \leq 0$ $\forall t \in \mathrm{Co}(Im_{\partial h}(U))$; this implies that $\alpha_h^T \frac{\partial h}{\partial v}(x_0) \leq 0$ $\forall v \in U$, $v \neq 0$. Being $\frac{\partial h}{\partial v}(x_0)$ continuous as a function of direction v due to the Hadamard directional differentiability of h, it then follows that $\alpha_h^T \frac{\partial h}{\partial v}(x_0) \leq 0$ $\forall v \in \mathrm{Cl}(U)$, $v \neq 0$. The whole result is then proved just assuming $\alpha_f = 0$ and $\alpha_g = 0$. □

Note that Lemma 4.1 points out that the case $\mathrm{Co}(Im_{\partial h}(U)) \neq \Re^p$ is trivial, since a support hyperplane for $\mathrm{Co}(Im_{\partial h}(U))$ exists without the need of any additional hypothesis, such as convexity ones, optimality assumptions on x_0, regularity conditions for the problem.

Lemma 4.2 *Consider problem P with $p \geq 1$, suppose (H_N) holds and let $U \subseteq \Re^n$ be a cone. If (C_N) is verified and* $\text{Co}(Im_{\partial h}(U)) = \Re^p$ *then* $(\alpha_f, \alpha_g) \neq 0$.

Proof Suppose by contradiction that $\alpha_f = 0$ and $\alpha_g = 0$, so that $\alpha_h \neq 0$. Then

$$\alpha_h^T \frac{\partial h}{\partial v}(x_0) \leq 0 \ \forall v \in \text{Cl}(U) \setminus \{0\},$$

and this yields $\alpha_h^T t \leq 0$ $\forall t \in Im_{\partial h}(U)$. Consequently it results $\alpha_h^T t \leq 0$ $\forall t \in \text{Co}(Im_{\partial h}(U)) = \Re^p$ which implies $\alpha_h = 0$, and this is a contradiction since $(\alpha_f, \alpha_g, \alpha_h) \neq 0$. □

It is now possible to fully characterize condition (C_N) in the image space.

Theorem 4.1 *Consider problem P, suppose (H_N) holds and let $U \subseteq \Re^n$ be a cone. Then condition (C_N) is verified if and only if the following implication holds:*

$$\left.\begin{array}{c} p = 0 \ \text{or} \\ \text{Co}(Im_{\partial h}(U)) = \Re^p \end{array}\right] \Rightarrow \quad \text{Co}(K_U) \cap (\text{Int}(C) \times \text{Int}(V) \times 0) = \emptyset$$

In particular, if $p = 0$ or $\text{Co}(Im_{\partial h}(U)) = \Re^p$ *then* $(\alpha_f, \alpha_g) \neq 0$.

Proof $\Rightarrow)$ Suppose (C_N) holds and first consider the case $p \geq 1$ and $\text{Co}(Im_{\partial h}(U)) = \Re^p$. By means of Lemma 4.2 it is $(\alpha_f, \alpha_g) \neq 0$. Suppose now by contradiction that $\exists (t_f, t_g, t_h) \in \text{Co}(K_U) \cap (\text{Int}(C) \times \text{Int}(V) \times 0) \neq \emptyset$; being $\alpha_f \in C^+$, $\alpha_g \in V^+$, $(\alpha_f, \alpha_g) \neq 0$, $t_f \in \text{Int}(C)$, $t_g \in \text{Int}(V)$ and $t_h = 0$ it is:

$$\alpha_f^T t_f + \alpha_g^T t_g + \alpha_h^T t_h > 0 \tag{4.1}$$

Since $(t_f, t_g, t_h) \in \text{Co}(K_U)$ $\exists q \in \mathbf{N}$, $q > 0$, $\exists v_1, \ldots, v_q \in U$, such that

$$(t_f, t_g, t_h) = \sum_{i=1}^{q} \left(\frac{\partial f}{\partial v_i}(x_0), \frac{\partial g}{\partial v_i}(x_0), \frac{\partial h}{\partial v_i}(x_0) \right)$$

hence

$$\alpha_f^T t_f + \alpha_g^T t_g + \alpha_h^T t_h = \sum_{i=1}^{q} \left(\alpha_f^T \frac{\partial f}{\partial v_i}(x_0) + \alpha_g^T \frac{\partial g}{\partial v_i}(x_0) + \alpha_h^T \frac{\partial h}{\partial v_i}(x_0) \right) \leq 0$$

and this contradicts (4.1). The proof for the case $p = 0$ is analogous.

$\Leftarrow)$ If $p \geq 1$ and $\text{Co}(Im_{\partial h}(U)) \neq \Re^p$ the result follows from Lemma 4.1. Consider now the case $p \geq 1$ and $\text{Co}(Im_{\partial h}(U)) = \Re^p$, so that $\text{Co}(K_U) \cap (\text{Int}(C) \times \text{Int}(V) \times 0) = \emptyset$; by means of a well known separation theorem between convex sets, $\exists(\alpha_f, \alpha_g, \alpha_h) \in (\text{Int}(C) \times \text{Int}(V) \times 0)^+$, $(\alpha_f, \alpha_g, \alpha_h) \neq 0$, such that $(\alpha_f, \alpha_g, \alpha_h)^T t \leq 0 \, \forall t \in \text{Co}(K_U) \supseteq K_U$. A known result on polar cones ([6]) implies that $(\text{Int}(C) \times \text{Int}(V) \times 0)^+ = \text{Int}(C)^+ \times \text{Int}(V)^+ \times \Re^p$ and hence, being C and V convex cones ([7]), $\exists \alpha_f \in C^+$, $\exists \alpha_g \in V^+$, $\exists \alpha_h \in \Re^p$, $(\alpha_f, \alpha_g, \alpha_h) \neq 0$, such that:

$$\alpha_f^T \frac{\partial f}{\partial v}(x_0) + \alpha_g^T \frac{\partial g}{\partial v}(x_0) + \alpha_h^T \frac{\partial h}{\partial v}(x_0) \leq 0 \;\; \forall v \in U, v \neq 0.$$

The directional derivatives $\frac{\partial f}{\partial v}(x_0)$, $\frac{\partial g}{\partial v}(x_0)$ and $\frac{\partial h}{\partial v}(x_0)$ are continuous as functions of direction, since f, g and h Hadamard directionally differentiable at x_0, hence (C_N) is verified. In particular for Lemma 4.2 it is $(\alpha_f, \alpha_g) \neq 0$.

