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Anna Marchi

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Via Cosimo Ridolfi, 10 – 56124 PISA – Tel. Segr. Amm. 050 945231 Segr. Stud. 050 945317 Fax 050 945375
Cod. Fisc. 80003670504 - P. IVA 00286820501 - Web http://statmat.ec.unipi.it/ - E-mail: dipstat@ec.unipi.it

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Anna Marchi

Dept. of Statistics and Applied Mathematics,

University of Pisa, Italy

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Abstract

In this paper the concept of Set Without Local Efficient Maxima is introduced. By means of this concept we will investigate the connectedness of the efficient frontier for vector maximization problems defined by functions whose local efficient maxima are global. Conditions under which the outcome of a vector function is a set without local efficient maxima are established. Applications for bicriteria and three criteria problems are given.

1 Introduction

The connectedness of the efficient frontier for vector maximization problems is an important field of research because of its applications. Let E be the set of all efficient points (Pareto optimal) of the problem $max\ F(x)$, $x \in X$ where $X \subset \Re^n$ is a nonempty compact set and $F : \Re^n \to \Re^m$ is a continuous vector function. In the literature the connectedness of both E and $F(E)$ has been obtained for the class of semistrictly quasi-concave functions in the bicriteria [1], [7] and tricriteria [2] cases. Hu and Sun extended these results to the general case [3], for the class of strictly quasi concave functions, under the condition that

$F(E)$ is closed. This hypothesis is necessary even if $F(X) \subset \Re^2_+$. It is well known that if E is a connected set, then $F(E)$ is connected too. In general, the viceversa isn't true. We can get examples (see [5]) where E isn't connected while $F(E)$ is connected. The aim of this paper is to state for which class of functions the property of the connectedness of $F(E)$ holds. In section 2 we will introduce the concept of set Without Local Efficient Maxima and study its properties. For this kind of sets we state in section 3 that the $F(E)$ is connected under condition of its closure. In section 3.1, in the bicriteria case, we are able to relax the assumption of closure and obtain a necessary and sufficient condition. In section 4, we state that the outcome $F(X)$ of vector functions, which does not have efficient local maxima different from the global ones, is a Set Without Local Efficient Maxima so we have the connectedness of the efficient frontier for this class of functions. Obviously this class is wider than the class of strictly quasi concave functions, considered by Hu and Sun in [3].

2 Sets Without Local Efficient Maxima

Let T be a nonempty set in $\Re^m$and $W_i : T \to \Re$ the Projection function $W_i(r) = r_i$, $i \in I = \{1, 2, \cdots, m\}$, where $r = (r_1, r_2, .., r_m)$. In [5], the concept of set without local maxima is introduced.

Definition 2.1 *$T \subseteq \Re^m$ is said to be a set Without Local Maxima on the i-th component (WLM$_i$) iff W_i is a function whose local maxima*[1] *are global on T.*

Definition 2.2 *$T \subseteq \Re^m$ is said to be a set Without Local Maxima (WLM) iff T is a set Without Local Maxima on the i-th component for any $i \in I$.*

Consider the upper level sets of functions W_i

$$L(\alpha_i) = \{r \in T \mid r_i \geq \alpha_i\}$$

[1] A point $r^0 \in T$ is a local maximum for W_i on T but not global if there exists a $\delta > 0$ such that $W_i(r) \leq W_i(r^0)$, $\forall r \in B_\delta(r^0) \cap T$ and there exists $\overline{r} \in T$ such that $W_i(\overline{r}) > W_i(r^0)$, where $B_\delta(r)$ denotes an open ball with radius δ centered around r.

for every $\alpha_i \in G_i = \{\alpha_i \in \Re \mid L(\alpha_i) \neq \emptyset\}$, $i \in I$.

