Universit degli Studi di Pisa
Dipartimento di Statistica e Matematica
Applicata all Economia

Report n. 213

On generalized convexity of quadratic fractional functions

Riccardo Cambini — Laura Carosi

Pisa, Ottobre 2001

- Stampato in Proprio -

Via Cosimo Ridolfi, 10 – 56124 PISA – Tel. Segr. Amm. 050 945231 Segr. Stud. 050 945317 Fax 050 945375
Cod. Fisc. 80003670504 - P. IVA 00286820501 - Web http://statmat.ec.unipi.it/ - E-mail: dipstat@ec.unipi.it

On generalized convexity of quadratic fractional functions.

Riccardo Cambini, Laura Carosi*
Department of Statistics and Applied Mathematics
University of Pisa

Abstract

In this paper the generalized convexity of quadratic fractional functions is studied. It is proved that, for this class of functions, pseudoconvexity is equivalent to quasiconvexity and some characterizations for both pseudoconvexity and strict pseudoconvexity are given. Furthermore, the stated results are specialized to some particular classes of quadratic fractional functions, obtaining conditions that can be easily checked.

Keywords Generalized Convexity, Fractional Programming, Quadratic Programming
AMS - 2000 Math. Subj. Class. 90C26, 90C32, 90C20
JEL - 1999 Class. Syst. C61

1 Introduction

Many problems in Management Science, such as maximization of productivity, maximization of return of investment, can be seen as applications of fractional programming; whenever we have a problem that describes some kind of efficiency measure of a system, we can formulate it as a fractional program. In particular, several applicative problems (Portfolio Theory, Risk Theory, Location Models) can be seen as quadratic fractional programs (see for example [2, 3]), that is problems where the objective function is the ratio of a quadratic and an affine one. For this reason, many papers about fractional programming have been published in the last decades and both theoretical and algorithmic point of views have been handled (see for all [1, 17]);

*The paper has been discussed jointly by the authors. In particular, sections 1 and 4 have been developed by Laura Carosi while sections 2 and 3 by Riccardo Cambini.

several of these results deal with the generalized convexity of the objective function, since this property plays a key role in the study of minimization problems (see for example [4, 5, 6, 7, 8, 14]).

In this paper we aim to analyze the various generalized convexity properties of quadratic fractional functions, studying them in a unified framework. We first deep on some results by Crouzeix [12] concerning the inertia of symmetric matrices and then we give a characterization of the pseudoconvexity based on the inertia of the quadratic form and the behavior of the gradient of the function. This result allows us also to prove that, for quadratic fractional functions, pseudoconvexity is equivalent to quasiconvexity.

The obtained necessary and sufficient conditions are then improved in order to state characterizations directly based on the elements which define the function; in this way the given conditions become easier to be checked.

Analogous results are also provided for the strict pseudoconvexity, which in particular comes out to be equivalent to strict quasiconvexity.

Finally, we specialize the obtained results for quadratic fractional functions whose quadratic form is the product of two affine ones.

2 Preliminary Results

In the next section the generalized convexity of quadratic fractional functions will be characterized using the inertia of symmetric matrices. With this regards, from now on the number of the negative eigenvalues of a symmetric matrix Q is denoted by $\nu_-(Q)$, similarly $\nu_+(Q)$ represents the number of the positive eigenvalues while $\nu_0(Q)$ is the algebraic multiplicity of the 0 eigenvalue. A key tool in our study is the following result given by Crouzeix (see [12]).

Theorem 1 *Let $h \in \Re^n$, $h \neq 0$, let $Q \in \Re^{n\times n}$ be a symmetric matrix and denote with $Q^\sharp$ the Moore-Penrose pseudoinverse matrix of Q ([1]). Then the following implication*

$$h^T v = 0 \quad \Rightarrow \quad v^T Q v \geq 0$$

is verified $\forall v \in \Re^n$ if and only if one of the following conditions holds:

i) $\nu_-(Q) = 0$,

ii) $\nu_-(Q) = 1$, $h \in Q(\Re^n)$ *and* $h^T Q^\sharp h \leq 0$.

[1]Let $Q \in \Re^{n\times n}$ be a symmetric matrix. The Moore-Penrose pseudoinverse matrix of Q is a matrix $Q^\sharp \in \Re^{n\times n}$ such that $QQ^\sharp Q = Q$.

The assumption $h \neq 0$ in Theorem 1 leads to some technical difficulties in the application of the theorem itself to particular problems where the vector h is not necessarily different from zero. For this reason we state the following corollary which improves the result by Crouzeix [12] not requiring in the assumptions that the vector h is different from zero.

Corollary 1 *Let $h \in \Re^n$ and let $Q \in \Re^{n\times n}$ be a symmetric matrix. Then the following implication*

$$h^T v = 0 \quad \Rightarrow \quad v^T Q v \geq 0 \tag{1}$$

is verified $\forall v \in \Re^n$ if and only if one of the following conditions holds:

i) $\nu_-(Q) = 0$,

ii) $\nu_-(Q) = 1$, $h \neq 0$, $h \in Q(\Re^n)$ *and* $u^T Q u \leq 0$ $\forall u \in \Re^n$ *s.t.* $Qu = h$.

Proof. *⇐)* If $\nu_-(Q) = 0$ then (1) is trivially verified; if *ii)* holds then the results follows from Theorem 1 since condition

$$u^T Q u \leq 0 \ \forall u \in \Re^n \text{ such that } Qu = h$$

implies $h^T Q^\sharp h \leq 0$.

⇒) If $\nu_-(Q) = 0$ then *i)* holds. Let us now assume $\nu_-(Q) \neq 0$ and suppose by contradiction $h = 0$; then $h^T v = 0$ $\forall v \in \Re^n$ and hence for condition (1) $v^T Q v \geq 0$ $\forall v \in \Re^n$, that is to say that Q is positive semidefinite which is a contradiction being $\nu_-(Q) \neq 0$. Hence it yields that $h \neq 0$ and *ii)* follows from *ii)* of Theorem 1. ■

The following result by Crouzeix [12] will be also useful in the development of the paper.

Theorem 2 *Let $h \in \Re^n$, $h \neq 0$, and let $Q \in \Re^{n\times n}$ be a symmetric matrix. Then the following implication*

$$h^T v = 0, v \neq 0 \quad \Rightarrow \quad v^T Q v > 0$$

is verified $\forall v \in \Re^n$ if and only if one of the following conditions holds:

i) $\nu_+(Q) = n$,

ii) $\nu_-(Q) = 0$, $\nu_+(Q) = n - 1$ *and* $h \notin Q(\Re^n)$,

iii) $\nu_-(Q) = 1$, $\nu_+(Q) = n - 1$ *and* $h^T Q^{-1} h < 0$.

Theorem 2 can be slightly improved too, showing that the assumption $h \neq 0$ is redundant.

