Università degli Studi di Pisa
Dipartimento di Statistica e Matematica
Applicata all'Economia

Report n. 223

The price-volatility feedback rate: an implementable mathematical indicator of market stability

Emilio Barucci, Paul Malliavin, M. Elvira Mancino,
Roberto Renò, Anton Thalmaier

Pisa, Aprile 2002

- Stampato in Proprio –

Via Cosimo Ridolfi, 10 - 56124 PISA - Tel. Segr. Amm. 050 945231 Segr. Stud. 050 945317 Fax 050 945375
Cod. Fisc. 80003670504 - P. IVA 00286820501 - Web http://statmat.ec.unipi.it/ - E-mail: dipstat@ec.unipi.it

The price-volatility feedback rate: an implementable mathematical indicator of market stability

Emilio Barucci
Dipartimento di Statistica e Matematica Applicata all'Economia
Università di Pisa
ebarucci@ec.unipi.it

Paul Malliavin
10 rue Saint Louis en l'Isle, 75004 Paris, France
sli@ccr.jussieu.fr

Maria Elvira Mancino
DIMAD, Università di Firenze
mancino@mail.dm.unipi.it

Roberto Renò
Dipartimento di Economia Politica
Università di Siena
reno@unisi.it

Anton Thalmaier
Institut für Angewandte Mathematik, Universität Bonn
anton@wiener.iam.uni-bonn.de

Abstract

Geometric analysis of iterated cross-volatilities of asset prices is adopted to assess the stability of the (risk free) measure under infinitesimal perturbations. Perturbations of asset prices evolve through time according to an ordinary linear differential equation (hedged transfer). The decay (feedback) rate is explicitly computed through a Fourier series method which is implemented on high frequency time series.

1 Introduction

Feedback effects of asset prices on volatilities are recognized in the financial markets literature both theoretically and empirically. Lack of *liquidity*, related to asset price movements, induces erratic motion of the asset price and therefore amplifies its volatility; trading by a large trader or by an agent employing a portfolio insurance-hedging strategy again can originate a feedback effect of the asset price on its volatility with destabilizing effects; on the other hand, there are traders (fundamentalists and arbitrageurs) who trade in order to exploit perturbations and therefore damping them (Friedman's conjecture, see Friedman (1953)).

In this paper we will construct a measure of these effects and we will show how to compute it. Our setting is very general, our main assumption will be of qualitative nature and model free: we shall assume that *instantaneous volatilities are functions of price levels*, these functions being time dependent and unknown; the only assumption made is that they are of class C^2.

Our setting includes stochastic volatility as well as level dependent volatility models; we shall not assume market completeness.

Working under this general assumption, we shall produce a time dependent *elasticity matrix*, implementable in real time from asset price observations. Eigenfunctions associated with positive eigenvalues of the elasticity matrix correspond to instability directions of the market, eigenfunctions associated with negative eigenvalues correspond to stability directions of the market.

The key mathematical tool to construct the elasticity matrix is a new methodology of transferring price perturbations through time, using *inertial frame transport*; this transport is designed so that through time the variation is governed by a first order linear *ordinary differential equation*. It is well known that perturbations evolution through time are governed by a linear stochastic differential equation; in some sense it can be said that our inertial frame transport approach allows us to eliminate partially that stochasticity (*hedged transfer*). The rate of variation through time of the initial perturbation is given by the elasticity matrix. The method can be developed for any finite number of assets. In this paper we shall deal with the univariate and with the bivariate case; the general n-variate case needs more lengthy computations and will be treated in a subsequent paper.

Volatility is a *problem* in financial markets for many reasons. First of all, it is very difficult to measure it, volatility of a diffusion process is well defined in terms of the quadratic variation, but then a finite sample of observations is used in order to compute it. Various methodologies can be adopted to this end, volatility estimate is sensitive to the methodology adopted, biases can be introduced in particular handling high frequency data, e.g. see Andersen and Bollerslev (1998); Barucci and Renò (2000, 2002). Financial time series volatility shows patterns-regularities that are hard to model and to justify theoretically. A partial list of stylized facts includes volatility clustering (volatility varies over time with a strong autoregressive component), asset returns and prices are negatively correlated with the conditional volatility, see Black (1976); Bekaert and Wu (2000), volatility is too high, i.e., not compatible with asset pricing models (*excess volatility*), volatility is linked to volume but is only in part explained by the arrival of information in the market and by macroeconomic-fundamental news-variables. Moreover, looking at the volatility implicit in option prices, it is observed that it is not constant as suggested by the Black & Scholes model as it depends on the asset price-strike ratio (*smile effect*). On these topics we refer to Ghysels, et al. (1996); Bates (1995). To these problems, we should add that volatility is a crucial quantity to price-hedge contingent claims and for risk management, e.g. VAR like methodologies.