The proof for the case $p = 0$ is analogous. □

It is now worth making a comparison between conditions (C_N) and (3.4) one. Condition (3.4) states that

$$K_L \cap (\text{Int}(C) \times \text{Int}(V) \times 0) = \emptyset$$

while, for a given cone U, (C_N) implies

$$\text{Co}(K_U) \cap (\text{Int}(C) \times \text{Int}(V) \times 0) = \emptyset.$$

It is then clear that, even when $K_U \subseteq K_L$, (C_N) condition is stronger than (3.4) since it requires the existence of a separating hyperplane between K_U and $(\text{Int}(C) \times \text{Int}(V) \times 0)$, while K_L in (3.4) is not convex in general and hence a separation hyperplane may not exists.

Note finally that in Example 4.1, where $U = T(X \cap S, x_0)$ is assumed and (C_N) does not hold, it results that $\text{Co}(K_U) = \Re^2$ and hence no separating hyperplane exists; note also that in Example 4.1 condition (3.4) holds without any convexity assumption regarding to the cones U, $T(X \cap S, x_0)$, K_U or K_L.

4.2 *U*-regularity conditions

As it has been pointed out in the previous subsection, condition

$$K_L \cap (\text{Int}(C) \times \text{Int}(V) \times 0) = \emptyset$$

[6]Let $C_1, \ldots, C_n$ be cones, then $(C_1 \times \ldots \times C_n)^+ = (C_1^+ \times \ldots \times C_n^+)$.
To prove this property it is sufficient to consider just the case $n = 2$. First verify that $(C_1^+ \times C_2^+) \subseteq (C_1 \times C_2)^+$; assuming $(\alpha_1, \alpha_2) \in (C_1^+ \times C_2^+)$ it yields that $\alpha_1^T c + \alpha_2^T v \geq 0$ $\forall c \in C_1$ and $\forall v \in C_2$ so that $(\alpha_1, \alpha_2) \in (C_1 \times C_2)^+$. Verify now that $(C_1 \times C_2)^+ \subseteq (C_1^+ \times C_2^+)$; assume $(\alpha_1, \alpha_2) \in (C_1 \times C_2)^+$ and suppose by contradiction that $\alpha_1 \notin C_1^+$ $[\alpha_2 \notin C_2^+]$, then $\exists \bar{c} \in C_1$ $[\exists \bar{v} \in C_2]$ such that $\alpha_1^T \bar{c} < 0$ $[\alpha_2^T \bar{v} < 0]$; since C_1 $[C_2]$ is a cone then $\lambda \bar{c} \in C_1$ $[\lambda \bar{v} \in C_2]$ $\forall \lambda > 0$ so that, given $v \in C_2$ $[c \in C_1]$, for $\lambda > 0$ great enough we have $\alpha_1^T(\lambda \bar{c}) + \alpha_2^T v < 0$ $[\alpha_1^T c + \alpha_2^T(\lambda \bar{v}) < 0]$ and this contradicts that $(\alpha_1, \alpha_2) \in (C_1 \times C_2)^+$.

[7]Let C be a cone; it is known (see for all [27]) that $C^+ = \text{Cl}(C)^+$ so that $\text{Int}(C)^+ = \text{Cl}(\text{Int}(C))^+$ too. If C is a convex cone we also have (see for instance [4]) that $\text{Cl}(\text{Int}(C)) = \text{Cl}(C)$ so that $\text{Int}(C)^+ = C^+$.

does not guarantee (C_N), since

$$\left.\begin{array}{c} p = 0 \text{ or} \\ \mathrm{Co}(Im_{\partial h}(U)) = \Re^p \end{array}\right] \Rightarrow \mathrm{Co}(K_U) \cap (\mathrm{Int}(C) \times \mathrm{Int}(V) \times 0) = \emptyset$$

is needed. This behaviour suggests the introduction of the following regularity condition ([8]).

Definition 4.1 Consider Problem P and suppose (H_N) holds. A cone $U \subseteq \Re^n$ verifies an *U-regularity condition* if the following implication holds:

$$\left.\begin{array}{c} K_L \cap (\mathrm{Int}(C) \times \mathrm{Int}(V) \times 0) = \emptyset \text{ and} \\ [\ p = 0 \text{ or } \mathrm{Co}(Im_{\partial h}(U)) = \Re^p\ ] \end{array}\right\} \Rightarrow \mathrm{Co}(K_U) \cap (\mathrm{Int}(C) \times \mathrm{Int}(V) \times 0) = \emptyset \tag{4.2}$$

The use of U-regularity conditions is focused on in the next theorem which follows directly from (4.2) and Theorem 4.1.

Theorem 4.2 *Consider Problem P and suppose (H_N) holds; the following properties hold:*

i) U verifies an U-regularity condition if and only if

$$K_L \cap (\mathrm{Int}(C) \times \mathrm{Int}(V) \times 0) = \emptyset \quad \Rightarrow \quad (C_N) \textit{ holds};$$

ii) if $x_0 \in X$ is a feasible local efficient point and $U \subseteq \Re^n$ is a cone then:

$$U \textit{ verifies an } U\textit{-regularity condition} \quad \Leftrightarrow \quad (C_N) \textit{ holds.}$$

In other words, an U-regularity condition is nothing but the additional hypothesis needed in order to commute condition (3.4) in the image space to condition (C_N) in the decision space. Hence, from now on, the study of (C_N) optimality conditions can be equivalently done in the image space by means of U-regularity conditions.