Corollary 2 *Let $h \in \Re^n$ and let $Q \in \Re^{n\times n}$ be a symmetric matrix. Then the following implication*

$$h^T v = 0, v \neq 0 \quad \Rightarrow \quad v^T Q v > 0 \tag{2}$$

is verified $\forall v \in \Re^n$ if and only if one of the following conditions holds:

i) $\nu_+(Q) = n$,

ii) $\nu_-(Q) = 0$, $\nu_+(Q) = n-1$ *and* $h \notin Q(\Re^n)$,

iii) $\nu_-(Q) = 1$, $\nu_+(Q) = n-1$ *and* $u^T Q u < 0$ *where* $u = Q^{-1}h$.

Proof. $\Leftarrow$) If $\nu_+(Q) = n$ then (2) is trivially verified; if *ii)* holds then the results follows from Theorem 2 since condition $h \notin Q(\Re^n)$ implies $h \neq 0$; if *iii)* holds then the results follows again from Theorem 2 since condition $u^T Q u < 0$ where $u = Q^{-1}h$ implies $h \neq 0$ and $h^T Q^{-1} h < 0$.

$\Rightarrow$) If $\nu_+(Q) = n$ then *i)* holds. Let us now assume $\nu_+(Q) \leq n-1$ and suppose by contradiction $h = 0$; then $h^T v = 0$ $\forall v \in \Re^n$ and hence for condition (2) $v^T Q v > 0$ $\forall v \in \Re^n \setminus \{0\}$, that is to say that Q is positive definite which is a contradiction being $\nu_+(Q) \leq n-1$. Hence it yields that $h \neq 0$ and *ii),iii)* follows from *ii),iii)* of Theorem 1, respectively. ■

3 Generalized Convexity

The aim of this section is to study the generalized convexity of quadratic fractional functions of the following kind:

$$f(x) = \frac{\frac{1}{2}x^T Q x + q^T x + q_0}{b^T x + b_0} \tag{3}$$

defined on the set $X = \{x \in \Re^n : b^T x + b_0 > 0\}$, where $Q \neq 0$ is a $n \times n$ symmetric matrix, with $n \geq 2$, $q, x, b \in \Re^n$, $b \neq 0$, and $q_0, b_0 \in \Re$. Note that being Q symmetric, it is $Q \neq 0$ if and only if $\nu_0(Q) \leq n-1$.

Remark 1 It is important to point out that function f in (3) is not constant. Suppose by contradiction that f is constant, that is $f(x) = k$ and $\nabla f(x) = \frac{Qx+q-f(x)b}{b^T x + b_0} = 0$ $\forall x \in X$. Consider an arbitrary $x_1 \in X$ and let $\alpha \in \Re$ be such that $\alpha \neq 0$ and $\alpha x_1 \in X$; it results $Qx_1 + q - kb = Q\alpha x_1 + q - kb$ and hence $Qx_1 = \alpha Q x_1$ which implies $Qx_1 = 0$. Being $x_1 \in X$ arbitrary it results $Qx = 0$ $\forall x \in X$. Consequently, since X is an n-dimensional halfspace it is $Q = 0$, which contradicts the definition of (3).

The following known characterizations of generalized convex functions, given by Diewert, Avriel and Zang [13], will play a key role in our study.

Theorem 3 *Let f be a differentiable function defined on the open convex set $C \subset \Re^n$. Then:*

i) f is quasiconvex if and only if $\forall x \in C$, $\forall v \in \Re^n \setminus \{0\}$, such that $\nabla f(x)^T v = 0$ the function $\phi_v(t) = f(x+tv)$ does not attain a semistrict local maximum at $t = 0$ ([2]);

ii) f is strictly quasiconvex if and only if $\forall x \in C$, $\forall v \in \Re^n \setminus \{0\}$, such that $\nabla f(x)^T v = 0$ the function $\phi_v(t) = f(x+tv)$ does not attain a local maximum at $t = 0$.

Suppose function f to be continuously differentiable, then:

iii) f is [strictly] pseudoconvex if and only if $\forall x \in C$, $\forall v \in \Re^n \setminus \{0\}$, such that $\nabla f(x)^T v = 0$ the function $\phi_v(t) = f(x+tv)$ attains a [strict] local minimum at $t = 0$.

By means of the results by Diewert, Avriel and Zang [13] and Corollaries 1 and 2 it is now possible to prove the following characterizations of [strictly] pseudoconvex and [strictly] quasiconvex quadratic fractional functions.

Theorem 4 *Consider function f defined in (3). Then the following conditions are equivalent:*

i) f is pseudoconvex on X,

ii) f is quasiconvex on X,

iii) $\forall x \in X$ $\forall v \in \Re^n \setminus \{0\}$ it holds:

$$\nabla f(x)^T v = 0 \quad \Rightarrow \quad v^T Q v \geq 0,$$

iv) one of the following conditions holds:

---

[2] Let f be defined on the open interval $(a,b) \subset \Re$. Then f is said to attain a *semistrict local maximum* at a point $x_0 \in (a,b)$ if there exists two points $x_1, x_2 \in (a,b)$, $x_1 < x_0 < x_2$, such that

$$f(x_0) \geq f(x_2 + \lambda(x_1 - x_2)) \quad \forall \lambda \in [0,1]$$

and $f(x_0) > \min\{f(x_1), f(x_2)\}$.

a) $\nu_-(Q) = 0$,

b) $\nu_-(Q) = 1$ *and for all* $x \in X$ *it is:* $\nabla f(x) \neq 0$, $\nabla f(x) \in Q(\Re^n)$ *and* $u^T Q u \leq 0$ $\forall u \in \Re^n$ *such that* $Qu = \nabla f(x)$.

Proof. First note that $\forall t \in \Re$ such that $x + tv \in X$ it is

$$\begin{aligned} \nabla f(x) &= \frac{Qx + q - f(x)b}{b^T x + b_0}, \\ \phi_v(t) &= f(x+tv) = f(x) + \frac{\frac{1}{2}t^2 v^T Q v + t(b^T x + b_0)\nabla f(x)^T v}{b^T x + b_0 + t b^T v}, \end{aligned}$$

and hence

$$\nabla f(x)^T v = 0 \quad \Rightarrow \quad \phi_v(t) = f(x) + \frac{\frac{1}{2}t^2 v^T Q v}{b^T(x+tv) + b_0}.$$

i)$\Rightarrow$*ii)* Trivial.

ii)$\Rightarrow$*iii)* By means of Theorem 3 when $\nabla f(x)^T v = 0$ the restriction

$$\phi_v(t) = f(x) + \frac{\frac{1}{2}t^2 v^T Q v}{b^T(x+tv) + b_0}$$

does not attain a semistrict local maximum at $t = 0$ and this is true only if $v^T Q v \geq 0$ since $x + tv \in X$ implies $b^T(x+tv) + b_0 > 0$.

iii)$\Rightarrow$*i)* When $\nabla f(x)^T v = 0$ it is

$$\phi_v(t) = f(x) + \frac{\frac{1}{2}t^2 v^T Q v}{b^T(x+tv) + b_0} \quad \text{with} \quad v^T Q v \geq 0,$$

hence $\phi_v(t)$ attains a local minimum at $t = 0$ since $x + tv \in X$ implies $b^T(x+tv) + b_0 > 0$. The result then follows from Theorem 3.

iii)$\Leftrightarrow$*iv)* For Corollary 1 condition *iii)* holds if and only if for any $x \in X$ one of the following conditions holds:

- $\nu_-(Q) = 0$,
- $\nu_-(Q) = 1$, $\nabla f(x) \neq 0$, $\nabla f(x) \in Q(\Re^n)$ and $u^T Q u \leq 0$ $\forall u \in \Re^n$ such that $Qu = \nabla f(x)$.