To cope with this empirical evidence, asset price evolution has been described by stochastic differential equations different from the classical Black and Scholes equation. As an example, volatility clustering has been modeled assuming an ARCH-GARCH model, i.e., volatility follows a stochastic process characterized by an autoregressive component, see Andersen and Bollerslev (1998). In other cases, volatility has been modeled as a level dependent quantity (i.e. an unknown time independent function of the asset price). This type of model is included in our setting. In what follows, we present three different motivations for this type of dependence.

First of all, this way to model asset price volatility is well suited to capture the relationship between volatility and asset price-return. Volatility changes are negatively correlated with stock returns and volatility decreases with the asset price. This phenomenon has been explained as an evidence of a leverage effect, i.e., a price drop increases financial leverage which makes the asset riskier inducing a higher volatility. A different interpretation relies on time varying risk premia. On the two interpretations see Bekaert and Wu (2000). The simplest way to model the negative relation between asset price-return and volatility is to assume a constant elasticity variance model, see Cox and Ross (1976); Cox (1997), see Example 2.4. Level dependent volatility has been also conjectured to reproduce the implied volatility smile, see Derman and Kani (1994); Dupire (1994); Hobson and Rogers (1998). A different perspective to introduce level dependent volatility is to build a model with heterogeneous agents (e.g. fundamentalist, rational, portfolio insurance traders) deriving the stochastic process for asset price in equilibrium, see Frey and Stremme (1997); Platen and Schweizer (1998).

Computation of the elasticity matrix is a three steps outcome, at each step it is necessary to compute volatilities of observed quantities or computed in a previous step. The matrix can be computed in real time by extending the methodology based on Fourier series developed in Malliavin and Mancino (2002) to compute volatility of diffusion processes. An interesting feature of the method based on Fourier series expansion is that the volatility is constructed as a function of time; this fact stabilizes the computation of the quantities needed in the computation of the matrix, obviously the three steps entail a loss in the precision of the computation. Explicit expressions in terms of Fourier coefficients of asset prices leading to real time determination of feedback rates are provided in Section 4.1.

We can describe our approach through a metaphor from Hamiltonian mechanics. The *state of the market* described by a stochastic process for asset prices is the outcome of a market equilibrium, its evolution through time is described by correlations between price changes and instantaneous volatility changes. In Hamiltonian mechanics, principal modes of oscillation of an Hamiltonian system nearby an equilibrium are determined by the eigenvalues of the Hessian which is the matrix given by the second derivatives of the Hamiltonian; the last two steps of our method correspond to computing the second derivatives. The elasticity matrix appears as counterpart of the Hessian; in mechanics computation of second derivatives in a moving frame introduces Coriolis forces; to work free of Coriolis forces differentiation in an inertial Galilean frame is needed.

We present below mathematical theorems with full proofs for the construction of a hedged transfer and of the elasticity matrix. The relevance of our construction to decipher the *market state* is an open question, the numerical implementation in Section 4 is intended to be a first step which needs a follow up. A basic objection against our construction is toward the assumption that a price variation is supposed to induce immediately a volatility variation; this is not realistic, the effect needs a delay to propagate. Nevertheless the Fourier series method which cuts down high frequency takes partially care of this effect.

The paper is organized as follows. In Section 2 we present the univariate case together with an example (constant elasticity variance model). In Section 3 we construct the inertial frames in the multivariate case and we compute the elasticity matrix. In Section 4.1 we provide a way to compute in real time the feedback effect rate from market observations. In Section 4.2 we implement the methodology applying it to stock and exchange rate high frequency time series.

2 The univariate case

Let $S(t)$ be the risky asset price at time t. Let us assume that $S(t)$ satisfies the following stochastic differential equation without drift:

$$dS(t) = S(t)a_1(S(t))\,dW(t),$$

where $a_1(\cdot)$ is a deterministic function of the asset price and W is a Brownian motion. Let $x(t)$ be the logarithm of the price, i.e. $S(t) = \exp(x(t))$, then it follows that $x(t)$ satisfies the following stochastic differential equation

$$dx(t) = a_1(S(t))\,dW(t) - \frac{1}{2}a_1^2(S(t))\,dt = a(x(t))\,dW(t) - \frac{1}{2}a^2(x(t))\,dt$$

where $a(x) \equiv a_1(S) = a_1(\exp(x))$. We assume that the function a is deterministic and that it does not depend on t. We assume that a belongs to $C^2(\mathbb{R})$.

2.1 The feedback effect rate

The data of an infinitesimal deformation $x_W(t) + \varepsilon\zeta(t)$ will transform the probability measure according to the following Girsanov factor

$$\int \frac{\zeta(t)}{a(t)}\,dW(t).$$

The rescaled variation is defined as

$$z(t) = \frac{\zeta(t)}{a(t)}.$$

The following results hold.