Theorem 4.3 *Consider Problem P, suppose (H_N) holds and let $x_0 \in X$ be a feasible local efficient point. Then for every cone $U \subseteq \Re^n$ verifying an U-regularity condition $\exists \alpha_f \in C^+$, $\exists \alpha_g \in V^+$, $\exists \alpha_h \in \Re^p$, $(\alpha_f, \alpha_g, \alpha_h) \neq 0$, such that:*

$$\alpha_f^T \frac{\partial f}{\partial v}(x_0) + \alpha_g^T \frac{\partial g}{\partial v}(x_0) + \alpha_h^T \frac{\partial h}{\partial v}(x_0) \leq 0 \ \forall v \in \mathrm{Cl}(U) \setminus \{0\}.$$

In particular, if $p = 0$ or $\mathrm{Co}(Im_{\partial h}(U)) = \Re^p$ then $(\alpha_f, \alpha_g) \neq 0$.

[8] A different definition of U-regularity condition, not characterizing conditions (C_N), has been already introduced in [13, 14].

Remark 4.1 Note that Theorem 4.3 cannot be applied to Example 4.1 when $U = T(X \cap S, x_0)$ is assumed, since being $\mathrm{Co}(K_U) = \Re^2$ the cone U verifies no U-regularity condition.

The aim of this paper is now moved in stating U-regularity conditions; the following trivial ones, which do not need of the optimality of x_0 in order to guarantee (C_N), can be obtained directly from (4.2):

i) $$\left.\begin{array}{c} p = 0 \text{ or} \\ \mathrm{Co}(Im_{\partial h}(U)) = \Re^p \end{array}\right] \Rightarrow \mathrm{Co}(K_U) \cap (\mathrm{Int}(C) \times \mathrm{Int}(V) \times 0) = \emptyset,$$

ii) $$\left.\begin{array}{c} p = 0 \text{ or} \\ \mathrm{Co}(Im_{\partial h}(\Re^n)) = \Re^p \end{array}\right] \Rightarrow \mathrm{Co}(K_U) \cap (\mathrm{Int}(C) \times \mathrm{Int}(V) \times 0) = \emptyset,$$

iii) $\mathrm{Co}(K_U) \cap (\mathrm{Int}(C) \times \mathrm{Int}(V) \times 0) = \emptyset$,

where *iii)*$\Rightarrow$*ii)*$\Rightarrow$*i)*. More interesting U-regularity conditions, based on the optimality of x_0, are stated in the next theorem; with this aim it is interesting to preliminary point out the following property.

Lemma 4.3 *Consider problem P, suppose (H_N) holds and let $U \subseteq \Re^n$ be a cone. The following conditions are equivalent:*

$$\mathrm{Co}(K_U) \cap (\mathrm{Int}(C) \times \mathrm{Int}(V) \times 0) = \emptyset \tag{4.3}$$
$$\mathrm{Co}(K_U - (C \times V \times 0)) \cap (\mathrm{Int}(C) \times \mathrm{Int}(V) \times 0) = \emptyset \tag{4.4}$$

Proof Being $\mathrm{Co}(K_U) \subseteq \mathrm{Co}(K_U - (C \times V \times 0))$, to prove the equivalence among (4.3) and (4.4) it must be shown that (4.3)$\Rightarrow$(4.4). It can be easily verified that (4.3) implies:

$$(\mathrm{Co}(K_U) - (C \times V \times 0)) \cap (\mathrm{Int}(C) \times \mathrm{Int}(V) \times 0) = \emptyset$$

so that the result follows since the convexity of C and V implies:
$$(\mathrm{Co}(K_U) - (C \times V \times 0)) = \mathrm{Co}(K_U - (C \times V \times 0)).$$
□

Theorem 4.4 *Consider problem P, suppose (H_N) holds and consider also a cone $U \subseteq \Re^n$. The following conditions are U-regularity ones:*

i) $$\left.\begin{array}{c} p = 0 \text{ or} \\ \mathrm{Co}(Im_{\partial h}(U)) = \Re^p \end{array}\right] \Rightarrow \mathrm{Co}(K_U) \subseteq (K_L - (C \times V \times 0)),$$

ii) $$\left.\begin{array}{c} p = 0 \text{ or} \\ \mathrm{Co}(Im_{\partial h}(U)) = \Re^p \end{array}\right] \Rightarrow \left\{\begin{array}{c} U \cap Ker_{\partial h} \subseteq T(X \cap S, x_0) \text{ and} \\ (K_U - (C \times V \times 0)) \text{ is a convex cone} \end{array}\right.$$

iii) $p = 0$, $U \subseteq T(X, x_0)$ *and* $(K_U - (C \times V \times 0))$ *is a convex cone* ([9]),

[9] Note that when $p = 0$ it is $S = A$ and $Ker_{\partial h} = \Re^n$.

where iii)⇒ii)⇒i).

Proof Let us first prove that *i)* is an U-regularity condition. Suppose that $K_L \cap (\text{Int}(C) \times \text{Int}(V) \times 0) = \emptyset$ and $[\, p = 0$ or $\text{Co}(Im_{\partial h}(U)) = \Re^p\,]$; then for *i)* it is $\text{Co}(K_U) \subseteq (K_L - (C \times V \times 0))$ while for Theorem 3.3 it is $(K_L - (C \times V \times 0)) \cap (\text{Int}(C) \times \text{Int}(V) \times 0) = \emptyset$ so that $\text{Co}(K_U) \cap (\text{Int}(C) \times \text{Int}(V) \times 0) = \emptyset$ and hence *i)* is an U-regularity condition.

Let us now prove that *ii)⇒i)*. Condition $U \cap Ker_{\partial h} \subseteq T(X \cap S, x_0)$ implies $K_U \subseteq K_L$, so that $(K_U - (C \times V \times 0)) \subseteq (K_L - (C \times V \times 0))$; being $(K_U - (C \times V \times 0))$ a convex cone it results

$$\text{Co}(K_U) \subseteq \text{Co}(K_U - (C \times V \times 0)) = (K_U - (C \times V \times 0)) \subseteq (K_L - (C \times V \times 0))$$

and hence *ii)⇒i)*. The whole result is then proved since *iii)⇒ii)* trivially. □

4.3 Subcones of $L(X, S, x_0)$

The U-regularity condition *ii)* stated in Theorem 4.4 points out the importance of the cones $U \subseteq \Re^n$ such that $U \cap Ker_{\partial h} \subseteq T(X \cap S, x_0)$; recall that this happens if and only if $U \subseteq L(X, S, x_0)$.

