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A note on a particular quadratic programming problem

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A note on a particular quadratic programming problem

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Abstract

In this paper a particular quadratic problem, with a matrix having at least n-1 positive eigenvalues, is studied. Some theoretical properties of the problem are given, and a characterization of the existence of minimum points is provided. It is finally shown that the problem can be seen as a particular D.C. quadratic program, thus suggesting how to solve it with a finite algorithm, even in the indefinite case.

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1. Introduction

In this paper we consider the following problem:

$$(P) \qquad \min f(x) = 1/2\, x^TQx + c^Tx$$
$$x \in X = \{x \in R^n : Ax \geq b\}$$

where Q is a symmetric nxn matrix with at least $n-1$ positive eigenvalue, $c \in R^n$, A is a mxn matrix, $b \in R^m$. Obviously, if the minimum eigenvalue of Q is positive then (P) is a strictly convex problem, if it is equal to zero then (P) is a convex problem, if the minimum eigenvalue of Q is negative then (P) is an indefinite problem.
In section 2 we will provide some theoretical properties of quadratic problems regarding to the existence of minimum points and the global optimality of local optima; with this aim a key role will be played by the so called copositivity of matrix Q. In section 3 we will point out that f(x) can be rewritten as the difference between two convex quadratic functions; in other words we will show that f(x) is a particular d.c. function (difference of convex functions, see [11, 12, 13, 16, 17]) and that problem (P) can be considered in the following equivalent form:

$$(P_{dc}) \qquad \min f(x) = 1/2\, x^TBx + c^Tx - (d^Tx)^2$$
$$x \in X = \{x \in R^n : Ax \geq b\}$$

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with appropriate vector $d \in R^n$ and symmetric positive definite matrix $B \in R^{nxn}$. In section 4 we provide some examples of the proposed transformation and, finally, in section 5 we deepen on the study of the vectors d which allow us to rewrite problem (P) in the form (P_{dc}). The studied transformation allows to solve the problem, even in the indefinite case, by means of the finite solving algorithm suggested in [6]. Let us now give an idea of such an algorithm, based on the optimal level solutions approach. If the constraint $d^Tx=\xi$, $\xi \in R$, is added to problem (P_{dc}) then the following strictly convex quadratic problem is obtained:

$$P_{dc}(\xi) \qquad z(\xi) = -\xi^2 + \min\ (1/2\ x^TBx + c^Tx)$$
$$x \in X(\xi)=X \cap \{x \in R^n:\ d^Tx=\xi\}$$

The parameter ξ is said to be a *feasible level* if the set $X(\xi)$ is nonempty. An optimal solution of problem $P_{dc}(\xi)$ is called an *optimal level solution* [4, 7,10, 14,15].

Clearly problem (P_{dc}) is equivalent (2) to problem $P_{dc}(\xi)$, when ξ is the level corresponding to an optimal solution of problem (P_{dc}).

Let $\xi_{min} = \inf\ \{d^Tx,\ x \in X\} \geq -\infty$ and $\xi_{max} = \sup\ \{d^Tx,\ x \in X\} \leq +\infty$. The algorithm, of the binding constraints type, through the resolution of a finite number of convex quadratic problems of the type $P_{dc}(\xi)$, describes implicitly the function $z(\xi)$ for $\xi_{min} \leq \xi \leq \xi_{max}$ and detects its global minimum ξ^* together with the corresponding optimal level solution x^*, which is also the global minimum of (P_{dc}). The optimal solution of each quadratic subproblem (except for the first) is obtained parametrically from the optimal solution of the preceding quadratic subproblem. The algorithm, together with the global minimum, finds all the possible local minimum and, clearly, if (P_{dc}) is convex the procedure can be stopped as soon a local minimum is reached.

Recall that the complete study of this solving algorithm can be found in [6].

2. Properties of quadratic problems

In order to approach problem (P), let us first point out some properties of generic quadratic problems, that is problems of the kind

$$(P_q) \qquad \min f(x) = 1/2\ x^TQx + c^Tx$$
$$x \in X=\{x \in R^n:\ Ax \geq b\}$$

where Q is a symmetric nxn matrix, $c \in R^n$, A is a mxn matrix, $b \in R^m$. Note that (P) is a particular quadratic problem of the kind (P_q) where Q has at least n–1 positive eigenvalues.

(2) In the sense that the optimal solutions coincide.

If Q is indefinite, then any minimum point of problem (P_q) belongs (if it exists) to the boundary of the feasible region X (no critical points in the interior of X may be minima).