It is seen that we have the following theorems of characterization for sets without local maxima

Theorem 2.1 *$L(\alpha_i)$ is lower semi-continuous*[2] *for any $\alpha_i \in G_i$ iff W_i is a function whose local maxima are global on T.*

Proof: Directly, taking into account the results obtained by Zang in [10] regarding functions whose local maxima are global.□

For the sake of simplicity, from now on we shall assume that T is a compact set. Set $\overline{r}_i = \max\{r_i \mid r \in T\}$, $S^i = \{r \in T \mid r_i = \overline{r}_i\}$ for any $i \in I$. Let us note, if T is a compact set then $S^i \neq \emptyset$ and $L(\alpha_i)$ is a compact set for any $i \in I$.

Theorem 2.2 *$T \subset \Re^m$ is a set WLM$_i$ iff $r \in T$ and $r \notin S_i$ implies the existence of a natural number M and a sequence $\{r^k\} \rightarrow r$ such that $r_i^k > r_i$, $k = M, M+1, \cdots$ for any $i \in I$.*

Proof: Directly, taking into account theorem 2.1 and the definition of local maxima for W_i on T.□

Now, let us consider the sets of maximal (or efficient) and weakly maximal (weakly efficient) elements of T denoted by $E(T, \Re^m_+)$ and $WE(T, \Re^m_+)$, respectively, i.e.

$$E(T, \Re^m_+) = \{\overline{r} \in T \mid \nexists\, r \in T \text{ such that } r \geq \overline{r}\}$$
$$WE(T, \Re^m_+) = \{\overline{r} \in T \mid \nexists\, r \in T \text{ such that } r > \overline{r}\}^3.$$

With the aim of studying the connectedness of $E(T, \Re^m_+)$ and extending the results given in [5], let us define the sets Without Local Efficient Maxima.

[2]The point to set mapping $L(\alpha_i)$ is said to be lower semi continuous (lsc) at the point $\alpha_i \in G_i$ if $r \in L(\alpha_i), \alpha_i^k \subset G_i, \{\alpha_i^k\} \rightarrow \alpha_i$ imply the existence of a natural number M and a sequence $\{r^k\}$ such that $r^k \in L(\alpha_i^k)$ (with $r_i^k \geq \alpha_i^k$) $k = M, M+1, \cdots$, and $\{r^k\} \rightarrow r$.

[3]For any $y^1, y^2 \subseteq \Re^m$, $y^1 \geqq y^2$ means $y^1 - y^2 \in \Re^m$, $y^1 \geq y^2$ means $y^1 - y^2 \in \Re^m \setminus \{0\}$, $y^1 > y^2$ means $y^1 - y^2 \in int\Re^m$.

Definition 2.3 *$T \subseteq \Re^m$ is said to be a set Without Local Efficient Maxima on the i-th component (WLEM$_i$) iff W_i is a function whose local efficient maxima*[4] *are global on T.*

Definition 2.4 *$T \subseteq \Re^m$ is said to be a set Without Local Efficient Maxima (WLEM) iff T is a set WLEM$_i$ for any $i \in I$.*

Obviously if T is a set *WLM* then it is a set *WLEM*, but the viceversa is not true. In a set *WLEM*, we may have local non efficient maxima while these points cannot exist in a set *WLM*. For sets *WLEM*, theorem 2.2 becomes:

Theorem 2.3 *$T \subset \Re^m$ is a set WLEM iff $r \in E(T, \Re^m_+)$ and $r \notin S_i$ implies the existence of a natural number M and a sequence $\{r^k\} \to r$ such that $r_i^k > r_i$ and there exists a $j \neq i$ such that $r_j^k < r_j$, $j \in I$, $k = M, M+1, \cdots$ for any $i \in I$.*

Proof: Directly, taking into account theorem 2.2 and definition 2.4.□

3 Connectedness of $E(T, \Re^m_+)$

Let $T \subseteq \Re^m$ be a nonempty set. Set $P_j(\alpha_i) : max\ \{ r_j \mid r \in L(\alpha_i)\}, j \neq i,\ j \in I, \alpha_i \in G_i$, the following lemma extends the results obtained in [5]:

Lemma 3.1 *Let $\hat{r} \in T$, $\hat{r} \notin S_j$. If T is a set WLEM$_j$ then the optimal solutions of problem $P_j(\hat{r}_i)$ are binding to the parametric constraint, $r_i \geq \hat{r}_i$.*

Proof: Let $r^0 \in T$ be an optimal solution of $P_j(\hat{r}_i)$. If r^0 is not binding to the parametric constraint (i.e. $r_i^0 > \hat{r}_i$), then there exists an $I_{r^0} \subset intL(\hat{r}_i)$ such that for every $r \in I_{r^0}$, $r_j^0 \geq r_j$, so r^0 is a local efficient maximum element of T for problem $W_j(r) = r_j$, since $\hat{r} \notin S_j$. This is absurd because T is a set WLEM$_j$ □

Lemma 3.2 *Let T be a set WLEM$_i$. $r \in E(T, \Re^m_+)$ and $r \notin S_i$, $i \in I$ implies the existence of a natural number N and a sequence $\{r^L\} \to r$ such that $r^L \in E(T, \Re^m_+)$, $L = N, N+1, \cdots$ with $r_i^L > r_i$.*

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[4] $r^0 \in T$ is a local efficient maximum point for W_i on T iff r^0 is a local maximum point for W_i on T and $r^0 \in E(T, R^m_+)$.

Proof: Taking into account theorem 2.3, for each $r \in E(T, \Re^m_+)$ and $r \notin S_i$ implies the existence of a natural number M and a sequence $\{r^s\} \to r$ such that $r_i^s > r_i$ and there exists a $j \neq i$ such that $r_j^s < r_j$, $s = M, M+1, \cdots, j \in I$. For each $r^s \in T$, we consider $P_j(r_i^s) = max\ \{\ r_j \mid r \in L(r_i^s)\}\ j \neq i$ so for lemma 3.1 the optimal solution of $P_j(r_i^s)$ are binding to the parametric constraint. Hence, there exists $r^L \in E(T, \Re^m_+)$ with $r_i^L = r_i^s$, because $L(r_i^s)$ is a compact set.□

Now, for the class of sets without local efficient maxima, we are able to prove that $E(T, \Re^m_+)$ is connected in hypothesis of closure of $E(T, \Re^m_+)$ and connectedness of $E(S_i, \Re^m_+)$.

Theorem 3.1 *Suppose $E(T, \Re^m_+)$ is a closed set and $E(S_i, \Re^m_+)$ is connected for some $i \in I$. If T is a set WLEM then $E(T, \Re^m_+)$ is connected.*

Proof: Suppose that $E(T, \Re^m_+)$ is disconnected, then there exist two closed sets E^1, E^2 of $E(T, \Re^m_+)$ such that $E(T, \Re^m_+) = E^1 \cup E^2$, $E^1 \cap E^2 = \emptyset$. Let $\hat{r}_i^1 = \max\{r_i | \ r \in E^1\}$ and $\hat{r}_i^2 = \max\{r_i | \ r \in E^2\}$ with $\hat{r}^1 \neq \hat{r}^2 \in T$. Since $E(S_i, \Re^m_+)$ is connected, then $\hat{r}_i^1 \neq \hat{r}_i^2$. Suppose $\hat{r}_i^1 > \hat{r}_i^2$, then for lemma 3.2 there exists a sequence $\{r^L\} \to \hat{r}^2$ such that $r^L \in E(T, \Re^m_+)$ with $r_i^L > \hat{r}_i^2$ so $r^L \in E^1$. This is absurd because E^1 is a closed set.□

3.1 Connectedness of $E(T, \Re^2_+)$ and $E(T, \Re^3_+)$

In the particular case of T subset of $\Re^2$, we will prove a necessary and sufficient condition for the closure and connectedness of $E(T, \Re^2_+)$. Set $\underline{r}_1 = \max\{r_1 \mid r \in L(\overline{r}_2)\}$, consider the following parametric problem:

$$P(\alpha_1) : z(\alpha_1) = \max\{r_2 | r \in L_{P_1}(\alpha_1), \alpha_1 \in [\underline{r}_1, \overline{r}_1]\}$$

Taking into account the results obtained in [6],[7], it is easy to prove the following lemmas regarding $z(\alpha_1)$ and $E(T, \Re^2_+)$.