The result then follows noticing that Q is independent from x and the two previous cases are disjoint. ■

Theorem 5 *Consider function f defined in (3). Then the following conditions are equivalent:*

i) f is strictly pseudoconvex on X,

ii) f is strictly quasiconvex on X,

iii) the following implication holds $\forall x \in X \ \forall v \in \Re^n \setminus \{0\}$:

$$\nabla f(x)^T v = 0 \quad \Rightarrow \quad v^T Q v > 0,$$

iv) one of the following conditions holds:

a) $\nu_+(Q) = n$,
b) $\nu_-(Q) = 0$, $\nu_+(Q) = n-1$ and $\nabla f(x) \notin Q(\Re^n) \ \forall x \in X$,
c) $\nu_-(Q) = 1$, $\nu_+(Q) = n-1$ and $u^T Q u < 0$, with $u = Q^{-1}\nabla f(x)$, $\forall x \in X$.

Proof. The result follows from Theorem 3 and Corollary 2, in a similar way to the one used in the proof of Theorem 4. ■

It can be easily seen that the characterizations provided in Theorems 4 and 5 are not so easy to be verified, since they are based on the behavior of $\nabla f(x)$ and not on the elements which define the objective function (that is Q, q, q_0, b, b_0). For this reason, we now prove the following lemmas which will allow us to state some more characterizations easier to be checked.

Lemma 1 *Consider function f defined in (3). Then:*

$$\nabla f(x) \in Q(\Re^n) \ \forall x \in X \quad \Leftrightarrow \quad \{b, q\} \subset Q(\Re^n)$$

Proof. Suppose that

$$\nabla f(x) = \frac{Qx + q - f(x)b}{b^T x + b_0} \in Q(\Re^n) \ \forall x \in X$$

and let us prove that $\{b, q\} \subset Q(\Re^n)$, that is to say that

$$\exists \overline{x}, \overline{y} \in \Re^n \text{ such that } Q\overline{x} = q \text{ and } Q\overline{y} = b.$$

Since f is not constant (see Remark 1), $\exists x_1, x_2 \in X$ such that $f(x_1) \neq f(x_2)$ and hence $\exists u_1, u_2 \in \Re^n$ such that

$$Qu_1 = Qx_1 + q - f(x_1)b \quad \text{and} \quad Qu_2 = Qx_2 + q - f(x_2)b.$$

This implies that

$$Q\left(\frac{u_1-u_2-x_1+x_2}{f(x_2)-f(x_1)}\right)=b$$

and hence $\exists\overline{y}\in\Re^n$ such that $Q\overline{y}=b$. It follows also that $Qu_1=Qx_1+q-f(x_1)Q\overline{y}$ which implies $q=Q(u_1-x_1+f(x_1)\overline{y})$ and hence $\exists\overline{x}\in\Re^n$ such that $Q\overline{x}=q$.

Suppose now that $\{b,q\}\subset Q(\Re^n)$, that is to say that $\exists\overline{x},\overline{y}\in\Re^n$ such that $Q\overline{x}=q$ and $Q\overline{y}=b$; then

$$\nabla f(x)=Q\left(\frac{x+\overline{x}-f(x)\overline{y}}{b^Tx+b_0}\right),$$

and hence $\nabla f(x)\in Q(\Re^n)\ \forall x\in X$. ■

Lemma 2 *Consider function f defined in (3), suppose $\nabla f(x)\in Q(\Re^n)\ \forall x\in X$ and let $\overline{x},\overline{y}\in\Re^n$ such that $Q\overline{x}=q$ and $Q\overline{y}=b$. Then for any given $x\in X$:*

i) $Qu=\nabla f(x)$ *if and only if* $u=\frac{x+\overline{x}-f(x)\overline{y}}{b^Tx+b_0}+k$ *with* $k\in\ker(Q)$,

ii) $b^Tx=b^T\overline{x}$ *and* $q^Tx=q^T\overline{x}$ *for all* $x\in\Re^n$ *such that* $Qx=q$,

iii) $b^Ty=b^T\overline{y}$ *and* $q^Ty=q^T\overline{y}$ *for all* $y\in\Re^n$ *such that* $Qy=b$,

iv) $u^TQu=\frac{p(x)}{(b^Tx+b_0)^2}$ *with:*

$$p(x)=(f(x))^2\,b^T\overline{y}+2f(x)\left[b_0-b^T\overline{x}\right]+\left(q^T\overline{x}-2q_0\right) \tag{4}$$

v) *if* $\nu_-(Q)=1$ *and* $\nu_+(Q)=0$ *then* $b^T\overline{y}<0$.

Proof. *i)* From Lemma 1 we have

$$\nabla f(x)=Q\left(\frac{x+\overline{x}-f(x)\overline{y}}{b^Tx+b_0}\right),$$

so that $Qu=\nabla f(x)$ if and only if

$$Q\left(u-\frac{x+\overline{x}-f(x)\overline{y}}{b^Tx+b_0}\right)=0$$

and this happens if and only if

$$\left(u-\frac{x+\overline{x}-f(x)\overline{y}}{b^Tx+b_0}\right)=k\in\ker(Q).$$

ii) Since $Qx = Q\overline{x} = q$ it is $Q(x-\overline{x}) = 0$ hence $x = \overline{x}+k$ with $k \in \ker(Q)$; the result then follows being $b^T k = \overline{y}^T Qk = 0 = \overline{x}^T Qk = q^T k \ \forall k \in \ker(Q)$.

iii) Analogous to *ii)*.

iv) Just note that:

$$\begin{aligned} u^T Qu &= \frac{1}{(b^T x + b_0)^2}\left[(x+\overline{x}-f(x)\overline{y})^T Q(x+\overline{x}-f(x)\overline{y})\right] \\ &= \frac{1}{(b^T x + b_0)^2}\left[(x+\overline{x}-f(x)\overline{y})^T (Qx+q-f(x)b)\right] \\ &= \frac{(f(x))^2 b^T\overline{y} + 2f(x)(b_0 - b^T\overline{x}) + (q^T\overline{x} - 2q_0)}{(b^T x + b_0)^2}. \end{aligned}$$

v) If $\nu_-(Q) = 1$ and $\nu_+(Q) = 0$ then Q can be written as $Q = [2\alpha a a^T]$ with $\alpha < 0$, eigenvalue of Q, and $a \in \Re^n$, eigenvector corresponding to α. Since $b \in Q(\Re^n)$ and $b \neq 0$, there exists $\beta \neq 0$ such that $b = \beta a$. Moreover $\overline{y} = \frac{\beta}{2\alpha\|a\|^2}a + k$ where $k \in \ker(Q)$. Consequently $a^T k = 0$ and hence $b^T\overline{y} = (\beta a)^T\left(\frac{\beta}{2\alpha\|a\|^2}a + k\right) = \frac{\beta^2}{2\alpha} < 0$. ∎