Theorem 2.1. *The rescaled variation is a function differentiable with respect to t; its logarithmic derivative $\lambda(t)$ is called the feedback effect rate. Therefore, we have*

$$z(t) = \exp\left(\int_s^t \lambda(\tau)\,d\tau\right) z(s), \quad s \leq t.$$

Proof. The variation equation has the following expression

$$d\zeta(t) = a'\zeta\,dW - a'a\,\zeta\,dt.$$

Using Itô calculus we have

$$da = a'a\,dW - \frac{1}{2}a'a^2\,dt + \frac{1}{2}a''a^2\,dt$$

$$d\left(\frac{1}{a}\right) = -\frac{a'}{a}\,dW + \frac{1}{2}a'\,dt - \frac{1}{2}a''\,dt + \frac{1}{a}(a')^2\,dt.$$

Therefore the rescaled variation has the following Itô differential

$$dz(t) = \zeta(t)\left(\left(\frac{a'}{a} - \frac{a'}{a}\right)dW - \frac{1}{2}(a' + a'')\,dt\right).$$

Then $z(t)$ is a differentiable function of t and

$$\dot{z}(t) = -\frac{1}{2}z(t)(a'a + a''a) = \lambda(t)z(t), \quad \lambda(t) = -\frac{1}{2}(a'a + a''a).$$

□

Note that in the standard Black and Scholes framework $\lambda = 0$.

Theorem 2.2. *Denoting by $*$ the Itô contraction, define the following cross-volatilities,*

$$dx * dx =: A\,dt, \quad dA * dx =: B\,dt, \quad dB * dx =: C\,dt,$$

then the feedback effect rate function λ has the following expression

$$\lambda = \frac{3}{8}\frac{B^2}{A^3} - \frac{1}{4}\frac{B}{A} - \frac{1}{4}\frac{C}{A^2}. \tag{2.1}$$

Proof. Consider the following Itô differential:

$$dx = a\,dW - \frac{1}{2}a^2\,dt;$$

then $A = a^2$; B is the cross-volatility of A and x and can be written as

$$B\,dt = 2aa'\,dx * dx = 2a^3 a'\,dt.$$

Therefore we get

$$aa' = \frac{B}{2a^2} = \frac{1}{2}\frac{B}{A}. \tag{2.2}$$

The cross-volatility of B and x is C, and we have

$$2\,d(aa') * dx = 2(aa'' + (a')^2)a^2\,dt.$$

On the other side we conclude from (2.2) that

$$2\,d(aa') * dx = \frac{1}{A^2}\Big(A(dB * dx) - B(dA * dx)\Big) = \frac{1}{A^2}(AC - B^2)\,dt,$$

which implies

$$2aa'' = \frac{C}{A^2} - \frac{3}{2}\frac{B^2}{A^3}.$$

Finally

$$\lambda = -\frac{C}{4A^2} - \frac{B}{4A} + \frac{3}{8}\frac{B^2}{A^3}.$$

□

Remark 2.3. As x is the logarithm of a price it is dimensionless. The function A appears in the parabolic operator

$$\frac{\partial}{\partial t} + \frac{1}{2}A^2\frac{\partial^2}{\partial x^2}.$$

Therefore A has the dimension of the inverse of time, $A \simeq T^{-1}$; by Itô calculus $dx * dx = A\,dt$ which means for the Itô differentiation:

$$d \simeq T^{-\frac{1}{2}}.$$

We get

$$B\,dt = dA * dx \simeq T^{-\frac{3}{2}}T^{-\frac{1}{2}} = T^{-2},$$
$$\frac{B}{A} \simeq T^{-1}, \quad \frac{B^2}{A^3} \simeq T^{-4}T^3 = T^{-1},$$
$$C\,dt = dB * dx \simeq T^{-\frac{5}{2}}T^{-\frac{1}{2}} = T^{-3},$$
$$\frac{C}{A^2} \simeq T^{-3}T^2 = T^{-1}.$$

Finally λ has the dimension of the inverse of a time, as it should be in the interest rate category.

Example 2.4. We explicitly compute the rescaled variation and the instantaneous feedback rate considering a constant elasticity variance model, see Cox and Ross (1976); Cox (1997). The price process $S(t)$ satisfies

$$dS(t) = S^{\delta}(t)\, dW(t)$$

where $\delta < 1$ is a fixed constant. As pointed out above, in this model volatility changes are negatively correlated with stock returns and price level. Empirical tests have shown that return distributions are less positively skewed than log-normal distribution when $\delta < 1$ and in many cases are negatively skewed when $\delta < 0$, see Bates (1995). The basic ingredient in the definition of the rescaled variation is to use as reference metric the metric defined by the volatility; as a result the computation of the instantaneous liquidity rate does not depend upon a choice of coordinate; taking, as in the previous paragraph, $\log S = x$ as new variable we get

$$a(x) = \exp((\delta - 1)\log x)$$

and

$$\lambda = \frac{a^2(1-\delta)}{2x^2}(x + \delta - 2).$$

If $\delta = 1$ we find the Black-Scholes model and $\lambda = 0$; in the others cases the instantaneous liquidity rate changes sign at $x = 2 - \delta$.