Lemma 3.3 *i. $z(\alpha_1)$ is a decreasing function $\forall \alpha_1 \in [\underline{r}_1, \overline{r}_1]$,*

ii. $z(\alpha_1)$ is an upper semicontinuous function.

Lemma 3.4 *If $z(\alpha_1)$ is a strictly decreasing function $\forall \alpha_1 \in [\underline{r}_1, \overline{r}_1]$, then $E(T, \Re^2_+) = \{(\alpha_1, z(\alpha_1)) \mid \alpha_1 \in [\underline{r}_1, \overline{r}_1]\}$.*

Lemma 3.5 *If $z(\alpha_1)$ is a continuous and strictly decreasing function $\forall \alpha_1 \in [\underline{r}_1, \overline{r}_1]$, then $E(T, \Re^2_+)$ is closed and connected.*

Now we are able to prove the following corollary:

Corollary 3.1 *If $T \subset \Re^2$ is a set WLM$_2$ then:*

1. *$z(\alpha_1)$ is a strictly decreasing function $\forall\ \alpha_1 \in [\underline{r}_1, \overline{r}_1]$,*
2. *$E(T, \Re^2_+) = \{(\alpha_1, z(\alpha_1)) \mid \alpha_1 \in [\underline{r}_1, \overline{r}_1]\}$.*

Proof: *1.* Directly from lemma 3.1. *2.* As a consequence of lemmas 3.2, 3.4. □

Theorem 3.2 *$T \subset \Re^2$ is a set WLEM if and only if $E(T, \Re^2_+)$ is closed and connected.*

Proof: For corollary 3.1, we know that $E(T, \Re^2_+) = \{(\alpha_1, z(\alpha_1)) \mid \alpha_1 \in [\underline{r}_1, \overline{r}_1]\}$ and z is a strictly decreasing function. We have to prove that z is also a continuous function because $E(T, \Re^2_+)$ is the graphics of z. Instead, we suppose there exists a $r^0 \in E(T, \Re^2_+)$, $r^0 \notin S_1$ such that $r^0_2 = \lim_{r_1 \to r_1^{0-}} z(r_1) > \lim_{r_1 \to r_1^{0+}} z(r_1)$. Hence, a sequence $\{r^k\} \to r^0$ such that $r^k_1 > r^0_1$ and $r^k_2 < r^0_2$ does not exist. For theorem 2.3, this is absurd because T is a set *WLEM*. Now we have to prove that if $E(T, \Re^2_+)$ is closed and connected then T is a set WLEM.We suppose there exists a local efficient maximum point for W_2 on T $r^0 \in E(T, \Re^2_+)$, $r^0 \notin S_2$. Hence, a sequence $\{r^k\} \to r^0$ such that $r^k_2 > r^0_2$ does not exist. This is absurd because $E(T, \Re^2_+)$ is a closed and connected set.□

In the particular case of $m = 3$ the following theorem gives us the condition for the closure of set $E(T, \Re^3_+)$.

Theorem 3.3 *Suppose $T \subset \Re^3$ is a set WLEM. If $y^1, y^2 \in WE(T, \Re^3_+)$ such that $y^1, y^2 \notin S_i$ with two equal components do not exist on T, then $E(T, \Re^3_+)$ is closed.*

Proof: We know that a sequence $\{r^k\} \to r$ such that $r^k \in E(T, \Re^3_+)$ implies $r \in WE(T, \Re^3_+)$. We have to prove that if T is a set *WLEM* then $r \in E(T, \Re^3_+)$. In fact, if $r \in T$ is not an efficient point but only a weakly efficient point, then there exists another weakly efficient point, but this is absurd for the hypothesis.□