The study of U-regularity conditions can then be deepened on looking for particular subcones of $T(X \cap S, x_0)$ ([10]). For the sake of simplicity, from now on we will use the following notations:

$$I_X = I(X, x_0), \quad T_X = T(X, x_0), \quad F_X = F(X, x_0)$$
$$I_S = I(S, x_0), \quad T_S = T(S, x_0), \quad F_S = F(S, x_0).$$

Lemma 4.4 *Let us consider Problem P and suppose* (H_N) *holds. It results:*

$$\text{Cl}(I_X \cap T_S) \cup \text{Cl}(T_X \cap I_S) \cup \text{Cl}(F_X \cap F_S) \subseteq T(X \cap S, x_0). \tag{4.5}$$

Proof We firstly prove that $I(X, x_0) \cap T(S, x_0) \subseteq T(X \cap S, x_0)$. If $I_X \cap T_S = \emptyset$ the result is trivial, otherwise let $t \in I(X, x_0) \cap T(S, x_0)$, $t \neq 0$ (note that if $t = 0$ then $t \in T(X \cap S, x_0)$ trivially), so that $\exists \{x_k\} \subset S$, $x_k \to x_0$, $\exists \{\lambda_k\} \subset \Re^{++}$, $\lambda_k \to +\infty$, such that $t = \lim_{k \to +\infty} \lambda_k (x_k - x_0)$. Since $t \in I(X, x_0)$ then $\exists \bar{k} > 0$, $\exists \delta > 0$ such that $\mu \in (0, \delta)$, $k > \bar{k}$ imply $x_0 + \mu(\lambda_k(x_k - x_0)) \in X$. Being $x_k = x_0 + \frac{1}{\lambda_k}(\lambda_k(x_k - x_0))$ and $\lambda_k \to +\infty$, then $\exists \tilde{k} > \bar{k}$ such that $\forall k > \tilde{k}$ it results $\frac{1}{\lambda_k} < \delta$ and $x_k = x_0 + \frac{1}{\lambda_k}(\lambda_k(x_k - x_0)) \in X$. This means that $\forall k > \tilde{k} > \bar{k} > 0$ we have $x_k \in X \cap S$ so that $t \in T(X \cap S, x_0)$ and hence $I(X, x_0) \cap T(S, x_0) \subseteq T(X \cap S, x_0)$. Being $T(X \cap S, x_0)$ a closed cone we finally have $\text{Cl}(I_X \cap T_S) \subseteq T(X \cap S, x_0)$. In the same way we can also prove that $\text{Cl}(T_X \cap I_S) \subseteq T(X \cap S, x_0)$. Since

[10] It is known (see for all [3]) that:

$$F(X, x_0) \cap F(S, x_0) = F(X \cap S, x_0) \subseteq T(X \cap S, x_0) \subseteq T(X, x_0) \cap T(S, x_0).$$

$F(X, x_0) \cap F(S, x_0) = F(X \cap S, x_0) \subseteq T(X \cap S, x_0)$ (see for example [3]) it results $\text{Cl}(F(X, x_0) \cap F(S, x_0)) \subseteq T(X \cap S, x_0)$ being $T(X \cap S, x_0)$ a closed cone. □

Corollary 4.1 *Consider problem P, suppose (H_N) holds and let $U \subseteq \Re^n$ be a cone such that:*

$$U \subseteq \text{Cl}(I_X \cap T_S) \cup \text{Cl}(T_X \cap I_S) \cup \text{Cl}(F_X \cap F_S) \cup Ker^C_{\partial h},$$

or such that:

$$U \subseteq I_X \cup \text{Cl}(T_X \cap I_S) \cup \text{Cl}(F_X \cap F_S) \cup Ker^C_{\partial h} \text{ with } T(S, x_0) = Ker_{\partial h}.$$

An U-regularity condition is verified if one of the following properties holds:

i) $(K_U - (C \times V \times 0))$ *is a convex cone,*

ii) K_U *is a convex cone.*

Proof For *ii)* of Theorem 4.4 we must verify that $U \cap Ker_{\partial h} \subseteq T(X \cap S, x_0)$. In the first case the result follows from Lemma 4.4 since

$$U \cap Ker_{\partial h} \subseteq \text{Cl}(I_X \cap T_S) \cup \text{Cl}(T_X \cap I_S) \cup \text{Cl}(F_X \cap F_S)$$

while in the second one we have:

$$U \cap Ker_{\partial h} \subseteq (I_X \cap Ker_{\partial h}) \cup \text{Cl}(T_X \cap I_S) \cup \text{Cl}(F_X \cap F_S)$$

and the result is proved being $T(S, x_0) = Ker_{\partial h}$. □

Recall that the above hypothesis $T(S, x_0) = Ker_{\partial h}$ is not trivial since in general it is just $T(S, x_0) \subseteq Ker_{\partial h}$, as it has been proved in Remark 3.1 and it is pointed out in the next Example 4.2. Example 4.2 points out also that assuming (H_N) it is possible to have $I(S, x_0) \neq \emptyset$ even when $\text{Co}(Im_{\partial h}(\Re^n)) = \Re^p$.

Example 4.2 Let us consider the point $x_0 = (0, 0)$ and the following function $h : \Re^2 \to \Re$:

$$h(x) = \begin{cases} 0 & \text{if } x_1 \geq 0, x_2 \leq 0 \\ \min(x_1, x_2) & \text{if } x_1 \geq 0, x_2 > 0 \\ x_1 x_2 & \text{if } x_1 < 0, x_2 \geq 0 \\ \max(x_1, x_2) & \text{if } x_1 < 0, x_2 < 0 \end{cases}$$

It results:

$$\frac{\partial h}{\partial v}(x_0) = \begin{cases} 0 & \text{if } x_1 x_2 \leq 0 \\ h(v) & \text{if } x_1 x_2 > 0 \end{cases}$$

so that $Im_{\partial h}(\Re^n) = \Re$, $Ker_{\partial h} = \{(x_1, x_2) : x_1 x_2 \leq 0\}$, $T(S, x_0) = S$ where:

$$S = \{(x_1, x_2) : x_1 = 0 \text{ or } x_2 = 0\} \cup \{(x_1, x_2) : x_1 > 0, x_2 < 0\},$$

and $I(S, x_0) = \{(x_1, x_2) : x_1 > 0, x_2 < 0\}$. Even if $\text{Co}(Im_{\partial h}(\Re^n)) = Im_{\partial h}(\Re^n) = \Re$, we then have $I(S, x_0) \neq \emptyset$ and $T(S, x_0) \subset Ker_{\partial h}$ but $T(S, x_0) \neq Ker_{\partial h}$, since for example $d = (-1, 1)^T \in Ker_{\partial h}$ but $d \notin T(S, x_0)$.