2.2 Substitution of cross volatilities by volatilities

In Theorem 2.2 we expressed the feedback rate in terms of cross-volatilities. For a given mathematical model, it can be computed in exact terms as demonstrated in Example 2.4.

As far as the econometrical estimation of the feedback rate is concerned, cross-volatilities have the inconvenience to be related to the speed of economic propagation of feedback effects and therefore to become instable at high frequency data.

Of course, by polarization we have the identity

$$4B = \text{Volatility}(A + x) - \text{Volatility}(A - x).$$

More interesting is the following statement:

Theorem 2.5. *Denoting*

$$D = \text{Volatility}(\log A), \quad E = \text{Volatility}(\log D),$$

then λ can be expressed by the following formula:

$$-4\lambda = \eta_1\sqrt{D}\left(\sqrt{A} + \frac{\eta_2}{2}\sqrt{E}\right) \tag{2.3}$$

where $\eta_1 = 1$ if price x and volatility A are positively correlated and $\eta_1 = -1$ in the other case, and where $\eta_2^2 = 1$.

Proof. Starting with the identities

$$\log A = 2\log a, \quad D = 4\left(\frac{a'}{a}\right)^2 a^2 = 4(a')^2, \quad E = 4\left(\frac{aa''}{a'}\right)^2$$

and taking square roots, we get

$$a = \sqrt{A}, \quad a' = \frac{\eta_1}{2}\sqrt{D}, \quad aa'' = a'\frac{\eta_2}{2}\sqrt{E}.$$

□

Remark 2.6. It is remarkable that the price-volatility feedback effect can be computed effectively measuring only a qualitative feedback, the sign of the η_i.

2.3 Stability criteria involving only a one step iterated volatility

Iterating twice the volatility operator induces numerical instability. From another point of view, whereas the rescaled variation is natural for dealing with the risk free process, it is not appropriate for the historical price process. Of course the historical price process contains a drift which is unknown and can only be estimated; but on a short period of time as a single day it can be assumed that the derivative of this drift is small and hence that the variational equation for the historical price process is the "same" as for the risk free process.

The non-rescaled variation satisfies the equation

$$d\zeta = a'\zeta\, dW - a'a\zeta\, dt.$$

Defining $\eta = \zeta^2$ then by Itô calculus

$$d\eta = \eta\left(2a'\, dW - (2a'a - (a')^2)\, dt\right).$$

The condition that for η being a supermartingale is

$$2aa' \geq (a')^2$$

which is equivalent to $0 \leq a' \leq 2a$, or

$$0 \leq B \leq 4A^2. \tag{2.4}$$

We can dominate B by applying Cauchy-Schwarz inequality:

$$|B| \leq A\sqrt{DA}$$

where D denotes again the volatility of $\log A$. Then (2.4) is guaranteed if we require

$$0 \leq B, \quad D \leq 16A. \tag{2.5}$$

Theorem 2.7. *If one of the conditions* (2.4) *or* (2.5) *holds true then*

$$E[\zeta_t^2] \leq \zeta_{t_0}^2. \tag{2.6}$$

Remark 2.8. Negativity of the feedback rate λ implies a pathwise inequality, whereas (2.6) is only an inequality in the quadratic mean. Denoting δS the variation of the asset price S we have

$$\zeta(t) = \frac{\delta S(t)}{S(t)}.$$

3 The bivariate case: elasticity matrix

We limit our attention to the bivariate setting. Our analysis is developed in four steps. In Section 3.1 we introduce stochastic calculus of variations in the instantaneous moving frame. In Section 3.2 we prove existence and we characterize a hedged transfer of the variation of the process by requiring that its evolution is described by an ordinary differential equation. In Section 3.3 we introduce the inertial derivative using the inertial frame. In Section 3.4 we fully compute the elasticity matrix (the equivalent of λ in the previous section).

3.1 Stochastic calculus of variations in the instantaneous moving frame

We consider the process describing the evolution of the logarithm of two asset prices. We get an $\mathbb{R}^2$-valued process $x^\alpha(t)$, $\alpha = 1, 2$. We assume again that there is no drift, or if you want we work under the risk neutral probability measure with a null risk free rate. The variance matrix is defined by the identities

$$\mathbf{A}^{\alpha,\beta}(t)\, dt = dx^\alpha(t) * dx^\beta(t)$$

where the r.h.s. denotes the Itô contraction. Note that the matrix can be measured through market data. Then the symmetric matrix $\mathbf{A}$ can be written as $\mathbf{A} = \mathbf{U}^*\mathbf{D}\mathbf{U}$ where $\mathbf{D}$ is a strictly positive diagonal matrix and $\mathbf{U}$ is an orthogonal matrix. Define

$$\sqrt{\mathbf{A}} = \mathbf{U}^*\sqrt{\mathbf{D}}\,\mathbf{U} = \mathbf{A}^{1/2}.$$