As a results we have the following corollary,

Corollary 3.2 *Suppose $T \subset \Re^3$ is a set WLEM. If $y^1, y^2 \in WE(T, \Re^3_+)$ such that $y^1, y^2 \notin S_i$ with two equal components do not exist on T, then $E(T, \Re^3_+)$ is connected.*

4 Vector Functions Without Local Efficient Maxima

Consider the vector maximization problem:

$$P_M : \max F(x),\ x \in X$$

where $X \subset \Re^n$ is a compact set and $F : X \to \Re^m$ is continuous function, $F(x) = (f_1(x), f_2(x), \cdots, f_m(x))$ and $F(X) \subset \Re^m$. Let $f_i(\overline{x}) = \max f_i(x)$, $x \in X$ and $S_i = \{x \in X \mid f_i(\overline{x}) = f_i(x)\}$ for any $i \in I$.

In this section we introduce, first of all, the following definition with the aim of stating for which class of vector functions F, the outcome $F(X)$ is a set *WLEM*.

Definition 4.1 *f_i is a function Without Local Efficient Maxima on X respect to F if and only if for every local maximum $\overline{x} \in X$ different from the global ones there exits $x \in X$ such that $F(x) \geq F(\overline{x})$.*

Theorem 4.1 *If $f_1, f_2, \cdots, f_m$ are functions Without Local Efficient Maxima on X respect to F, then $F(X)$ is a set WLEM.*

Proof: If f_i does not have local efficient maxima on X, then for any $\overline{x} \in E(F(X), \Re^m_+)$ and $\overline{x} \notin \overline{S_i}$, there is a sequence $\{x^k\} \to \overline{x}$ with $x^k \in X$ such that $f_i(x^k) > f_i(\overline{x})$ and $\{f_i(x^k)\} \to f_i(\overline{x})$ for the continuity of f_i. Now, if we consider $F(\overline{x})$ and $\{F(x^k)\}$ we have that $\{F(x^k)\} \to F(\overline{x})$ with $f_i(x^k) > f_i(\overline{x})$, hence, $F(\overline{x})$ is not a local efficient maximum for function W_i on $F(X)$. For theorem 2.3, $F(X)$ is a set without local efficient maxima.□

Obviously, this class of functions is wider than the class considered by Hu and Sun in [3], it is sufficient to note that a strictly quasiconcave function does not have local maxima different from the global ones. Hence, the outcome of these functions $F(X)$ is a set *WLM* and it is not *WLEM*.

Moreover, in the following lemma, we will note that $F(X)$ may be a set *WLEM* even when a function f_i has local efficient maxima on X respect to F.

Lemma 4.1 *Suppose f_i has a local efficient maximum point on X, $x^{loc} \in X$ different from the global ones. If $\overline{x} \in X$ such that $F(\overline{x}) = F(x^{loc})$ exists and $\overline{x} \in X$ is not a local maximum for f_i on X then $F(x^{loc})$ is not a local efficient maximum for function W_i on $F(X)$.*

Proof: If $\overline{x} \in X$ is not a local optimal solution for f_i on X then there is a sequence $\{x^k\} \to \overline{x}$ with $x^k \in X$ such that $f_i(x^k) > f_i(\overline{x})$ and, for the continuity of f_i, $\{f_i(x^k)\} \to f_i(\overline{x})$. Now, if we consider $F(\overline{x})$ and $F(x^k)$ we have that $\{F(x^k)\} \to F(\overline{x}) = F(x^{loc})$ with $f_i(x^k) > f_i(\overline{x})$, then $F(x^{loc})$ is not a local maximum for W_i on $F(X)$.□

Example 4.1 *Consider the following bicriteria problem:*

$$P_B : \max F(x) = (f_1(x), f_2(x)), \ \ x \in X = [-1, 4],$$

where $f_1(x) = x^2(x-3)$ and $f_2(x) = -x(x-3)^2$.