Being the feasible region X a polyhedron, it can be decomposed as $X=K_p+C_p$, where K_p is a polyhedral compact set and C_p is a polyhedral cone which coincides with the so called *recession cone* of the feasible region X, defined in general as follows:

$$\begin{aligned} rec(X) &= \{y: \exists\{x_n\}\subset X, \exists\{t_n\}\subset R, t_n\to+\infty, (x_n/t_n)\to y\} \\ &= \{0\}\cup\{y: \exists\{x_n\}\subset X, \|x_n\|\to+\infty, (x_n/\|x_n\|)\to v, y=\lambda v, \lambda\geq 0\}. \end{aligned}$$

It is worth reminding that if a set X is closed and convex then its recession cone is closed and convex too and can be rewritten as follows:

$$\begin{aligned} rec(X) &= \{y: \exists x\in X \text{ such that } x+\lambda y\in X\ \ \forall\lambda>0\} \\ &= \{y: x+\lambda y\in X\ \ \forall x\in X\ \ \forall\lambda>0\}. \end{aligned}$$

Also the concept of copositivity of a matrix will be useful in the rest of the paper [9]; remind that a symmetric matrix Q is said to be *[strictly] copositive with respect to a cone V* if and only if $v^TQv\geq 0$ [>0] $\forall v\in V$, $v\neq 0$.

Theorem 2.1

The minimum exists for problem (P_q) if at least one of the following conditions holds:

i) $v^TQv>0$ $\forall v\in rec(X)$, $v\neq 0$ (that is to say that Q is strictly copositive with respect to the cone rec(X))

ii) $\min_{\|y\|=1, y\in rec(X)} y^TQy>0$.

Proof. i) Let $\{x_n\}\subset X$ be the sequence such that $f(x_n)\to\inf\{f(x)\}$ and let $w_n=(x_n/\|x_n\|)\to y$. Let us now prove that $\lim_{n\to+\infty}\|x_n\|<+\infty$; suppose by contradiction that $\lim_{n\to+\infty}\|x_n\|=+\infty$, then $y\in rec(X)$ so that, being Q strictly copositive with respect to the cone rec(X), we have:

$$\begin{aligned} \lim_{n\to+\infty} f(x_n) &= \lim_{n\to+\infty} (1/2\, x_n^TQx_n+c^Tx_n) = \\ &= \lim_{n\to+\infty}\|x_n\|^2 \lim_{n\to+\infty} (1/2\, w_n^TQw_n+(1/\|x_n\|)\, c^Tw_n) = +\infty \end{aligned}$$

which is impossible. Being $\lim_{n\to+\infty}\|x_n\|<+\infty$ then, by means of the closure of X,

$x_n \to x^* \in X$ so that for the continuity of f(x) the infimum is reached as a minimum and x* is a minimum point.

ii) Follows directly from i). ♦

The following further lemma will be helpful in stating a necessary and sufficient condition for the existence of a minimum for problem (P_q) [5].

Lemma 2.1

Let $Q \in R^{nxn}$ be any matrix, $c \in R^n$ be any vector and X any closed subset of R^n; consider also the function $f(x)=1/2\, x^TQx+c^Tx$. Suppose Q to be copositive with respect to the cone rec(X) and define the following auxiliary function: $g_n(x)=f(x)+(1/n)\, x^Tx$ n=1,2,3,...

Then the following properties hold:

i) $\forall n$ the function $g_n(x)$ attains a minimum over X, say $x_n \in \mathrm{argmin}\{g_n(x)\}$

ii) the sequence $\{f(x_n)\}$ is decreasing and $f(x_n) \to \inf\{f(x) \text{ over } X\}$

The next result follows from the definition of the feasible set X [1, 2, 3].

Lemma 2.2

Let us consider the feasible region X of problem (P_q); since $X=K_p+C_p$, where K_p is a polyhedral compact set and C_p is a polyhedral cone, then for every sequence $\{x_n\} \subset X$ such that $\|x_n\| \to +\infty$ and $(x_n/\|x_n\|) \to d \in \mathrm{rec}(X)$ the following property holds:

$$\forall \rho>0 \quad x_n-\rho d \in X \quad \text{for } n \text{ sufficiently large.}$$

The following theorem provides a necessary and sufficient condition for the existence of the minimum for problem (P_q).

Theorem 2.2

The minimum exists for problem (P_q) if and only if both the two following conditions hold:

i) $v^TQv \geq 0 \quad \forall v \in \mathrm{rec}(X)$, that is to say that Q is copositive with respect to the cone rec(X);

ii) $\forall v \in \mathrm{rec}(X)$ such that $v^TQv=0$, it results $v^TQx+v^Tc \geq 0 \quad \forall x \in X$, that is to say that the function f(x) along the direction v is linear and nondecreasing.