Then $\sqrt{\mathbf{A}}$ can be measured by market data as the unique positive matrix which has $\mathbf{A}$ as its square. We denote by $A_1(t), A_2(t)$ the two column vectors constituting the two columns of the matrix $\sqrt{\mathbf{A}}$; they are linearly independent: at time t, $A_1(t), A_2(t)$ constitute *the instantaneous frame* of the market; as $(A_1\ A_2)$ is a symmetric matrix we have the identity $A_1^2(t) = A_2^1(t)$.

The (risk free) measure has the following infinitesimal generator:

$$\mathcal{L} = \frac{1}{2}\sum_{k=1,2} A_k^\alpha A_k^\beta\, \partial^2_{\alpha,\beta} + A_0^\gamma \partial_\gamma \quad \text{where } A_0^\gamma(t) = -\frac{1}{2}\sum_{k=1,2}(A_k^\gamma)^2(t). \tag{3.1}$$

The purpose of this paragraph is to introduce the classical stochastic calculus of variations for (3.1) in a geometric formalism which is apt to introduce the *inertial* stochastic calculus of variations of the next paragraph. Our procedure of the previous section in dimension one is generalized verbatim to dimension two.

We denote by ω^s, $s = 1, 2$, the differential forms dual to the vector fields A_k: $< A_k, \omega_s > = \delta_k^s$ where δ_*^* is the Kronecker symbol; denoting by ω the matrix having ω^k as k^{th} line we get

$$\omega = \mathbf{A}^{-1/2}.$$

The forms ω^s are therefore measurable by market data; their expression in coordinates is

$$\omega^s = \sum_{\alpha=1,2} b_\alpha^s\, dx^\alpha.$$

The co-boundaries

$$d\omega^s = \left(\frac{\partial b_2^s}{\partial x^1} - \frac{\partial b_1^s}{\partial x^2}\right) dx^1 \wedge dx^2$$

can be computed through the inversion of the identities

$$db_\alpha^s * dx^\beta = \sum_\gamma \frac{\partial b_\alpha^s}{\partial x^\gamma}\mathbf{A}^{\beta,\gamma}\, dt \quad \text{or} \quad \frac{\partial \omega}{\partial x^\gamma}\, dt = -\frac{1}{2}\mathbf{A}^{-3/2}\, d\mathbf{A} * \mathbf{A}^{-1/2} dx^\gamma. \tag{3.2}$$

There exist two functions $u^s(t)$, $s = 1, 2$, called *structural functions* of the instantaneous frame such that

$$d\omega^s = -u^s\, \omega^1 \wedge \omega^2. \tag{3.3}$$

Theorem 3.1. *The structural functions u^s, $s = 1, 2$, determine the stochastic calculus of variations in the instantaneous frame: the martingale part of the variation process in the instantaneous frame is given in* (3.8).

Proof. The risk free process is given by the Itô SDE

$$dx_W(t) = A_1\, dW^1 + A_2\, dW^2 + A_0\, dt, \tag{3.4}$$

where W^* is a two dimensional Brownian motion; denote

$$\mathcal{A}_k\big|^{\alpha}_{\beta} = \partial_\beta A^\alpha_k.$$

The usual variation transfer satisfies the Itô SDE

$$dz_W = \mathcal{A}_1(z_W)\, dW^1 + \mathcal{A}_2(z_W)\, dW^2 + \mathcal{A}_0(z_W)\, dt. \tag{3.5}$$

Writing the vector $z(t)$ in the instantaneous frame gives

$$z(t) = \sum_{s=1,2} \zeta^s(t)\, A_s(t) \tag{3.6}$$

where $\zeta^s(t) := < z(t), \omega^s(t) >$. Computing the martingale part of the Itô differential of the r.h.s. of (3.6), we get up to terms of bounded variation:

$$(\mathcal{A}_1(A_1)\, dW^1 + \mathcal{A}_1(A_2)\, dW^2)\ \zeta^1 + (\mathcal{A}_2(A_1)\, dW^1 + \mathcal{A}_2(A_2)\, dW^2)\ \zeta^2 + A_1\, d\zeta^1 + A_2\, d\zeta^2. \tag{3.7}$$

Expressing the differential of the l.h.s. of (3.6) by (3.5), the Lie bracket appears and we have

$$[A_1, A_2]\, (\zeta^1\, dW^2 - \zeta^2\, dW^1) \simeq A_1\, d\zeta^1 + A_2\, d\zeta^2,$$

where given two semi-martingales S_1, S_2 we write $dS_1 \simeq dS_2$ for the fact that $S_1 - S_2$ is a bounded variation process. By means of the well known identity

$$< A_1 \wedge A_2\, ,\, d\omega^s > + < [A_1, A_2]\, ,\, \omega^s > = \partial_{A_1}(< \omega^s\, , A_2 >) - \partial_{A_2}(< \omega^s\, , A_1 >),$$

as the r.h.s. vanishes, we get $[A_1, A_2] = u^1 A_1 + u^2 A_2$.