Lemma 3 *Consider function f defined in (3) and suppose $Q\overline{x} = q$ and $Q\overline{y} = b$. Then the following statements are equivalent:*

i) $\nabla f(x) \neq 0 \ \forall x \in X$;

ii) $\nexists \mu \in \Re$ *such that* $\left\{\begin{array}{l} \mu b^T\overline{y} - b^T\overline{x} + b_0 > 0 \\ \mu^2 b^T\overline{y} + 2\mu(b_0 - b^T\overline{x}) + (q^T\overline{x} - 2q_0) = 0 \end{array}\right.$;

iii) one of the following conditions holds:

a) $b^T\overline{y} = 0$ *and* $b^T\overline{x} \geq b_0$,

b) $b^T\overline{y} \neq 0$ *and* $\frac{\Delta}{4} = (b_0 - b^T\overline{x})^2 - b^T\overline{y}(q^T\overline{x} - 2q_0) \leq 0$.

Proof. First note that, being $q^T\overline{y} = b^T\overline{x}$, if $Qx = \mu b - q$ then:

$$\begin{aligned} \frac{1}{2}x^T Qx + \mu b_0 &= \frac{1}{2}\left[(Qx)^T x + 2\mu b_0\right] = \frac{1}{2}\left[(\mu b - q)^T x + 2\mu b_0\right] = \\ &= \frac{1}{2}\left[(\mu\overline{y} - \overline{x})^T Qx + 2\mu b_0\right] = \\ &= \frac{1}{2}\left[(\mu\overline{y} - \overline{x})^T(\mu b - q) + 2\mu b_0\right] = \\ &= \frac{1}{2}\left[\mu^2 b^T\overline{y} + 2\mu(b_0 - b^T\overline{x}) + q^T\overline{x}\right] \qquad (5) \end{aligned}$$

$i) \Rightarrow ii)$ Suppose by contradiction that $\exists \mu_* \in \Re$ such that $b^T(\mu_*\overline{y}-\overline{x})+b_0 > 0$ and $\frac{1}{2}\left[\mu_*^2 b^T\overline{y} + 2\mu_*(b_0 - b^T\overline{x}) + q^T\overline{x}\right] = q_0$. Defined $x_* = \mu_*\overline{y} - \overline{x}$ it results $b^T x_* + b_0 > 0$, hence $x_* \in X$, and $Qx_* = Q(\mu_*\overline{y} - \overline{x}) = \mu_* b - q$; then for equation (5) it is $q_0 = \frac{1}{2}x_*^T Q x_* + \mu_* b_0$ and hence

$$\begin{aligned} f(x_*) &= \frac{\frac{1}{2}x_*^T Q x_* + q^T x_* + q_0}{b^T x_* + b_0} = \\ &= \frac{\frac{1}{2}x_*^T Q x_* + (\mu_* b - Qx_*)^T x_* + (\frac{1}{2}x_*^T Q x_* + \mu_* b_0)}{b^T x_* + b_0} = \\ &= \frac{\mu_*(b^T x_* + b_0)}{b^T x_* + b_0} = \mu_* \end{aligned}$$

then $\nabla f(x_*) = \frac{Qx_* + q - f(x_*)b}{b^T x_* + b_0} = \frac{Qx_* + q - \mu_* b}{b^T x_* + b_0} = 0$ which is a contradiction.
$ii) \Rightarrow i)$ Suppose by contradiction that $\exists x_* \in X$ such that

$$\nabla f(x_*) = \frac{Qx_* + q - f(x_*)b}{b^T x_* + b_0} = 0,$$

then it is $Qx_* + q - f(x_*)b = 0$ since $b^T x_* + b_0 > 0$. Defined $\mu_* = f(x_*)$ it results $Q(x_* + \overline{x} - \mu_*\overline{y}) = 0$ and hence $x_* = \mu_*\overline{y} - \overline{x} + k$ where $k \in \ker(Q)$. It then results $0 < b^T x_* + b_0 = b^T(\mu_*\overline{y} - \overline{x}) + b^T k + b_0$ so that, being $b^T k = \overline{y}^T Q k = 0$, it is

$$\mu_* b^T\overline{y} - b^T\overline{x} + b_0 > 0, \quad Qx_* = \mu_* b - q \quad \text{and} \quad \mu_* = f(x_*)$$

Note that:

$$\begin{aligned} \mu_* &= f(x_*) = \frac{\frac{1}{2}x_*^T Q x_* + q^T x_* + q_0}{b^T x_* + b_0} = \\ &= \frac{\frac{1}{2}x_*^T Q x_* + (\mu_* b - Qx_*)^T x_* + q_0 + \mu_* b_0 - \mu_* b_0}{b^T x_* + b_0} = \\ &= \mu_* - \frac{(\frac{1}{2}x_*^T Q x_* + \mu_* b_0) - q_0}{b^T x_* + b_0} \end{aligned}$$

hence $\frac{1}{2}x_*^T Q x_* + \mu_* b_0 - q_0 = 0$ since $b^T x_* + b_0 > 0$. From equation (5) it then results that

$$\mu_* b^T\overline{y} - b^T\overline{x} + b_0 > 0 \quad \text{and} \quad \frac{1}{2}\left[\mu_*^2 b^T\overline{y} + 2\mu_*(b_0 - b^T\overline{x}) + (q^T\overline{x} - 2q_0)\right] = 0$$

and this is a contradiction.
$ii) \Rightarrow iii)$ In the case $b^T\overline{y} = 0$ condition $ii)$ can be specified as follows

$$\nexists \mu \in \Re \text{ such that } \begin{cases} b_0 > b^T\overline{x} \\ 2\mu(b_0 - b^T\overline{x}) + (q^T\overline{x} - 2q_0) = 0 \end{cases},$$

and this happens only if $b_0 \leq b^T\overline{x}$.