Proof. ⇒) *i)* Let us suppose by contradiction that $\exists v \in rec(X)$ such that $v^TQv<0$; being $v \neq 0$ we have also $y^TQy<0$ where $y=v/\|v\|$. Being $y \in rec(X)$ there exists a sequence $\{x_n\} \subset X$, $\|x_n\| \to +\infty$, such that $(x_n/\|x_n\|) \to y$; denoting with $w_n=(x_n/\|x_n\|)$ it then results:

$$\lim_{n\to+\infty} f(x_n) = \lim_{n\to+\infty} (1/2\ x_n{}^TQx_n + c^Tx_n) =$$
$$= \lim_{n\to+\infty} \|x_n\|^2 \lim_{n\to+\infty} (1/2\ w_n{}^TQw_n + (1/\|x_n\|)\ c^Tw_n) = -\infty$$

which is a contradiction.
ii) Suppose on the contrary that $\exists v \in rec(X)$, $v^TQv=0$, $\exists x \in X$ such that $v^TQx+v^Tc<0$ and consider the function $f(x)$ restricted to the halfline $x+\lambda v \in X$ $\forall \lambda>0$. It results $f(x+\lambda v) = 1/2\ x^TQx + \lambda\ v^TQx + 1/2\ \lambda^2\ v^TQv + c^Tx + \lambda\ v^Tc$ so that $f(x+\lambda v) = \lambda\ (v^TQx + v^Tc) + 1/2\ x^TQx + c^Tx$. We then have, being $v^TQx+v^Tc<0$, that for $\lambda \to +\infty$, $f(x+\lambda v) \to -\infty$ which is a contradiction.
⇐) Let $g_n(x)=f(x)+ (1/n)\ x^Tx$, $n=1,2,3,\ldots$; being Q copositive with respect to the cone rec(X) then for Lemma 2.1 the function $g_n(x)$ attains a minimum over X $\forall n$, say $x_n \in \mathrm{argmin}\{g_n(x)\}$, and the sequence $\{f(x_n)\}$ is decreasing with $f(x_n) \to \inf\{f(x)$ over $X\}$. Let us now prove that $\lim_{n\to+\infty} \|x_n\| < +\infty$; suppose by contradiction that $\lim_{n\to+\infty} \|x_n\| = +\infty$ and let $v \in rec(X)$ such that $w_n=(x_n/\|x_n\|) \to v$. It results:

$$+\infty > \lim_{n\to+\infty} f(x_n) = \lim_{n\to+\infty} (1/2\ x_n{}^TQx_n + c^Tx_n) =$$
$$= \lim_{n\to+\infty} \|x_n\|^2 \lim_{n\to+\infty} (1/2\ w_n{}^TQw_n + (1/\|x_n\|)\ c^Tw_n)$$

so that $v^TQv \le 0$; being Q copositive with respect to the cone rec(X) it then follows $v^TQv=0$. For condition ii), being $f(x_n+\lambda v)=\lambda(v^TQx_n+v^Tc)$, we have that f(x) is nondecreasing along the direction v; this along with Lemma 2.2 implies that

$\forall \rho>0$ and for n sufficiently large $x_n-\rho v \in X$ and $f(x_n-\rho v) \le f(x_n)$;

note also that for n sufficiently large $\|x_n-\rho v\|^2<\|x_n\|^2$. Being $x_n \in \mathrm{argmin}\{g_n(x)\}$ we have $g_n(x_n) \le g_n(x_n-\rho v)$ so that:

$$f(x_n)+(1/n)\|x_n\|^2=g_n(x_n) \le g_n(x_n-\rho v)=f(x_n-\rho v)+(1/n)\|x_n-\rho v\|^2<f(x_n)+(1/n)\|x_n\|^2$$

which is a contradiction. Being $\lim_{n\to+\infty} \|x_n\| < +\infty$ then, by means of the closure of X, $x_n \to x^* \in X$ so that for the continuity of f(x) the infimum is reached as a minimum and x^* is a minimum point. ♦

Corollary 2.1

The following properties hold:

i) if problem (P_q) has no minimum then inf{f(x) over X}=-∞;

ii) if inf{f(x) over X}>-∞ then problem (P_q) admits minimum points.

Proof. i) By means of the previous theorem if problem (P_q) has no minimum then $\exists d \in rec(X)$ such that $v^TQv<0$ or $\exists d \in rec(X)$, $\exists x \in X$ such that $v^TQv=0$ and $d^TQx+d^Tc<0$.

In both cases, being v a feasible direction for problem (P_q), we have that $f(x+\lambda v) \to -\infty$.

ii) Follows trivially from i). ♦

Remark 2.1

Note that the previous theorem gives us some useful stop criterions for solving algorithms, since if one of the following conditions holds:

(2.1) if a feasible direction v is found such that $v^TQv<0$,

(2.2) if a feasible direction v and a feasible point x are found such that $v^TQv=0$ and $v^TQx+v^Tc<0$,

then there is no minimum for problem (P_q) and inf{f(x) over X}=-∞.

The concept of copositivity allow us to state the following global optimality conditions for problem (P_q), which is helpful in solving algorithms.

Theorem 2.3

Let us consider problem (P_q), a feasible point $x_0 \in X$ and a convex cone $V \subseteq R^n$; let us also define the following subset of the feasible region $Y=X \cap (x_0+V) \subseteq X$. Suppose finally that $v^TQv \geq 0$ $\forall v \in V$, that is to say that matrix Q of problem (P_q) is copositive with respect to the cone V. Then the following properties hold:

i) if $v^T\nabla f(x_0) = v^TQx_0 + v^Tc \geq 0$ $\forall v \in V$ then x_0 is a global minimum point over Y;

ii) if x_0 is a local minimum point over Y then it is also a global minimum point over Y.