We get the martingale part of variation transfer in the instantaneous frame:

$$d\zeta^1 \simeq u^1(\zeta^1\, dW^2 - \zeta^2\, dW^1), \quad d\zeta^2 \simeq u^2(\zeta^1 dW^2 - \zeta^2 dW^1). \tag{3.8}$$

□

3.2 Characterization and existence of the hedged variation

The infinitesimal Radon-Nikodym derivative effect of the (risk free) measure of a perturbation $\zeta^*(t_0)$ up to time T is given by the infinitesimal Girsanov factor

$$\Gamma = \frac{1}{T - t_0} \int_{t_0}^{T} (\zeta^1\, dW^1 + \zeta^2\, dW^2);$$

therefore

$$E(\Gamma^2) = \frac{1}{(T - t_0)^2} E\left(\int_{t_0}^{T} \|\zeta\|^2\, dt \right) \quad \text{where } \|\zeta\|^2 = \sum_{s=1,2} (\zeta^s)^2. \tag{3.9}$$

Definition 3.2. A *hedged transfer* of a perturbation is a transfer such that the variation evolution satisfies an ordinary differential equation.

Since by Eq. (3.8),

$$\frac{1}{2}d\,\|\zeta\|^2 \simeq a(\zeta^1 dW^2 - \zeta^2 dW^1) \quad \text{where} \quad a = u^1\zeta^1 + u^2\zeta^2,$$

choosing as starting value $\zeta(t_0)$ such that $a \neq 0$, we see that *the transfer along the instantaneous moving frame is not a hedged transfer.*

To get a hedged transfer we need to introduce a supplementary degree of freedom by introducing a properly chosen random rotation, continuously depending upon time, of the instantaneous moving frame. In dimension one, the group of rotations consists of the multiplication by 1 or by -1; therefore this extra degree of freedom does not exist. In Section 2 we have shown that in dimension one the transfer in the instantaneous moving frame is a hedged transfer.

Denote $\mathcal{R}_\vartheta$ the rotation by an angle ϑ. Then the vector fields obtained from the fields A_k by the rotation $\mathcal{R}_\vartheta$ are

$$A_1^\vartheta = A_1 \cos\vartheta + A_2 \sin\vartheta,\ A_2^\vartheta = A_2\cos\vartheta - A_1 \sin\vartheta, \quad \rho = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\mathcal{R}_\varepsilon. \tag{3.10}$$

We take $\vartheta_W(t)$ the semi-martingale,

$$d\vartheta_W = \phi^1(x_W(t))\,dW^1 + \phi^2(x_W(t))\,dW^2 + \phi^0(x_W(t))\,dt, \quad \vartheta(t_0) = 0 \tag{3.11}$$

where the coefficients ϕ^* will be determined later.

The following Itô SDE differs from the infinitesimal generator of the risk free measure by a first order term

$$dx_W^\vartheta(t) = A_1^\vartheta(x_W^\vartheta(t))\,dW^1 + A_2^\vartheta(x_W^\vartheta(t))\,dW^2 + A_0(x_W^\vartheta(t))\,dt. \tag{3.12}$$

In fact, for the second order terms we have $\frac{1}{2}(\partial^2_{A_1^\vartheta} + \partial^2_{A_2^\vartheta}) = \frac{1}{2}(\partial^2_{A_1} + \partial^2_{A_2})$. We consider the new frame $A_1^\vartheta, A_2^\vartheta$. The differential forms

$$\omega_\vartheta^1 = \omega^1 \cos\vartheta + \omega^2 \sin\vartheta, \quad \omega_\vartheta^2 = \omega^2 \cos\vartheta - \omega^1 \sin\vartheta$$

constitute the dual basis; we have $\omega^1 \wedge \omega^2 = \omega_\vartheta^1 \wedge \omega_\vartheta^2$; hence the structural equation for the new frame is given by the vector $\mathcal{R}_\vartheta(u)$.