Remark 4.2 Note that in Lemma 4.4 no particular properties at all are required for the sets X and S. Note also the difficulty of stating a subcone of $T(X \cap S, x_0)$ greater than $\text{Cl}(I_X \cap T_S) \cup \text{Cl}(T_X \cap I_S) \cup \text{Cl}(F_X \cap F_S)$ since in general it results (see Example 4.3):

$$\text{Int}(F(X, x_0)) \cap T(S, x_0) \nsubseteq T(X \cap S, x_0),$$

and even if X and S are convex sets it results (see Example 4.4):

$$\text{Cl}(I(X, x_0)) \cap T(S, x_0) \nsubseteq T(X \cap S, x_0),$$
$$\text{Cl}(F(X, x_0)) \cap \text{Cl}(F(S, x_0)) \nsubseteq T(X \cap S, x_0).$$

Example 4.3 Let $X = X_1 \cup X_2 \subset \Re^2$, $X_1 = \{(x_1, x_2) : 0 \leq x_2 \leq \sqrt{|x_1|}\}$ and $X_2 = \{(x_1, x_2) : x_1 = 0\}$, let $x_0 = (0, 0)$ and let $S = \{(x_1, x_2) : h(x_1, x_2) = x_2^2 - 4x_1 = 0\}$ so that $X \cap S = \{x_0\}$ and $\frac{\partial h}{\partial v}(x_0) = \nabla h(x_0)^T v = -4v_1$. It then results $T(X \cap S, x_0) = \{0\}$ and $T(S, x_0) = X_2$ so that:

$$\text{Int}(F_X) \cap T_S = \text{Cl}(I_X) \cap T_S = T_S = X_2 \nsubseteq \{0\} = T(X \cap S, x_0)$$

Example 4.4 Consider the convex set with nonempty interior

$$X = \{(x_1, x_2) \in \Re^2 : x_2 \geq x_1^2\},$$

let $x_0 = (0, 0)$ and let $h(x_1, x_2) = x_2$, so that $S = \{(x_1, x_2) \in \Re^2 : x_2 = 0\}$, $X \cap S = \{x_0\}$, $\frac{\partial h}{\partial v}(x_0) = \nabla h(x_0)^T v = v_2$ and $Ker_{\partial h} = S$. It results:

$$\begin{aligned} \text{Cl}(I(X, x_0)) &= \{(x_1, x_2) \in \Re^2 : x_2 \geq 0\}, \\ T(X \cap S, x_0) &= \{0\}, \\ L(X, S, x_0) &= \{0\} \cup \{(x_1, x_2) \in \Re^2 : x_2 \neq 0\}, \end{aligned}$$

hence $\text{Cl}(I_X) \cap T_S = \text{Cl}(F_X) \cap \text{Cl}(F_S) = T_S = S \nsubseteq \{0\} = T(X \cap S, x_0)$.

5 Optimality conditions in the differentiable case

In this section the optimality conditions in the decision space are deepened on assuming functions f, g and h to be differentiable.

(H_G) *Gâteaux Differentiability Assumptions*

- Functions f, g and h are Gâteaux differentiable at $x_0 \in X$ ([11]).

Note that (H_G) implies that the directional derivatives are linear with respect to the direction, hence the following properties hold ([12]):

$$\text{Co}(K_U) = K_{\text{Co}(U)} = \text{Co}(K_{\text{Co}(U)}) \quad \text{and} \quad \text{Co}(Im_{\partial h}(U)) = J_h(x_0)[\text{Co}(U)].$$

It results also that:

$$Ker_{\partial h} = \text{Ker}(J_h(x_0)) \quad \text{and} \quad Im_{\partial h}(\Re^n) = \text{Img}(J_h(x_0)) = J_h(x_0)[\Re^n],$$

that is to say that $Ker_{\partial h}$ is the kernel of the Jacobian matrix $J_h(x_0)$ while $Im_{\partial h}(\Re^n)$ is its image. Note finally that when $p \geq 1$:

$$\text{Co}(Im_{\partial h}(\Re^n)) = J_h(x_0)[\Re^n] = \Re^p \iff J_h(x_0) \text{ is surjective},$$

hence when $p \geq 1$ and $J_h(x_0)$ is not surjective the trivial case already discussed in Lemma 4.1 occurs with $U = \Re^n$. It is worth noticing that when $J_h(x_0)$ is surjective then assumptions (H_G) imply that $I(S, x_0) = \emptyset$.

Theorem 5.1 *Let $h : X \to \Re^p$, $X \subseteq \Re^n$, be a given mapping, let $x_0 \in S = \{x \in \Re^n : h(x) = 0\}$ and let $h(x)$ be Gâteaux differentiable at x_0.*
i) If $\exists \bar{d} \in \Re^n$ such that $J_h(x_0)\bar{d} \neq 0$ then $I(S, x_0) = \emptyset$,
ii) if $I(S, x_0) \neq \emptyset$ then $\text{Img}(J_h(x_0)) = \{0\}$ and $\text{Ker}(J_h(x_0)) = \Re^n$,
iii) if $J_h(x_0)$ is surjective then $I(S, x_0) = \emptyset$.