Proof. i) We will prove the result by contradiction. Suppose on the contrary that $\exists y \in Y$ such that $f(y)<f(x_0)$ and define $v=y-x_0$. Firstly note that Y is a convex set (being the intersection of two convex sets) and that, being $y \in Y$, $v=y-x_0 \in V$ is a feasible direction. It then results:

$$f(y) = f(x_0+v) = [\ 1/2\ x_0^TQx_0 + c^Tx_0\ ] + [\ v^TQx_0 + v^Tc\ ] + 1/2\ v^TQv\ <$$
$$< [\ 1/2\ x_0^TQx_0 + c^Tx_0\ ] = f(x_0)$$

so that it follows, being $\nabla f(x_0)^Tv = v^TQx_0 + v^Tc \geq 0\ \ \forall v \in V$:

$$0 > [\ v^TQx_0 + v^Tc\ ] + 1/2\ v^TQv \geq 1/2\ v^TQv$$

which is a contradiction since Q is copositive with respect to the cone V.
ii) The thesis follows directly from property i) since if x_0 is a local minimum point over Y then $\nabla f(x_0)^Tv \geq 0\ \ \forall v \in V$. ♦

3. Particular properties of problem (P)

In order to study some particular properties of problem (P), let us first define the following notation for the smaller real eigenvalue of matrix Q in problem (P):

let $k \in \{1,\ldots,n\}$ be such that λ_k is the smaller eigenvalue of Q

Another key tool for our study is the canonical form $Q=UDU^T$ of the symmetric matrix Q, where D is a diagonal matrix having as main diagonal entries the eigenvalues λ_i of Q, while U is a unitary real matrix ($U \in R^{nxn}$ s.t. $UU^T=U^TU=I$) having as columns the eigenvectors u_i of Q corresponding to the eigenvalues λ_i, respectively.

First of all, let us specify Theorem 2.2 in the case Q is positive semidefinite but not positive definite, that is the case where the minimum eigenvalue of Q is zero.

Corollary 3.1
Let us consider problem (P); if matrix Q has one zero eigenvalue then the minimum exists if and only if

$$\forall v \in rec(X) \text{ such that } Qv=0 \text{ it results } v^Tc \geq 0,$$

that is if and only if it results $v^Tc \geq 0$ for every vector $v \in rec(X)$ which is also eigenvector of Q corresponding to the zero eigenvalue.
Proof. Follows directly from Theorem 2.2 taking care that, being Q positive semidefinite, it is $v^TQv=0$ if and only if $Qv=0$. ♦

The following result points out that the objective function f(x) of problem (P) is a particular D.C. function of the type $f(x) = 1/2\ x^TBx + c^Tx - (d^Tx)^2$, that is to say that there exists an appropriate vector d and a positive definite matrix B such that $Q=B-2dd^T$.

Theorem 3.1

Let $Q\in R^{nxn}$; the following conditions are equivalent:

i) Q is symmetric with at least n-1 positive eigenvalues,

ii) $\forall\alpha>0\ \exists d\in R^n$ and $\exists B\in R^{nxn}$, B symmetric and positive definite, s.t. $Q=B-\alpha dd^T$.

Proof. i)⇒ii) If Q has all positive eigenvalues then the result follows assuming B=Q and d=0. Suppose now that Q has one nonpositive eigenvalue $\lambda_k \le 0$ and consider its canonical form $Q=UDU^T$, where the diagonal elements of D are the eigenvalues of Q while the columns of U are eigenvectors of Q. Let now $\eta\in R$ be any positive number, let D_+ be the nonnegative diagonal matrix obtained from D just by substituting λ_k with η, and let D_- be the nonnegative diagonal matrix obtained from D substituting the positive elements with 0 and λ_k with $\sqrt{\frac{\eta-\lambda_k}{\alpha}} > 0$; then:

$$D = D_+ - \alpha\,(D_-)^2 = D_+ - \alpha\, D_-D_-^T$$

and hence:

$$Q = UDU^T = U(D_+ - \alpha\, D_-D_-^T)U^T = UD_+\, U^T - \alpha\,(UD_-)(UD_-)^T.$$

Let $B=UD_+U^T$ and let d be the unique nonzero column of (UD_-); B is symmetric and positive definite since all its eigenvalues are positive, the results then follows being

$$\alpha\,(UD_-)(UD_-)^T = \alpha dd^T.$$

ii)⇒i) Suppose by contradiction that $Q=B-\alpha dd^T$ has two nonpositive eigenvalues, say $\lambda_1 \le 0$ and $\lambda_2 \le 0$; then for all eigenvectors $x_1,x_2\in R^n\backslash\{0\}$, corresponding to λ_1 and λ_2 respectively, it results:

$$Bx_1-\alpha dd^Tx_1 = \lambda_1x_1 \quad\text{and}\quad Bx_2-\alpha dd^Tx_2 = \lambda_2\, x_2\,.$$

Note that $d^Tx_1 \ne 0$ and $d^Tx_2 \ne 0$, otherwise $Bx_1=\lambda_1x_1$ and/or $Bx_2=\lambda_2x_2$ which implies that $\lambda_1 \le 0$ and/or $\lambda_2 \le 0$ is an eigenvalue of B and this is a contradiction being B positive definite.