Theorem 3.3. *The transfer given by* (3.12) *is a hedged transfer in an infinitesimal neighborhood of* t_0 *if and only if condition* (3.13) *below holds true.*

Proof. The infinitesimal variation of the Brownian path W is given by $W \mapsto W + \varepsilon h$ where $h \in H^1$ is in the Cameron-Martin space. To get the initial variation ζ_0 we take a sequence $h_n \in H^1$ such that $\dot{h}_n \to \zeta_0 \times \delta_0$, the Dirac mass at t_0. From (3.11) we get

$$\lim_n \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{R}_{\vartheta_{W+\varepsilon h_n}} = (\zeta^1\phi^1 + \zeta^2\phi^2)\,\rho(\mathcal{R}_{\vartheta_W});$$

therefore the variation of ϑ induces on (3.12) for t close to t_0 the variation

$$B\,\rho(W(t)), \quad B = \zeta^1\phi^1 + \zeta^2\phi^2.$$

Eq. (3.6) holds true in the frame $(A_1^\vartheta, A_2^\vartheta)$. Taking into account that $\vartheta(t) \to 0$ as $t \searrow t_0$, we get for t close to t_0 the variation

$$\zeta(t) - \zeta(t_0) \simeq u\,(\zeta_0^1 W^2(t) - \zeta_0^2 W^1(t)) + B\,\rho(W(t));$$

the condition of a hedged transfer means that the coefficients of $W^1(t)$ and $W^2(t)$ vanish in the following expression:

$$(\zeta_0^1 u^1 + \zeta_0^2 u^2)\,(\zeta_0^1 W^2(t) - \zeta_0^2 W^1(t)) - B\zeta^1 W^2(t) + B\zeta^2 W^1(t).$$

We get in this way the condition

$$\zeta_0^1 u^1 + \zeta_0^2 u^2 = \zeta_0^1 \phi^1 + \zeta_0^2 \phi^2 = 0;$$

this equation holding true for all initial values (ζ_0^1, ζ_0^2) implies

$$\phi^1 = u^1, \quad \phi^2 = u^2. \tag{3.13}$$

□

3.3 Inertial frame and inertial derivative

Under the hypothesis ensuring existence of a hedged transfer, the norm $\|\zeta\|$ satisfies a first order ordinary differential equation, completely determined by an instantaneous elasticity matrix. This section and the next one are devoted to its computation. To this end, one approach could be, as we have done in Section 2, to resume exactly the same steps as in the last paragraph, at each step the full Itô calculus being applied. This approach does not require any new concepts; it is from a computational point of view relatively heavy.

We prefer a more conceptual road, based on a machinery reminiscent of celestial mechanics where the Newton principle of inertia stays valid only in Galilean frames, that is frames having for origin the center of mass of solar system, their axis being fixed relatively to the stars.

Given a vector field Y along the trajectory we express it in the instantaneous frame (A_1, A_2) as $Y = \sum_{i=1,2} \eta^i A_i$; therefore the data of a vector field is equivalent to the data of the two process η^i. The *inertial derivative* of a vector field Y is the 2×2 matrix

$$\nabla_k Y = (\partial_{A_k}\eta^1)A_1 + (\partial_{A_k}\eta^2)A_2 - u^k \rho(Y), \quad k = 1, 2; \tag{3.14}$$

this matrix can be measured through market data, as a matter of fact:

$$\partial_{A_\alpha}\eta = \sum_{\beta=1,2} \left[\mathbf{A}^{-1/2}\right]_\beta^\alpha (d\eta * dx^\beta); \tag{3.15}$$

given a vector field $Z = \sum_i \zeta^i A_i$, then $\nabla_Z Y := \sum_i \zeta^i \nabla_i Y$.

Considering the process defined by (3.12), the *inertial stochastic differential*

$$\nabla_t(Y) = A_1\, d\eta_1 + A_2\, d\eta_2 - (u \mid dx^\vartheta)\, \rho(Y) \tag{3.16}$$

where $(u|v) = \sum_i u^i v^i$ denotes the scalar product.

Theorem 3.4. *The choice of ϑ made by imposing* (3.13) *implies that for $t \to t_0$,*

$$\nabla_t A_i^\vartheta \simeq 0, \quad i = 1, 2. \tag{3.17}$$

Proof. The components of the vector field A_1^ϑ in the instantaneous frame are $(\cos\vartheta, \sin\vartheta)$, therefore the second component of $\nabla_t A_1^\vartheta$ is, computing $d\vartheta$ by (3.11),

$$\begin{aligned} \simeq \quad & -\cos\vartheta\,(\phi_1\,dW^1 + \phi_2\,dW^2) \\ & +\cos\vartheta\,[(u^1\cos\vartheta + u^2\sin\vartheta)\,dW^1 + (-u^1\sin\vartheta + u^2\cos\vartheta)\,dW^2]. \end{aligned}$$

As $t \to t_0$ we get $-\phi_1 + u^1 = 0$ and $-\phi_2 + u^2 = 0$. □

3.4 Differential Geometry in the instantaneous frame

In this paragraph we shall replace the computations on semi-martingales by computations on their associated infinitesimal generators, leading to manipulations of PDEs familiar to differential geometers.