Consider now the case $b^T\overline{y} \neq 0$ and suppose by contradiction that $\frac{\Delta}{4} > 0$. Then $\mu^2 b^T\overline{y} + 2\mu(b_0 - b^T\overline{x}) + (q^T\overline{x} - 2q_0) = 0$ for $\mu_{1,2} = \frac{-b_0 + b^T\overline{x} \pm \sqrt{\frac{\Delta}{4}}}{b^T\overline{y}}$ and hence either μ_1 or μ_2 satisfies $\mu b^T\overline{y} - b^T\overline{x} + b_0 > 0$ which is a contradiction. *iii)* $\Rightarrow$ *ii)* If case a) occurs the result trivially follows. Suppose now that condition b) holds; if $\frac{\Delta}{4} < 0$ $\mu^2 b^T\overline{y} + 2\mu(b_0 - b^T\overline{x}) + (q^T\overline{x} - 2q_0) = 0$ is never verified while when $\frac{\Delta}{4} = 0$ the unique solution $\mu = \frac{-b_0 + b^T\overline{x}}{b^T\overline{y}}$ does not verify $\mu b^T\overline{y} - b^T\overline{x} + b_0 > 0$. ■

Lemma 4 *Consider function f defined in (3) and suppose $Q\overline{x} = q$, $Q\overline{y} = b$, $\nu_-(Q) = 1$ and $\nu_+(Q) > 0$.*

i) Condition

$$u^T Q u \leq 0 \; [< 0] \; \forall u \in \Re^n \text{ such that } Qu = \nabla f(x) \tag{6}$$

is verified $\forall x \in X$ if and only if one of the followings holds:

a) $b^T\overline{y} < 0$ and $\frac{\Delta}{4} = (b_0 - b^T\overline{x})^2 - b^T\overline{y}(q^T\overline{x} - 2q_0) \leq 0$ $[< 0]$;

b) $b^T\overline{y} = 0$, $b^T\overline{x} = b_0$ and $q^T\overline{x} \leq 2q_0$ $[< 2q_0]$;

ii) if condition (6) holds $\forall\, x \in X$ then $\nabla f(x) \neq 0$ $\forall x \in X$.

Proof. First recall that, from Lemma 2, $u^T Q u = \frac{p(x)}{(b^T x + b_0)^2}$ with:

$$p(x) = (f(x))^2 b^T\overline{y} + 2f(x)\left[b_0 - b^T\overline{x}\right] + \left(q^T\overline{x} - 2q_0\right)$$

Let $\lambda_1 > 0, \lambda_2 < 0$ be eigenvalues of Q with $x_1, x_2 \in X$ corresponding eigenvectors. Then $x_1^T Q x_1 = \lambda_1 \|x_1\|^2 > 0$ and $x_2^T Q x_2 = \lambda_2 \|x_2\|^2 < 0$. This implies that $\sup_{x\in X} f(x) = +\infty$ and $\inf_{x \in X} f(x) = -\infty$.

i) Consider the case $b^T\overline{y} < 0$. From the above facts, it is $u^T Q u \leq 0$ $[< 0]$ $\forall u \in \Re^n$ such that $Qu = \nabla f(x)$ if and only if

$$\mu^2 b^T\overline{y} + 2\mu\left[b_0 - b^T\overline{x}\right] + \left(q^T\overline{x} - 2q_0\right) \leq 0 \; [< 0] \; \forall \mu \in \Re,$$

and this happens if and only if $\frac{\Delta}{4} \leq 0$ $[< 0]$. Consider now the case $b^T\overline{y} = 0$. Then $u^T Q u \leq 0$ $[< 0]$ $\forall u \in \Re^n$ such that $Qu = \nabla f(x)$ is equivalent to

$$2\mu\left[b_0 - b^T\overline{x}\right] + \left(q^T\overline{x} - 2q_0\right) \leq 0 \; [< 0] \; \forall \mu \in \Re$$

and this holds if and only if $b^T\overline{x} = b_0$ and $q^T\overline{x} \leq 2q_0$ [$< 2q_0$]. Finally, whenever $u^TQu \leq 0$ [< 0] $\forall u \in \Re^n$ such that $Qu = \nabla f(x)$ it cannot be $b^T\overline{y} > 0$ since otherwise

$$\mu^2 b^T\overline{y} + 2\mu\left[b_0 - b^T\overline{x}\right] + \left(q^T\overline{x} - 2q_0\right) > 0$$

for some $\mu \in \Re$ large enough. The proof is now complete.

ii) Follows trivially from *i)* and Lemma 3. ∎

The previous lemmas allow us to state the following further characterizations of pseudoconvex and strictly pseudoconvex quadratic fractional functions.

Corollary 3 *Consider function f defined in (3). Then the following properties hold:*

i) f is pseudoconvex (quasiconvex) on X if and only if one of the following conditions holds:

a) $\nu_-(Q) = 0$,

b) $\nu_-(Q) = 1$, $\exists \overline{x}, \overline{y} \in \Re^n$ such that $Q\overline{x} = q$ and $Q\overline{y} = b$, $b^T\overline{y} = 0$, $b^T\overline{x} = b_0$ and $q^T\overline{x} \leq 2q_0$;

c) $\nu_-(Q) = 1$, $\exists \overline{x}, \overline{y} \in \Re^n$ such that $Q\overline{x} = q$ and $Q\overline{y} = b$, $b^T\overline{y} < 0$ and $\frac{\Delta}{4} = (b_0 - b^T\overline{x})^2 - b^T\overline{y}(q^T\overline{x} - 2q_0) \leq 0$.

ii) f is strictly pseudoconvex (strictly quasiconvex) on X if and only if one of the following conditions holds:

a) $\nu_+(Q) = n$,

b) $\nu_-(Q) = 0$, $\nu_+(Q) = n - 1$ and $\{b, q\} \not\subset Q(\Re^n)$,

c) $\nu_-(Q) = 1$, $\nu_+(Q) = n - 1$ and $b^T\overline{y} = 0$, $b^T\overline{x} = b_0$, $q^T\overline{x} < 2q_0$ where $\overline{x} = Q^{-1}q$ and $\overline{y} = Q^{-1}b$;

d) $\nu_-(Q) = 1$, $\nu_+(Q) = n - 1$ and $b^T\overline{y} < 0$, $\frac{\Delta}{4} < 0$ where $\overline{x} = Q^{-1}q$ and $\overline{y} = Q^{-1}b$.

Proof. *i)* By means of Theorem 4, function f is pseudoconvex if and only if one of the followings is verified:

1) $\nu_-(Q) = 0$,

2) $\nu_-(Q) = 1$ and for all $x \in X$ it is: $\nabla f(x) \neq 0$, $\nabla f(x) \in Q(\Re^n)$ and $u^TQu \leq 0$ $\forall u \in \Re^n$ such that $Qu = \nabla f(x)$.

From Lemma 1 it is:

$$\nabla f(x) \in Q(\Re^n)\ \forall x \in X \quad \Leftrightarrow \quad \exists \overline{x}, \overline{y} \in \Re^n \text{ such that } Q\overline{x} = q \text{ and } Q\overline{y} = b$$

while from Lemma 4 condition (6) is verified for any $x \in X$ if and only if one of the following conditions holds:

- $b^T\overline{y} = 0$, $b^T\overline{x} = b_0$ and $q^T\overline{x} \leq 2q_0$;
- $b^T\overline{y} < 0$ and $\frac{\Delta}{4} \leq 0$;

The result then follows from *ii)* of Lemma 4.

ii) By means of Theorem 5, function f is strictly pseudoconvex if and only if one of the followings is verified:

1) $\nu_+(Q) = n$,
2) $\nu_-(Q) = 0$, $\nu_+(Q) = n - 1$ and $\nabla f(x) \notin Q(\Re^n)\ \forall x \in X$,
3) $\nu_-(Q) = 1$, $\nu_+(Q) = n - 1$ and $u^TQu < 0$, with $u = Q^{-1}\nabla f(x)$, $\forall x \in X$.