First consider the case $\lambda_1 = \lambda_2$ and let $y_1,y_2\in R^n\backslash\{0\}$ be eigenvectors of Q corresponding to λ_1 such that $y_1 \ne y_2$, $y_1{}^Ty_2 \ne 0$ and $d^Ty_1 = d^Ty_2 = 1$; then $By_1=\lambda_1y_1+\alpha d$ and $By_2=\lambda_1y_2+\alpha d$ so that $B(y_1-y_2)=\lambda_1(y_1-y_2)$ and hence $\lambda_1 \le 0$ is an eigenvalue of B, which is a contradiction being B positive definite.

Consider now the case $\lambda_1 \ne \lambda_2$ and let $y_1,y_2\in R^n\backslash\{0\}$ be eigenvectors of Q corresponding to λ_1 and λ_2, respectively, such that $y_1 \neq y_2$, $y_1^T y_2 = 0$ and $d^T y_1 = d^T y_2 = 1$; then

$$By_1 = \lambda_1 y_1 + \alpha d \text{ and } By_2 = \lambda_2 y_2 + \alpha d$$

and hence,

$$B(y_1 - y_2) = \lambda_1 y_1 - \lambda_2 y_2 .$$

Being B positive definite, $\lambda_1 \leq 0$ and $\lambda_2 \leq 0$ it follows that:

$$0 < (y_1 - y_2)^T B (y_1 - y_2) = \lambda_1 \|y_1\|^2 + \lambda_2 \|y_2\|^2 - (\lambda_1 + \lambda_2) y_1^T y_2 =$$
$$= \lambda_1 \|y_1\|^2 + \lambda_2 \|y_2\|^2 \leq 0$$

which is a contradiction. ♦

The previous theorem points out that, when the matrix Q is not positive definite, choosing $\alpha=2$ the objective function of problem (P) can be rewritten as

$$f(x) = 1/2\, x^T Q x + c^T x = 1/2\, x^T B x + c^T x - (d^T x)^2$$

where, in particular, $d \in R^n \setminus \{0\}$ is an eigenvector corresponding to the nonpositive eigenvalue $\lambda_k \leq 0$ of Q while B is symmetric and positive definite; the objective function is then a d.c. function given by the difference between the two convex functions $1/2\, x^T B x + c^T x$ and $(d^T x)^2$. This suggests to solve problem (P) by means of the algorithm proposed in [6], and recalled in Section 1, for the particular d.c. quadratic problems of the type (P_{dc}). A possible resolution method is the following:

i) if Q is positive definite then the problem can be solved by means of any of the well known algorithms for positive definite quadratic problems;

ii) if Q has a zero eigenvalue $\lambda_k = 0$ then, being $Q = UDU^T$, let D_+ be the nonnegative diagonal matrix obtained from D just by substituting λ_k with 2 (in other words, assume $\eta=2$ in *i)⇒ii)* of the proof of Theorem 3.1), let $B=UD_+U^T$ and let d be the column of U corresponding to λ_k (d is an eigenvector of Q corresponding to λ_k); then $Q=B-2dd^T$ and the problem can be rewritten in the form (P_{dc}) and solved with the algorithm suggested in [6]; note that since this is a convex problem, the algorithm can be stopped as soon as a local minimum is reached;

iii) if Q has a negative eigenvalue $\lambda_k < 0$ then, being $Q = UDU^T$, let D_+ be the nonnegative diagonal matrix obtained from D just by substituting λ_k with $-\lambda_k$ (that is assume $\eta=-\lambda_k$ in *i)⇒ii)* of the proof of Theorem 3.1), let $B=UD_+U^T$, let u_k be the column of U corresponding to λ_k (u_k is an eigenvector of Q corresponding to λ_k) and let $d = \mu\, u_k$ where $\mu = \sqrt{-\lambda_k}$; then $Q=B-2dd^T$ and again the problem can be rewritten in the form (P_{dc}) and solved with the algorithm suggested in [6].

4. Numerical examples

Let us show, for the sake of completeness, two examples of the transformation proposed in the previous section; both the semidefinite and the indefinite cases are going to be considered.