Proof *i)* Let $d \in I(S, x_0) \neq \emptyset$; if $d = 0$ then $x_0 \in \text{Int}(S)$ ([13]), so that there exists a suitable neighbourhood of x_0, say I_{x_0}, such that $h(x) = 0 \; \forall x \in I_{x_0}$ and this implies that $J_h(x_0) = 0$ which contradicts $J_h(x_0)\bar{d} \neq 0$. Suppose now $d \neq 0$; then there exists a suitable neighbourhood of d, say I_d, such that all the directions $v \in I_d$ are feasible for the set S, this implies that $h(x_0 + tv) = 0$ in a neighbourhood of $t = 0 \; \forall v \in I_d$ and hence $J_h(x_0)v = 0$

[11] Let $F : A \to \Re^m$, with $A \subseteq \Re^n$ open set, and let $J_F(x_0)$ be the Jacobian matrix of F at x_0. Recall that $F(x)$ is called *Gâteaux differentiable at $x_0 \in A$* if for all directions v it yields $\lim_{\lambda \to 0^+} \frac{F(x_0+\lambda v) - F(x_0)}{\lambda} = J_F(x_0)^T v$, while $F(x)$ is called *Fréchet differentiable at $x_0 \in A$* if for all directions v it yields $\lim_{\|v\| \to 0^+} \frac{F(x_0+v) - F(x_0) - J_F(x_0)^T v}{\|v\|} = 0$.

[12] Let $U \in \Re^n$ be any cone; from now on the following notation is used:

$$J_h(x_0)[U] = \{t \in \Re^p : t = J_h(x_0)v, v \in U\}.$$

[13] It is known that the following conditions i), ii) and iii) are equivalent for any set $S \in \Re^n$ (see [18]): i) $0 \in I(S, x_0)$ ii) $x_0 \in \text{Int}(S)$ iii) $I(S, x_0) = \Re^n$

$\forall v \in I_d$; since n linearly independent directions d_i exist in I_d we then have that $J_h(x_0)v = 0\ \forall v \in \Re^n$ which is a contradiction.
ii),iii) Follow directly from the previous result i). □

By means of the above discussed properties it can be easily proved that, given a cone $U \subseteq \Re^n$ and assuming (H_G), condition (C_N) is equivalent to the following one:

(C_G) $\exists \alpha_f \in C^+$, $\exists \alpha_g \in V^+$, $\exists \alpha_h \in \Re^p$, $(\alpha_f, \alpha_g, \alpha_h) \neq 0$, such that:

$$[\alpha_f^T J_f(x_0) + \alpha_g^T J_g(x_0) + \alpha_h^T J_h(x_0)]v \leq 0\ \ \forall v \in \mathrm{Cl}(\mathrm{Co}(U)).$$

Note that (C_G) refers to any direction of $\mathrm{Cl}(\mathrm{Co}(U))$, while (C_N) considers just the directions of $\mathrm{Cl}(U)$. Conditions like (C_G) are known in the literature as "maximum principle conditions".

Theorem 4.3 can now be specified in the differentiable case as follows.

Theorem 5.2 *Consider Problem P, suppose (H_G) holds and let $x_0 \in X$ be a feasible local efficient point. Then for every cone $U \subseteq \Re^n$ verifying an U-regularity condition $\exists \alpha_f \in C^+$, $\exists \alpha_g \in V^+$, $\exists \alpha_h \in \Re^p$, $(\alpha_f, \alpha_g, \alpha_h) \neq 0$, such that:*

$$[\alpha_f^T J_f(x_0) + \alpha_g^T J_g(x_0) + \alpha_h^T J_h(x_0)]v \leq 0\ \ \forall v \in \mathrm{Cl}(\mathrm{Co}(U)).$$

In particular, if $p = 0$ or $J_h(x_0)[\mathrm{Co}(U)] = \Re^p$ then $(\alpha_f, \alpha_g) \neq 0$.

Let us now determine some more U-regularity conditions based on the differentiability of the functions f, g and h. The next trivial ones, not needing of the optimality of x_0, follows directly from (4.2):

i) $\left.\begin{array}{c} p = 0 \text{ or} \\ J_h(x_0)[\mathrm{Co}(U)] = \Re^p \end{array}\right] \Rightarrow K_{\mathrm{Co}(U)} \cap (\mathrm{Int}(C) \times \mathrm{Int}(V) \times 0) = \emptyset,$

ii) $\left.\begin{array}{c} p = 0 \text{ or} \\ J_h(x_0) \text{ is surjective} \end{array}\right] \Rightarrow K_{\mathrm{Co}(U)} \cap (\mathrm{Int}(C) \times \mathrm{Int}(V) \times 0) = \emptyset,$

iii) $K_{\mathrm{Co}(U)} \cap (\mathrm{Int}(C) \times \mathrm{Int}(V) \times 0) = \emptyset,$

Further U-regularity conditions, based on the optimality of x_0, are stated in the next theorem.

Theorem 5.3 *Consider problem P, suppose (H_G) holds and consider also a cone $U \subseteq \Re^n$. The following conditions are U-regularity conditions:*

i) $\left.\begin{array}{c} p = 0 \text{ or} \\ J_h(x_0)[\mathrm{Co}(U)] = \Re^p \end{array}\right] \Rightarrow K_{\mathrm{Co}(U)} \subseteq (K_L - (C \times V \times 0)),$

$$ii) \quad \left.\begin{array}{c} p=0 \text{ or} \\ J_h(x_0)[\mathrm{Co}(U)] = \Re^p \end{array}\right] \Rightarrow \mathrm{Co}(U) \cap \mathrm{Ker}(J_h(x_0)) \subseteq T(X \cap S, x_0),$$

iii) U *is a convex cone and* $U \cap \mathrm{Ker}(J_h(x_0)) \subseteq T(X \cap S, x_0)$*,*

iv) $U = I(X, x_0)$ *is a convex cone and* $T(S, x_0) = \mathrm{Ker}(J_h(x_0))$*,*

where iv)⇒iii)⇒ii)⇒i).

Proof *i)* is an U-regularity condition since it is equivalent to *i)* of Theorem 4.4; *ii)⇒i)* since $\mathrm{Co}(U) \cap \mathrm{Ker}(J_h(x_0)) \subseteq T(X \cap S, x_0)$ implies that $K_{\mathrm{Co}(U)} \subseteq K_L$; *iii)⇒ii)* trivially; *iv)⇒iii)* for Lemma 4.4. □

Some more U-regularity conditions can be found when $T(S, x_0) = Ker_{\partial h}$; with this aim let us recall the following result, which is a generalization of the well known Lyusternik theorem (see for all [21, 22]).