It is easy to verify that $x = 0$ is a local efficient maximum while $x = 3$ is not a local maximum for f_1 on X. Since $F(0) = F(3) = (0,0)$ then $(0,0)$ is not a local maximum for W_1 on $F(X)$. Moreover (0,0) is an efficient point for $F(X)$ because $x = 0$ is not a local maximum for W_2 on $F(X)$.

Theorem 4.2 *Suppose $E(F(X), \Re^m_+)$ is closed and $E(S_i, \Re^m_+)$ is connected for some $i \in I$. If $f_1, f_2, \cdots, f_m$ are functions Without Local Efficient Maxima on X respect to F then $E(F(X), \Re^m_+)$ is connected.*

Proof: Directly from theorems 3.1,4.1.□

Taking into account the previous results, in the bicriteria case, we obtain a necessary and sufficient condition for connectedness and closure of the efficient frontier of $F(X)$. These results extend the ones given in [5], [3]. Let $F : X \to \Re^2$, $F(x) = (f_1(x), f_2(x))$

Corollary 4.1 *Suppose f_1, f_2 are functions without local efficient maxima on X. Then $E(F(X), \Re^2_+)$ is closed and connected.*

Proof: For theorems 3.2 and 4.1.□

Let $F : X \to \Re^3$, $F(x) = (f_1(x), f_2(x), f_3(x))$. Let us consider $\bar{x}, \hat{x} \in X$ such that $\bar{x}, \hat{x} \notin S_i$ and $F(\bar{x}), F(\hat{x}) \in WE(T, \Re^3_+)$ with two equal components.

Theorem 4.3 *Suppose f_1, f_2, f_3 are functions without local efficient maxima on X and $\bar{x}, \hat{x} \in X$ do not exist, then $E(F(X), \Re^3_+)$ is closed and connected.*

Proof: For corollary 3.2 and theorems 3.3,4.1.□

References

[1] Choo, E.U., Schaible, S., and Chew, K.P., Connectedness of the Efficient Set in Tree Criteria Quasiconcave Programming, *Cahier du CERO*, vol.27, pp. 213-220, 1985.

[2] Daniilidis A., Hadjisavvas N. and S. Schaible, Connectedness of the Efficient Set for the tree Objective Quasiconcave Maximization Problems, *Journal of Optimiz. Theory and Applic.*, vol.93, n. 3 pp. 517-524, 1999.

[3] Hu, Y.D. and Sun, E.J., Connectedness of the Efficient Set in Strictly Quasiconcave Vector Maximization, *Journal of Optimiz. Theory and Applic.*, vol.78, pp. 613-622, 1993.

[4] Luc D.T., Theory of Vector Optimization, *Springer-Verlag*, 1989.

[5] Marchi, A., Sulla connessione della frontiera efficiente di un problema bicriteria, *Proceed. of XIV Convegno AMASES* (in italian), pp. 439-449, 1995.

[6] Martein, A., On the Bicriteria Maximization Problem, *Generalized Convexity and Fractional Programming with Economic Application, Lecture notes in Economics and Matematical Systems*, pp. 27-35, 1990.

[7] Schaible, S., Bicriteria Quasiconcave Programs, *Cahier du CERO*, vol.25, pp. 93-101, 1983.

[8] Sun, E.J., Connectedness of the Efficient Set in Strictly Quasiconcave Vector Minimization Problems, *Journal of Optimiz. Theory and Applic.*, vol.89, pp. 475-481, 1996.

[9] Warburton, A.R., Quasiconcave Vector Maximization: Connectedness of the Set of Pareto-Optimal and Weak Pareto-Optimal Alternative, *Journal of Optimiz. Theory and Applic.*, vol.40, pp. 537-557, 1983.

[10] Zang, I and M. Avriel, On Functions whose Local Minima are Global, *Journal of Optimiz. Theory and Applic.*, vol.16, pp. 183-190, 1975.