Semidefinite case

Let us consider the following symmetric matrix

$$Q = \begin{bmatrix} 4/3 & 0 & -2/3 \\ 0 & 2/3 & 2/3 \\ -2/3 & 2/3 & 1 \end{bmatrix}$$

The matrix Q is semidefinite positive and its eigenvalues are $\lambda_1=1$, $\lambda_2=2$ and $\lambda_3=0$. The eigenvectors corresponding to λ_1 are $[\ 2\alpha \quad 2\alpha \quad \alpha\ ]$, $\alpha \in R$, the eigenvectors corresponding to λ_2 are $\left[\ -\beta \quad \frac{1}{2}\beta \quad \beta\ \right]$, $\beta \in R$, and the eigenvectors corresponding to λ_3 are $\left[\ \frac{1}{2}\gamma \quad -\gamma \quad \gamma\ \right]$, $\gamma \in R$. Setting $\alpha = \frac{1}{3}$, $\beta = \frac{2}{3}$ and $\gamma = \frac{2}{3}$ we obtain the unitary eigenvectors $[\ 2/3 \quad 2/3 \quad 1/3\ ]$, $[\ -2/3 \quad 1/3 \quad 2/3\ ]$ and $[\ 1/3 \quad -2/3 \quad 2/3\ ]$ and the following unitary matrix

$$U = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix}$$

such that $UU^T=U^TU=I$. Matrix Q can then be rewritten in the form

$$Q = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}$$

and from ii) of section 3 it results

$$B = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 14/9 & -4/9 & -2/9 \\ -4/9 & 14/9 & -2/9 \\ -2/9 & -2/9 & 17/9 \end{bmatrix}$$

$$d = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix} \quad \text{so that} \quad dd^T = \begin{bmatrix} 1/9 & -2/9 & 2/9 \\ -2/9 & 4/9 & -4/9 \\ 2/9 & -4/9 & 4/9 \end{bmatrix}$$

and hence $Q=B-2dd^T$.

Indefinite case

Let us consider now the following symmetric matrix

$$Q = \begin{bmatrix} 8/9 & 8/9 & -14/9 \\ 8/9 & -10/9 & 22/9 \\ -14/9 & 22/9 & -7/9 \end{bmatrix}$$

The matrix Q is indefinite and its eigenvalues are $\lambda_1=1$, $\lambda_2=2$ and $\lambda_3=-4$. The eigenvectors corresponding to λ_1 are $[\ 2\alpha \quad 2\alpha \quad \alpha\ ]$, $\alpha \in R$, the eigenvectors corresponding to λ_2 are $\left[\ -\beta \quad \frac{1}{2}\beta \quad \beta\ \right]$, $\beta \in R$, and the eigenvectors corresponding to λ_3 are $\left[\ \frac{1}{2}\gamma \quad -\gamma \quad \gamma\ \right]$, $\gamma \in R$. Setting $\alpha = \frac{1}{3}$, $\beta = \frac{2}{3}$ and $\gamma = \frac{2}{3}$ we obtain the unitary eigenvectors $[\ 2/3 \quad 2/3 \quad 1/3\ ]$, $[\ -2/3 \quad 1/3 \quad 2/3\ ]$ and $[\ 1/3 \quad -2/3 \quad 2/3\ ]$ and the unitary matrix

$$U = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix}$$

such that $UU^T=U^TU=I$. Matrix Q can then be rewritten in the form

$$Q = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}$$

and from iii) of section 3 it results

$$B = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 16/9 & -8/9 & 2/9 \\ -8/9 & 22/9 & -10/9 \\ 2/9 & -10/9 & 25/9 \end{bmatrix}$$

$$d = 2\begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -4/3 \\ 4/3 \end{bmatrix} \text{ so that } dd^T = \begin{bmatrix} 4/9 & -8/9 & 8/9 \\ -8/9 & 16/9 & -16/9 \\ 8/9 & -16/9 & 16/9 \end{bmatrix}$$

and hence $Q=B-2dd^T$.

5. Concluding Remarks

Let us conclude our study stating conditions, concerning the vector d and the positive value α, which guarantee that matrix Q can be rewritten in the form $Q=B-\alpha dd^T$, with $B=Q+\alpha dd^T$ positive definite. With this aim the following preliminary result is necessary.

Lemma 5.1
The determinant of the following nxn matrix:

$$\begin{bmatrix} \lambda_1 & & & & & \gamma_1 \\ & \ddots & & 0 & & \vdots \\ & & \lambda_i & & & \gamma_i \\ & 0 & & \ddots & & \vdots \\ & & & & \lambda_{n-1} & \gamma_{n-1} \\ \gamma_1 & \cdots & \gamma_i & \cdots & \gamma_{n-1} & \omega \end{bmatrix}$$

is equal to $\prod_{i=1}^{n-1} \lambda_i \left(\omega - \sum_{i=1}^{n-1} \frac{\gamma_i^2}{\lambda_i} \right)$.

The following sufficient condition can now be proved.