Theorem 5.4 **[21]** *Let* $h : X \to \Re^p$, $X \subseteq \Re^n$, *be a given mapping and let* $x_0 \in S = \{x \in \Re^n : h(x) = 0\}$. *Let also* h *be locally Fréchet differentiable on a neighbourhood of* x_0, *let* $J_h(x)$ *be continuous at* x_0 *and let* $J_h(x_0)$ *be surjective. Then it follows:*

$$T(S, x_0) = \mathrm{Ker}(J_h(x_0)) = \{d \in \Re^n : J_h(x_0)d = 0\} = Ker_{\partial h}$$

From *ii)* of Theorem 5.3 we obtain the following U-regularity conditions.

Theorem 5.5 *Consider problem* P, *suppose* (H_G) *holds and consider a cone* $U \subseteq \Re^n$. *Suppose also that* h *is locally Fréchet differentiable on a neighbourhood of* x_0 *and the Jacobian matrix* $J_h(x)$ *is continuous at* x_0. *The following conditions are* U*-regularity conditions:*

$$i) \quad \left.\begin{array}{c} p=0 \text{ or} \\ J_h(x_0)[\mathrm{Co}(U)] = \Re^p \end{array}\right] \Rightarrow \mathrm{Co}(U) \cap T(S, x_0) \subseteq T(X \cap S, x_0),$$

ii) U *is a convex cone and* $U \cap T(S, x_0) \subseteq T(X \cap S, x_0)$*,*

iii) $U = I(X, x_0)$ *is a convex cone,*

iv) $U = I(X, x_0)$*,* X *is locally convex at* x_0 *([14]),*

v) $U = I(X, x_0)$*,* X *is convex with* $\mathrm{Int}(X) \neq \emptyset$*,*

where v)⇒iv)⇒iii)⇒ii)⇒i).

[14] $X \subseteq \Re^n$ is a locally convex set at x_0 if $\exists I_{x_0}$, arbitrary open ball about x_0, such that $X \cap I_{x_0}$ is convex

Proof Being *v)*⇒*iv)*⇒*iii)*⇒*ii)*⇒*i)* it must be proved just that *i)* is an U-regularity condition. If $J_h(x_0)$ is not surjective, that is $J_h(x_0)[\Re^n] = \text{Co}(Im_{\partial h}(\Re^n)) \neq \Re^p$, the result follows from (4.2); if $J_h(x_0)$ is surjective then for Theorem 5.4 it is $T(S, x_0) = Ker_{\partial h}$ and hence the results follows from *ii)* of Theorem 5.3. □

Finally, it is now shown that some "maximum principle conditions" given in the literature are nothing but particular cases of the results stated in this paper. First it is necessary to recall the following known results.

Theorem 5.6 *Consider problem P with a scalar objective function f, suppose (H_G) holds and assume that $x_0 \in X$ is a local maximizer. Suppose also that h is locally Fréchet differentiable on a neighbourhood of x_0 and the Jacobian matrix $J_h(x)$ is continuous at x_0.*

i) **[21, 23]** *If the following condition holds:*

$$X \text{ is convex, with } \text{Int}(X) \neq \emptyset$$

then $\exists \alpha_f \geq 0$, $\exists \alpha_g \in V^+$, $\exists \alpha_h \in \Re^p$, $(\alpha_f, \alpha_g, \alpha_h) \neq 0$, *such that:*

$$[\alpha_f \nabla f(x_0) + \alpha_g^T J_g(x_0) + \alpha_h^T J_h(x_0)](x - x_0) \leq 0 \quad \forall x \in X$$

and hence:

$$[\alpha_f \nabla f(x_0) + \alpha_g^T J_g(x_0) + \alpha_h^T J_h(x_0)]v \leq 0 \quad \forall v \in F(X, x_0)$$

ii) **[3]** *If the following condition holds* ([15]):

$$I(X, x_0) \text{ is a convex cone}$$

then $\exists \alpha_f \geq 0$, $\exists \alpha_g \in V^+$, $\exists \alpha_h \in \Re^p$, $(\alpha_f, \alpha_g, \alpha_h) \neq 0$, *such that:*

$$[\alpha_f \nabla f(x_0) + \alpha_g^T J_g(x_0) + \alpha_h^T J_h(x_0)]v \leq 0 \quad \forall v \in I(X, x_0)$$

Note that both the previous results are based on a sort of convexity hypothesis regarding to problem P, since the convexity of the set X or of the cone $I(X, x_0)$ is required. In the light of the stated results, these assumptions are nothing but U-regularity conditions, needed in order to have necessary optimality conditions in the decision space.

It is now worth comparing Theorem 5.2 with the results recalled in Theorem 5.6.

- Theorem 5.2 refers to a multiobjective problem, while the results in Theorem 5.6 deal with a scalar objective function;

[15] As it has been pointed out in [3] by its author, if $I(X, x_0) = \emptyset$ the result is trivial.

- case *ii)* of Theorem 5.6 [3] can be obtained by means of Theorem 5.2 using the U-regularity condition *iii)* of Theorem 5.5;

- assuming the U-regularity condition *v)* of Theorem 5.5, the maximum principle condition of Theorem 5.2 holds $\forall v \in T(X, x_0) = \text{Cl}(I(X, x_0))$ ([16]), generalizing case *i)* of Theorem 5.6 where the thesis is verified for a scalar optimization problem $\forall v \in F(X, x_0)$. Note also that this is the first time that we require $\text{Int}(X) \neq \emptyset$.

References

[1] Averbukh V.I. and O.G. Smolyanov, *The theory of differentiation in linear topological spaces*, Russian Mathematical Surveys, vol.22, pp.201-258, 1967.

[2] Averbukh V.I. and O.G. Smolyanov, *The various definitions of the derivative in linear topological spaces*, Russian Mathematical Surveys, vol.23, pp.67-113, 1968.