The result follows since for Lemma 1 it is:

$$\exists x \in X \text{ such that } \nabla f(x) \notin Q(\Re^n) \quad \Leftrightarrow \quad \{b, q\} \not\subset Q(\Re^n)$$

while from Lemma 4 for any $x \in X$ it results $u^TQu < 0\ \forall u \in \Re^n$ such that $Qu = \nabla f(x)$ if and only if one of the following conditions holds:

- $b^T\overline{y} = 0$, $b^T\overline{x} = b_0$ and $q^T\overline{x} < 2q_0$;
- $b^T\overline{y} < 0$ and $\frac{\Delta}{4} < 0$;

where $\overline{x} = Q^{-1}q$ and $\overline{y} = Q^{-1}b$. ■

Note that *i)* of Corollary 3 has been already proved in [6] in a different way. Note also that a complete study of the pseudoaffinity of quadratic fractional functions has been given in [9].

4 Particular cases

In this section we deal with a quadratic fractional function f whose quadratic form is the product of two affine ones. We aim to specialize the characterizations given in Corollary 3, in order to determine conditions which allow us to easily recognize the pseudoconvexity (strict pseudoconvexity) of f. More precisely we consider the following class of functions

$$f(x) = \frac{(a^T x + a_0)(c^T x + c_0) + (d^T x + d_0)}{b^T x + b_0} \tag{7}$$

on the set $X = \{x \in \Re^n : b^T x + b_0 > 0\}$, where $a, b, c, d, x \in \Re^n$, $a, b, c \neq 0$, $n \geq 2$, and $a_0, b_0, c_0, d_0 \in \Re$. Observe that when $a = 0$ or $c = 0$, f becomes a linear fractional function which is known to be pseudolinear (see for example [4], [8]), while when $b = 0$ f is a quadratic function whose properties are very well known.

4.1 a and c linearly dependent

Before dealing with the general case, we first deep on the pseudoconvexity of function f, given in (7), when the vectors a and c are linearly dependent (i.e. $c = ka$ with $k \neq 0$ since $a, c \neq 0$), that is

$$f(x) = \frac{(a^T x + a_0)(ka^T x + c_0) + (d^T x + d_0)}{b^T x + b_0} \tag{8}$$

where $a, b \neq 0$ and $k \neq 0$. It is worth noticing that we can rewrite function f in (8) in the form (3) with

$$Q = 2kaa^T, \qquad q = a(a_0 k + c_0) + d, \qquad q_0 = a_0 c_0 + d_0.$$

Matrix Q is a $n \times n$ symmetric matrix ($n \geq 2$) which results to have $\nu_0(Q) = n - 1$ (since $\dim(\ker(Q)) = n - 1$) and one only nonzero eigenvalue $2k\|a\|^2$ with corresponding eigenvector a (since $[2kaa^T]a = (2k\|a\|^2)a$). As a consequence, Q is positive semidefinite when $k > 0$ while it is negative semidefinite for $k < 0$; in particular when $k < 0$ the numerator belongs to the class of the D.C. functions, (i.e. it is the difference of convex functions), which is widely studied in Global Optimization.

The following theorem characterizes the pseudoconvexity of f in (8).

Theorem 6 *Function f in (8) is pseudoconvex on X if and only if one of the following conditions holds:*

i) $k > 0$;

ii) $k < 0$, $\exists \delta, \beta \in \Re$ *such that* $d = \delta a$, $b = \beta a$, *and*

$$k(a_0\beta - b_0)^2 \geq \beta(b_0\delta - d_0\beta) + \beta(a_0 k - c_0)(a_0\beta - b_0). \tag{9}$$

Proof. Since $Q(\Re^n) = \{\mu a, \mu \in \Re\}$ it results $q \in Q(\Re^n)$ if and only if $\exists \delta \in \Re$ such that $d = \delta a$; moreover it is $b \in Q(\Re^n)$ if and only if $\exists \beta \in \Re$, $\beta \neq 0$, such that $b = \beta a$ ($\beta \neq 0$ since $b \neq 0$). As a consequence, condition

$$\exists \overline{x}, \overline{y} \in \Re^n \text{ such that } Q\overline{x} = q \text{ and } Q\overline{y} = b$$

is equivalent to the following one:

$$\exists \delta, \beta \in \Re, \beta \neq 0, \text{ such that } d = \delta a \text{ and } b = \beta a. \tag{10}$$

In particular it is also

$$Q\overline{x} = q \quad \Leftrightarrow \quad \overline{x} = \frac{(a_0 k + c_0) + \delta}{2k\,\|a\|^2} a + k_{\overline{x}} \text{ with } k_{\overline{x}} \in \ker(Q),$$

$$Q\overline{y} = b \quad \Leftrightarrow \quad \overline{y} = \frac{\beta}{2k\,\|a\|^2} a + k_{\overline{y}} \text{ with } k_{\overline{y}} \in \ker(Q). \tag{11}$$

Let us now apply *i)* of Corollary 3.

If $k > 0$ then Q is positive semidefinite and the results follows from condition *i-a)*. Suppose now $k < 0$; whenever (10) and (11) hold it results

$$b^T\overline{y} = \frac{\beta}{2k\,\|a\|^2} b^T a + b^T k_{\overline{y}} = \frac{\beta^2}{2k\,\|a\|^2} a^T a = \frac{\beta^2}{2k} < 0$$

and hence case *i-b)* of Corollary 3 never occurs. As a consequence when $k < 0$ function f in (8) is pseudoconvex if and only if case *i-c)* of Corollary 3 holds, that is to say if and only if $\frac{\Delta}{4} \leq 0$. By means of simple calculations we have

$$b^T\overline{x} = \beta\,\frac{a_0 k + c_0 + \delta}{2k}, \quad q^T\overline{x} = \frac{(a_0 k + c_0 + \delta)^2}{2k}$$

so that

$$\begin{aligned}\frac{\Delta}{4} &= (b_0 - b^T\overline{x})^2 - b^T\overline{y}(q^T\overline{x} - 2q_0) = \\ &= (b_0 - \beta\,\frac{a_0k + c_0 + \delta}{2k})^2 - \frac{\beta^2}{2k}(\frac{(a_0k + c_0 + \delta)^2}{2k} - 2q_0) = \\ &= \frac{kb_0^2 - k\beta a_0 b_0 - \beta b_0 c_0 - \beta\delta b_0 + \beta^2 d_0 + \beta^2 a_0 c_0}{k} = \\ &= \frac{k(a_0\beta - b_0)^2 - \beta(b_0\delta - d_0\beta) - \beta(a_0k - c_0)(a_0\beta - b_0)}{k}\end{aligned}$$

and the result is proved. ■

The following example shows how Theorem 6 can be applied in order to study the pseudoconvexity of a function of the kind (8).