Theorem 5.1
Consider problem (P), suppose that matrix Q has one nonpositive eigenvalue $\lambda_k \le 0$ and let $Q=UDU^T$ be the canonical form of Q. If a vector $d \in R^n$ and a number $\alpha>0$ verify the following conditions:

(5.1) $$u_k^T d \ne 0 \text{ and } \sum_{i \ne k} \frac{(u_i^T d)^2}{\lambda_i} < \frac{1}{\alpha^2}\left(\alpha + \frac{\lambda_k}{(u_k^T d)^2} \right)$$

then $Q=B-\alpha dd^T$, where $B=Q+\alpha dd^T$ is a positive definite matrix. In particular it results:

$$\alpha > -\frac{\lambda_k}{(u_k^T d)^2} \ge 0.$$

Proof. Define $v=U^T d$, so that $v_i = u_i^T d \ \forall i = 1,\dots,n$ and $d=Uv$; we just have to prove that, under the hypothesis, the matrix $B=Q+\alpha dd^T$ is positive definite. First note that it results $B=U[D+\alpha vv^T]U^T$, so that we can equivalently prove the positive definiteness of $D+\alpha vv^T$. Let e_k be the k-th vector of R^n canonical basis and partition v in the following way:

$$v = \begin{pmatrix} v_A \\ v_k \\ v_B \end{pmatrix} = \begin{pmatrix} 0 \\ v_k \\ 0 \end{pmatrix} + \begin{pmatrix} v_A \\ 0 \\ v_B \end{pmatrix} = v_k e_k + w \ ,$$

hence $vv^T = v_k^2 e_k e_k^T + v_k(e_k w^T + w e_k^T) + ww^T$ and:

$$D + \alpha vv^T = \left[D + \alpha v_k^2 e_k e_k^T + \alpha v_k(e_k w^T + w e_k^T)\right] + \alpha ww^T = M + \alpha ww^T$$

Since $\alpha>0$ then αww^T is a positive semidefinite matrix, hence a sufficient condition for $D + \alpha vv^T$ to be positive definite is that M is positive definite. By means of simple calculations M results to have the following structure:

$$M = \begin{bmatrix} \lambda_1 & 0 & 0 & & & & \\ 0 & \ddots & 0 & \alpha v_k v_A & & 0 & \\ 0 & 0 & \lambda_{k-1} & & & & \\ & \alpha v_k v_A^T & & \lambda_k + \alpha v_k^2 & & \alpha v_k v_B^T & \\ & & & & \lambda_{k+1} & 0 & 0 \\ & 0 & & \alpha v_k v_B & 0 & \ddots & 0 \\ & & & & 0 & 0 & \lambda_n \end{bmatrix}$$

and results to be positive definite if and only if the following matrix (obtained by means of a permutation of the rows and the corresponding columns) is positive definite:

$$\overline{M} = \begin{bmatrix} \lambda_1 & & & & & & \\ & \ddots & & & 0 & & \alpha v_k v_A \\ & & \lambda_{k-1} & & & & \\ & & & \lambda_{k+1} & & & \\ & 0 & & & \ddots & & \alpha v_k v_B \\ & & & & & \lambda_n & \\ & \alpha v_k v_A^T & & & \alpha v_k v_B^T & & \lambda_k + \alpha v_k^2 \end{bmatrix}$$

Since $\lambda_i > 0$ for all $i \neq k$ matrix $\overline{M}$ is positive definite if and only if $\det(\overline{M}) > 0$. For Lemma 5.1 it is:

$$\det(\overline{M}) = \prod_{i \neq k} \lambda_i \left(\lambda_k + \alpha v_k^2 - \sum_{i \neq k} \frac{(\alpha v_k v_i)^2}{\lambda_i} \right)$$

hence $\det(\overline{M}) > 0$ if and only if $\lambda_k + \alpha v_k^2 > \alpha^2 v_k^2 \sum_{i \neq k} \frac{v_i^2}{\lambda_i}$. This condition is equivalent to:

$$v_k \neq 0 \quad \text{and} \quad \sum_{i \neq k} \frac{v_i^2}{\lambda_i} < \frac{1}{\alpha^2}\left(\alpha + \frac{\lambda_k}{v_k^2} \right)$$

and the result follows being $v_i = u_i^T d \;\; \forall i = 1, \ldots, n$, and $v_i^2 \geq 0$, $\lambda_i > 0 \;\; \forall i \neq k$. ♦

It is worth considering the particular case where $d = \varepsilon u_k$, with $\varepsilon \neq 0$.

Corollary 5.1

Consider problem (P), suppose that matrix Q has one nonpositive eigenvalue $\lambda_k \leq 0$ and let $Q=UDU^T$ be the canonical form of Q. If the numbers $\varepsilon \neq 0$ and $\alpha>0$ are such that:

(5.2) $$\alpha\varepsilon^2 > -\lambda_k$$

then $Q=B-\alpha dd^T$, where $d = \varepsilon u_k$ and $B=Q+\alpha dd^T$ is a positive definite matrix.

Proof. Follows from the previous Theorem 5.1 noticing that assuming $d = \varepsilon u_k$ it is $u_i^T d = 0 \ \forall i \neq k$ and $u_k^T d = \varepsilon$. ♦

Remark 5.1

Note that in section 3, when the resolution method has been proposed, we choosed $\alpha = 2$ and a vector $d = \varepsilon u_k$; in particular:

- in the case $\lambda_k = 0$ we assumed $\varepsilon = 1$, which verify condition (5.2),
- in the case $\lambda_k < 0$ we assumed $\varepsilon = \sqrt{-\lambda_k}$, which again verify (5.2).