[3] Bazaraa M.S. and C.M. Shetty, *Foundations of Optimization*, Lecture Notes in Economics and Mathematical Systems, vol.122, Springer-Verlag, 1976.

[4] Bazaraa M.S., Sherali H.D. and C.M. Shetty, *Nonlinear Programming: Theory and Algorithms*, John Wiley & Sons, New York, 1993.

[5] Cambini A. and L. Martein, *Tangent cones in optimization*, in "Generalized Concavity for Economic Applications", edited by P. Mazzoleni, Tecnoprint, Bologna, pp.29-39, 1992.

[6] Cambini A. and L. Martein, *Generalized concavity and optimality conditions in vector and scalar optimization*, in "Generalized Convexity", edited by S. Komlósi, T. Rapcsák and S. Schaible, Lecture Notes in Economics and Mathematical Systems, vol.405, Springer-Verlag, Berlin, pp.337-357, 1994.

[16] Recall (see for all [17, 18]) that if X is locally convex at x_0 then the cones $I(X, x_0)$, $F(X, x_0)$ and $T(X, x_0)$ are convex and:

$$I(X, x_0) = \text{cone}(\text{Int}(X), x_0), \qquad F(X, x_0) = \text{cone}(X, x_0), \qquad T(X, x_0) = \text{Cl}(F(X, x_0)),$$

where $\text{cone}(X, x_0) = \{y : y = \lambda(x - x_0), \lambda \geq 0, x \in X\}$. If X is locally convex at x_0 and $\text{Int}(X) \neq \emptyset$ then:

$$I(X, x_0) = \text{Int}(T(X, x_0)) \qquad \text{and} \qquad T(X, x_0) = \text{Cl}(I(X, x_0)).$$

[7] Cambini A. and L. Martein, *Second order necessary optimality conditions in the image space: preliminary results*, in "Scalar and Vector Optimization in Economic and Financial Problems", edited by E. Castagnoli and G. Giorgi, pp.27-38, 1995.

[8] Cambini A., Martein L. and R. Cambini, *A new approach to second order optimality conditions in vector optimization*, in "Advances in Multiple Objective and Goal Programming", edited by R. Caballero, F. Ruiz and R. Steuer, Lecture Notes in Economics and Mathematical Systems, vol.455, Springer-Verlag, Berlin, pp.219-227, 1997.

[9] Cambini A., Komlósi S. and L. Martein, *Recent developments in second order necessary optimality conditions*, in "Generalized Convexity, Generalized Monotonicity: Recent Results", edited by J.-P. Crouzeix, J.-E. Martinez-Legaz and M. Volle, Nonconvex Optimization and Its Applications, vol.27, Kluwer Academic Publishers, Dordrecht, pp.347-356, 1998.

[10] Cambini R., *Second order optimality conditions in Multiobjective Programming*, Optimization, vol.44, pp.139-160, 1998.

[11] Cambini R., *Generalized concavity and optimality conditions in vector optimization*, in "Operations research and its applications", edited by D.Z. Du, X.S. Zhang and K. Cheng, Lecture Notes in Operations Research, vol.2, World Publishing Corporation, Beijing, pp.172-180, 1996.

[12] Cambini R., *Minimum Principle Type Necessary Optimality Conditions: Preliminary Results*, in "Convessitá e Calcolo Parallelo", edited by G. Giorgi and F.A. Rossi, Libreria Universitaria Editrice, Verona, pp.65-79, 1998.

[13] Cambini R., *Minimum principle type necessary optimality conditions with equality constraints: preliminary results*, in "Generalized Convexity and Optimization for Economic and Financial Decisions", edited by G. Giorgi and F.A. Rossi, Pitagora Editrice, Bologna, pp.63-79, 1999.

[14] Cambini R., *Minimum Principle Type Optimality Conditions*, Report n.180, Department of Statistics and Applied Mathematics, University of Pisa, June 2000.

[15] Canon M., Cullum C. and E. Polak, *Constrained Minimization Problems in Finite-dimensional Spaces*, SIAM Journal on Control, vol.4, pp.528-547, 1966.

[16] Demyanov V.F. and A.M. Rubinov, *Constructive Nonsmooth Analysis*, Peter-Lang, Berlin, 1995.

[17] Giorgi G. and A. Guerraggio, *On the notion of Tangent Cone in Mathematical Programming*, Optimization, vol.25, pp.11-23, 1992.

[18] Giorgi G. and A. Guerraggio, *Approssimazioni Coniche Locali: Proprietá Algebriche e Topologiche*, Technical Report n.14, Istituto di Metodi Quantitativi, Bocconi University of Milan (Italy), 1992.

[19] Halkin H., *A Maximum Principle of the Pontryagin Type for Systems Described by Nonlinear Difference Equations*, SIAM Journal on Control, vol.4, pp.90-111, 1966.

[20] Hestenes M.R., *Optimization Theory: The Finite Dimensional Case*, John Wiley & Sons, New York, 1975.

[21] Jahn J., *Introduction to the Theory of Nonlinear Optimization*, Springer-Verlag, Berlin, 1994.

[22] Lyusternik L.A., *Conditional extrema of functionals*, Math. Sb., vol.41, pp.390-401, 1934.

[23] Mangasarian O.L., *Nonlinear Programming*, McGraw-Hill, New York, 1969.

[24] Mangasarian O.L. and S. Fromovitz, *A Maximum Principle in Mathematical Programming*, in "Mathematical Theory of Control", edited by A.V. Balakrishnan and L.W. Neustadt, Academic Press Inc., New York, pp.85-95, 1967.

[25] Massam H. and S. Zlobec, *Various definitions of the derivative in mathematical programming*, Mathematical Programming, vol.7, pp.144-161, 1974.

[26] Pontryagin L.S., Boltyanskii V.G., Gamkrelidze R.V. and E.F. Mishchenko, *The Mathematical Theory of Optimal Processes*, John Wiley and Sons Inc., New York, 1962.

[27] Sawaragi Y., Nakayama H. and T. Tanino, *Theory of Multiobjective Optimization*, Academic Press, London, 1985.

[28] Shapiro A., *On concepts of directional differentiability*, Journal of Optimization Theory and Applications, vol.66, n.3, pp.477-487, 1990.