Example 1 Consider $f(x) = \frac{-(2x_1+3x_2+1)^2+x_1+3/2x_2+5}{4x_1+6x_2+1}$ and $X = \{(x_1, x_2) \in \Re^2 : 4x_1 + 6x_2 + 1 > 0\}$. Since the gradient of f never vanishes, f is pseudoconvex. We obtain the same result by means of Theorem 6 . Observe that $k = -1 < 0$, $a = (2,3)$, $b = (4,6)$ $d = (1,3/2)$ hence $b = \beta a$ with $\beta = 2$ and $d = \delta a$ with $\delta = 1/2$. By simple calculation $b^T\overline{y} = -2 < 0$ and $k(b_0 - a_0\beta)^2 = -1(1-2)^2 = -1$, $\beta(b_0\delta - d_0\beta) = 2(1/2-10)$ and consequently conditions *ii)* of Theorem 6 are verified.

As regards to the strict pseudoconvexity we get the following result.

Theorem 7 *Function f in (8) is strictly pseudoconvex on X if and only if $n = 2$, $k > 0$ and either b or d is not multiple of a.*

Proof. Being $n \geq 2$ and being Q semidefinite but not definite, by means of *ii)* of Corollary 3 function f results to be strictly pseudoconvex if and only if *ii-b)* holds, that is

$$\nu_-(Q) = 0,\ \ \nu_+(Q) = n - 1 \quad \text{and} \quad \{b, q\} \not\subset Q(\Re^n)$$

Since Q has only one nonzero eigenvalue this condition holds if and only if

$$n = 2,\ \ k > 0 \quad \text{and} \quad \{b, q\} \not\subset Q(\Re^n).$$

As it has been shown in (10) of Theorem 6, $\{b, q\} \not\subset Q(\Re^n)$ is equivalent to "either b or d is not multiple of a" and hence we are done. ■

4.2 a and c linearly independent

Let us now study the pseudoconvexity of function (7) when a and c are linearly independent that is:

$$f(x) = \frac{(a^T x + a_0)(c^T x + c_0) + (d^T x + d_0)}{b^T x + b_0} \quad \text{with } a, c \text{ l.i.} \tag{12}$$

where $a, b, c \neq 0$. Firstly note that function f in (12) can be rewritten in the form (3) with

$$Q = ac^T + ca^T, \qquad q = a_0 c + c_0 a + d, \qquad q_0 = a_0 c_0 + d_0,$$

Matrix Q is a $n \times n$ symmetric matrix ($n \geq 2$) which results to be indefinite ([3]) with $\nu_-(Q) = 1 = \nu_+(Q)$ and $\nu_0(Q) = n - 2$ (since $\dim(\ker(Q)) = n - 2$).

The pseudoconvexity of f in (12) is characterized in the next theorem.

Theorem 8 *Function f in (12) is pseudoconvex on X if and only if the following conditions hold:*

i) $\exists \delta_1, \delta_2 \in \Re$ *such that* $d = \delta_1 a + \delta_2 c$;

ii) $\exists \beta_1, \beta_2 \in \Re$ *such that* $b = \beta_1 a + \beta_2 c$;

iii) defining $\gamma_1 = a_0 + \delta_2$ *and* $\gamma_2 = c_0 + \delta_1$ *one of the following conditions holds:*
iii-a) $\beta_1 \beta_2 = 0$, $b_0 = \beta_1 \gamma_1 + \beta_2 \gamma_2$, $a_0 c_0 + d_0 \geq \gamma_1 \gamma_2$,
iii-b) $\beta_1 \beta_2 < 0$, $(b_0 - \beta_1 \gamma_1 - \beta_2 \gamma_2)^2 - 4\beta_1 \beta_2 (\gamma_1 \gamma_2 - a_0 c_0 - d_0) \leq 0$

Proof. First note that, being a and c linearly independent, it is $Q(\Re^n) = \{\mu_1 a + \mu_2 c, \mu_1, \mu_2 \in \Re\}$ so that

$$\begin{aligned} q \in Q(\Re^n) &\Leftrightarrow \exists \delta_1, \delta_2 \in \Re \text{ such that } d = \delta_1 a + \delta_2 c \\ b \in Q(\Re^n) &\Leftrightarrow \exists \beta_1, \beta_2 \in \Re \text{ such that } b = \beta_1 a + \beta_2 c \end{aligned}$$

In particular, assuming $\gamma_1 = a_0 + \delta_2$ and $\gamma_2 = c_0 + \delta_1$, it is

$$\begin{aligned} Q\overline{x} = q &\Leftrightarrow \overline{x} = \gamma_1 u + \gamma_2 v + k_{\overline{x}} \text{ with } k_{\overline{x}} \in \ker(Q), \\ Q\overline{y} = b &\Leftrightarrow \overline{y} = \beta_2 u + \beta_1 v + k_{\overline{y}} \text{ with } k_{\overline{y}} \in \ker(Q). \end{aligned}$$

[3] Being a and c linearly independent it is possible to determine a vector $u \in \Re^n$ such that $c^T u = 0$ and $a^T u \neq 0$. Then we have $Qu = (a^T u)c \neq 0$ and $u^T Q u = 0$ and this implies, for a known property of semidefinite matrices, that Q is indefinite.

where $u, v \in \Re^n$ are such that $c^T u = 0$, $a^T u = 1$, $a^T v = 0$ and $c^T v = 1$ (hence $Qu = c$ and $Qv = a$). As a consequence, condition

$$\exists \overline{x}, \overline{y} \in \Re^n \text{ such that } Q\overline{x} = q \text{ and } Q\overline{y} = b$$

is equivalent to conditions *i)* and *ii)*.

Let us now apply *i)* of Corollary 3. Being Q indefinite f results to be pseudoconvex if and only if *i-b)* or *i-c)* of Corollary 3 holds; the result then follows noticing that

$$b^T \overline{y} = 2\beta_1\beta_2, \quad b^T \overline{x} = \beta_1\gamma_1 + \beta_2\gamma_2, \quad q^T \overline{x} = 2\gamma_1\gamma_2$$

and hence $\frac{\Delta}{4} = (b_0 - \beta_1\gamma_1 - \beta_2\gamma_2)^2 - 4\beta_1\beta_2(\gamma_1\gamma_2 - a_0c_0 - d_0)$. ■

The following Theorem specify necessary and sufficient conditions for the strict pseudoconvexity of f when a and c are lineraly independent.