Let us now consider the semidefinite case.

Corollary 5.2

Consider problem (P), suppose that matrix Q has one zero eigenvalue $\lambda_k = 0$ and let $Q=UDU^T$ be the canonical form of Q. If a vector $d \in R^n$ and a number $\alpha>0$ verify the following conditions:

(5.3) $$u_k^T d \neq 0 \quad \text{and} \quad \sum_{i \neq k} \frac{(u_i^T d)^2}{\lambda_i} < \frac{1}{\alpha}$$

then $Q=B-\alpha dd^T$, where $B=Q+\alpha dd^T$ is a positive definite matrix.

Example 5.1

Let us consider again the positive semidefinite matrix

$$Q = \begin{bmatrix} 4/3 & 0 & -2/3 \\ 0 & 2/3 & 2/3 \\ -2/3 & 2/3 & 1 \end{bmatrix}$$

If we choose $d = [\ 0 \quad 1/2 \quad 0\ ]$ then, assuming $\alpha=2$, we have that

$$u_k^T d = -\frac{1}{3} \neq 0 \quad \text{and} \quad \sum_{i \neq k} \frac{(u_i^T d)^2}{\lambda_i} = \frac{1}{8}$$

and hence condition (5.3) is verified. In particular $Q=B-2dd^T$ where

$$B = Q+2dd^T = \begin{bmatrix} 4/3 & 0 & -2/3 \\ 0 & 7/6 & 2/3 \\ -2/3 & 2/3 & 1 \end{bmatrix}$$

is a positive definite matrix.

Note that if $\lambda_k = 0$ then for any vector $d \in R^n$ such that $u_k^T d \neq 0$ there always exists a number $\alpha>0$ verifying condition (5.3). The same does not hold when $\lambda_k < 0$, as it is proved in the next Theorem.

Theorem 5.2
Consider problem (P), suppose that matrix Q has one negative eigenvalue $\lambda_k < 0$ and let $Q=UDU^T$ be the canonical form of Q. For all vectors $d \in R^n$ such that:

(5.4) $$u_k^T d \neq 0 \quad \text{and} \quad 0 \leq \sum_{i \neq k} \frac{(u_i^T d)^2}{\lambda_i} < -\frac{1}{4}\frac{(u_k^T d)^2}{\lambda_k}$$

then there exist numbers $\alpha > -\frac{\lambda_k}{(u_k^T d)^2} > 0$ such that condition (5.1) is verified.

Proof. Consider the behaviour, depicted in the following Figure 1, of the function $h(\alpha) = \frac{\alpha + \mu}{\alpha^2}$ defined for $\alpha>0$. The result follows just assuming $\mu = \frac{\lambda_k}{(u_k^T d)^2}$. ♦

Example 5.2
Let us consider again the indefinite matrix

$$Q = \begin{bmatrix} 8/9 & 8/9 & -14/9 \\ 8/9 & -10/9 & 22/9 \\ -14/9 & 22/9 & -7/9 \end{bmatrix}$$

If we choose $d=[\ 1\ \ -1\ \ 1\ ]$ then, assuming $\alpha=2$, we have that

$$u_k^T d = \frac{5}{3} \neq 0 \, , \quad \frac{(u_k^T d)^2}{\lambda_k} = -\frac{25}{36} \quad \text{and} \quad \sum_{i \neq k} \frac{(u_i^T d)^2}{\lambda_i} = \frac{1}{6}$$

and hence both condition (5.1) and (5.4) are verified. In particular $Q=B-2dd^T$ where

$$B = Q+2dd^T = \begin{bmatrix} 26/9 & -10/9 & 4/9 \\ -10/9 & 8/9 & 4/9 \\ 4/9 & 4/9 & 11/9 \end{bmatrix}$$

is a positive definite matrix.

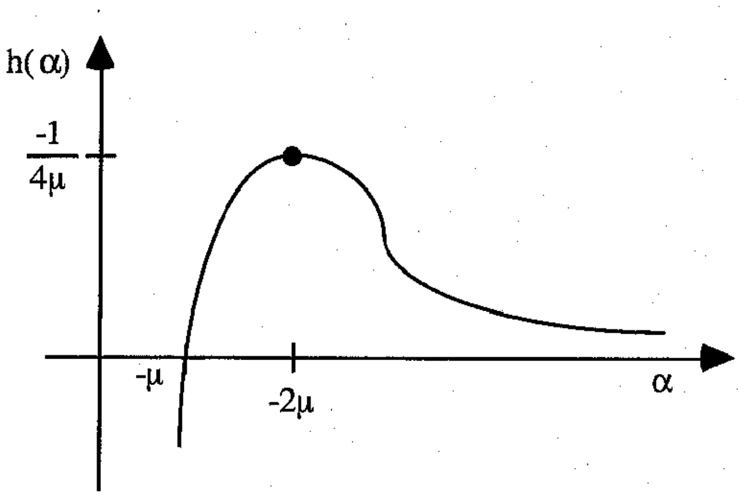

Figure 1

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