Theorem 9 *Function f in (12) is strictly pseudoconvex on X if and only if the following conditions hold:*

i) $n = 2$;

ii) $\exists \delta_1, \delta_2 \in \Re$ *such that* $d = \delta_1 a + \delta_2 c$;

iii) $\exists \beta_1, \beta_2 \in \Re$ *such that* $b = \beta_1 a + \beta_2 c$;

iv) defining $\gamma_1 = a_0 + \delta_2$ *and* $\gamma_2 = c_0 + \delta_1$ *one of the following conditions holds:*
iii-a) $\beta_1\beta_2 = 0$, $b_0 = \beta_1\gamma_1 + \beta_2\gamma_2$, $a_0c_0 + d_0 > \gamma_1\gamma_2$,
iii-b) $\beta_1\beta_2 < 0$, $(b_0 - \beta_1\gamma_1 - \beta_2\gamma_2)^2 - 4\beta_1\beta_2(\gamma_1\gamma_2 - a_0c_0 - d_0) < 0$

Proof. Being Q indefinite with $\nu_-(Q) = 1 = \nu_+(Q)$, by means of *ii)* of Corollary 3 function f is strictly pseudoconvex if and only if *ii-c)* or *ii-d)* holds; in particular it yields $n = 2$ and Q is non singular. The result then follows analogously to the one of Theorem 8. ■

4.3 The sum of an affine function and a linear fractional one

Using the previous results we are able to recover as a corollary the result given by [6] related to the following function

$$f(x) = a^T x + \frac{d^T x + d_0}{b^T x + b_0} \tag{13}$$

on the set $X = \{x \in \Re^n : b^T x + b_0 > 0\}$, where $a, d, b \in \Re^n$, $a, b \neq 0$, $d_0, b_0 \in \Re$. Function f can be easily seen as a particular case of function (7) when $c = b$, $c_0 = b_0$ and $a_0 = 0$. Recall that if $a = 0$ then f is a linear fractional function and hence it is pseudolinear.

Corollary 4 *Function f in (13) is pseudoconvex on X if and only if one of the following conditions holds:*

i) $\exists \beta > 0$ such that $b = \beta a$;

ii) $\exists \delta \in \Re$ such that $d = \delta b$ and $d_0 \geq \delta b_0$.

Proof. First note that f can be rewritten as

$$f(x) = \frac{a^T x(b^T x + b_0) + (d^T x + d_0)}{b^T x + b_0}$$

so that it is a function of the kind (8) or (12) where $a_0 = 0$, $c = b$ and $c_0 = b_0$.

Consider now the case a and b linearly dependent, that is $b = \beta a$ with $\beta \neq 0$ (since $a, b \neq 0$). By means of Theorem 6 function f is pseudoconvex if and only if one of the following conditions holds:

a) $\beta > 0$;

b) $\beta < 0$, $\exists \delta \in \Re$ such that $d = \delta a$ and $\beta b_0^2 \geq \beta(b_0 \delta - d_0 \beta) + \beta b_0^2$,

Since $b = \beta a$ it is $d = \frac{\delta}{\beta} b = \overline{\delta} b$, with $\overline{\delta} = \frac{\delta}{\beta}$, hence b) can be rewritten as

b') $\beta < 0$, $\exists \overline{\delta} \in \Re$ such that $d = \overline{\delta} b$ and $d_0 \geq \overline{\delta} b_0$

so that, in the case a and b linearly independent, the result is proved.

To complete the proof we are left to deal with the case a and b linearly independent. Since $c = b$ in *i)* of Theorem 8 we have $d = \delta_1 a + \delta_2 b$, while in *ii)* we get $\beta_1 = 0$, $\beta_2 = 1$. This implies that condition $\beta_1 \beta_2 < 0$ in *iii-b)* never occurs and that in *iii-a)* we have $\gamma_1 = \delta_2$, $\gamma_2 = b_0 + \delta_1$ and $b_0 = \gamma_2$; consequently $\delta_1 = 0$ and condition $a_0 c_0 + d_0 \geq \gamma_1 \gamma_2$ is specified as $d_0 \geq \delta_2 b_0$. Therefore, by means of Theorem 8, f is pseudoconvex if and only if $\exists \delta_2 \in \Re$ such that $d = \delta_2 b$, $d_0 \geq \delta_2 b_0$. The proof is now complete. ∎

Furthermore we use the previous results in order to characterize the strictly pseudoconvexity of function f in (13).

Corollary 5 *Function f in (13) is strictly pseudoconvex on X if and only if one of the following conditions holds:*

i) $n = 2$, $\exists \beta > 0$ *such that* $b = \beta a$, d *is not multiple of* a;

ii) $n = 2$, $\exists \delta \in \Re$ *such that* $d = \delta b$, $d_0 > \delta b_0$, b *is not multiple of* a.

Proof. In the case a and b are linearly dependent, condition *i)* follows directly from Theorem 7; if a and b are linearly independent condition *ii)* follows from Theorem 9 analogously to the proof of Corollary 4. ■

References

[1] Avriel M., Diewert W.E., Schaible S., Zang I., (1988), *Generalized Concavity, Mathematical Concepts and Methods in Science and Engineering*, vol.36, Plenum Press, New York.

[2] Barros A. I., (1998), *Discrete Fractional Programming Techniques for Location Models*, Kluwer Academic Publishers, Dordrecht.

[3] Bazaraa M.S., Sheraly H. D., Shetty C.M, (1993), *Nonlinear programming*, Second edition John Wiley & Sons, Inc., New York.

[4] Cambini A., Martein L., (1986), "A modified version of Martos' algorithm", *Methods of Operation Research*, vol. 53, pp. 33-44.

[5] Cambini A., Carosi L., Martein L., (1999), "On the supremum in fractional programming ", *Report* n. 153, Department of Statistics and Applied Mathematics, University of Pisa.

[6] Cambini A., Crouzeix J.P., Martein L., (2000), "On the pseudoconvexity of a quadratic fractional function", *Report* n. 189, Department of Statistics and Applied Mathematics, University of Pisa.

[7] Cambini R., (1994), "A class of non-linear programs: theoretical and algorithmical results", in *Generalized Convexity*, S. Komlósi, T. Rapcsák and S. Schaible eds., Lecture Notes in Economics and Mathematical Systems, vol.405, Springer-Verlag, Berlin, pp.294-310.

[8] Cambini R., (1998), "Composition Theorems for Generalized Concave Vector Valued Functions", *Journal of Information & Optimization Sciences*, vol.19, n.1, pp.133-150.

[9] Cambini R., Carosi L., (2001), On the pseudoaffinity of a class of quadratic fractional functions, *Report* n. 208, Department of Statistics and Applied Mathematics, University of Pisa.

[10] Chew K.L. and E.U. Choo, (1984), "Pseudolinearity and efficiency", *Mathematical Programming*, vol.28, pp.226-239.

[11] Craven D.B., (1988), *Fractional Programming,* Heldermann-Berlin.

[12] Crouzeix J.P., (1998), "Characterizations of Generalized Convexity and Monotonicity, a survey." in *Generalized Convexity, Generalized Monotonicity*, Crouzeix J.P. et Al. eds., Kluwer Academic Publisher, Dordrecht, pp.237-256.

[13] Diewert W.E., Avriel M. and I. Zang, (1981), "Nine kinds of quasiconcavity and concavity", *J. Econ. Theory*, vol.25, pp.397-420.

[14] Ellero A., (1996), "The optimal level solutions method", *Journal of Information & Optimization Sciences*, vol. 17, n. 2, pp. 355-372.

[15] Komlósi S., (1993), " First and Second Order Characterizations of Pseudolinear Functions", *European Journal of Operation Research*, Vol. 67, pp.278-286.

[16] Martos B., (1975), *Nonlinear programming theory and methods,* North Holland, Amsterdam.

[17] Schaible S., (1995), "Fractional Programming" in *Handbook of Global Optimization*, Edited by R. Horst, P.M. Pardalos, Kluwer Academic Publishers, Dordrecht, 495-608.