Given a function of time $\phi(t)$, its *gradient* is the vector field

$$Y(t) = A_1(\partial_{A_1}\phi) + A_2(\partial_{A_2}\phi),$$

where the partial derivatives are implementable through (3.15); given a vector field $Z = \sum_i \zeta^i A_i$, its *divergence* is the function defined by

$$-(\nabla_1 Z)^1 - (\nabla_2 Z)^2 = -(\partial_{A_1}\zeta^1 + \partial_{A_2}\zeta^2) + (u^2\zeta^1 - u^1\zeta^2);$$

the Laplacian $\Delta\phi = -\frac{1}{2}\operatorname{div}\operatorname{grad}\phi$ has the following expression:

$$2\Delta = \partial^2_{A_1} + \partial^2_{A_2} + u^1\partial_{A_2} - u^2\partial_{A_1}. \tag{3.18}$$

The expression of 2Δ in coordinates x^α (constituted by the logarithms of the prices) is

$$2\Delta = \sum_{k,\alpha,\beta} A_k^\alpha A_k^\beta \partial^2_{\alpha,\beta} + \partial_{Z_1}, \quad Z_1 = \Big(\sum_{k,\alpha} A_k^\alpha \partial_\alpha(A_k^1), \sum_{k,\alpha} A_k^\alpha \partial_\alpha(A_k^2)\Big);$$

finally using (3.1) and (3.18) we get

$$\mathcal{L} = \Delta + \partial_Q, \quad 2Q = u^2 A_1 - u^1 A_2 - Z_1 + 2A_0, \tag{3.19}$$

where A_0 has been defined in (3.1); the vector field Q encodes the information needed to go from Finance to Geometry.

We shall now proceed with some commutation relations between geometric operators:

$$\nabla_1 A_2 - \nabla_2 A_1 = u^1 A_1 + u^2 A_2 = [A_1, A_2]. \tag{3.20}$$

The curvature is defined by

$$\mathcal{C} = (\nabla_1\nabla_2 - \nabla_2\nabla_1 - u^1\nabla_1 - u^2\nabla_2)Y. \tag{3.21}$$

In its expression all derivatives on the components η_* disappear; therefore it is an endomorphism of $\mathbb{R}^2$; denote by ε_i the basis $(1,0)$, $(0,1)$ of $\mathbb{R}^2$,

$$\nabla_2\varepsilon_1 = -u^2\varepsilon_2, \quad \nabla_1\varepsilon_1 = -u^1\varepsilon_2, \quad \nabla_1\varepsilon_2 = u^1\varepsilon_1, \quad \nabla_2\varepsilon_2 = u^2\varepsilon_1.$$

We deduce

$$\mathcal{C}(\varepsilon_1) = -R\varepsilon_2, \quad R = \partial_{A_1}u^2 - \partial_{A_2}u^1 + \|u\|^2, \quad \|u\|^2 = (u^1)^2 + (u^2)^2.$$

Resuming the computation for ε_2 we obtain

$$\mathcal{C} = -R\rho\,. \tag{3.22}$$

Decompose the drift $\mathcal{Q}$ in the frame to obtain $\mathcal{Q} = c_1A_1 + c_2A_2$, and denote $\mathcal{D}$ its inertial derivative in the sense of (3.14):

$$\mathcal{D} = \begin{pmatrix} \partial_{A_1}c_1 + u^1c_2 & \partial_{A_1}c_2 - u^1c_1 \\ \partial_{A_2}c_1 + u^2c_2 & \partial_{A_2}c_2 - u^2c_1 \end{pmatrix}. \tag{3.23}$$

The *elasticity matrix* $\mathcal{E}$ is the following implementable symmetric 2×2 matrix:

$$2\mathcal{E} = -R \times \text{Identity} - \mathcal{D} - \mathcal{D}^*. \tag{3.24}$$

The feedback rate of a vector $\eta = (\eta_1, \eta_2)$ at time s is the homogeneous function of degree zero defined as

$$\lambda(s, Y_s) = \frac{1}{\|\eta\|^2}\left(\mathcal{E}_s(\eta)\,|\,\eta\right), \quad Y_s = \eta_1A_1(s) + \eta_2A_2(s). \tag{3.25}$$

Theorem 3.5 (Transfer of a variation and Greeks computation). *Define the angle ϑ by the following Itô* SDE

$$d\vartheta(s) = (\mathcal{R}_\vartheta(u)\,|\,dW) + \left(\mathcal{R}_\vartheta(u)\,|\,\rho(u) + \gamma\right)dt, \quad \vartheta(t_0) = 0, \tag{3.26}$$

where γ is defined by the relation $\mathcal{D} - \mathcal{D}^* = 2\gamma\rho$. Given t_0 and a vector Y_{t_0} at $x_W(t_0)$, define its rolling as

$$Y_s = \eta^1(s)A_1(s) + \eta^2(s)A_2(s), \quad \eta(s) = \mathcal{R}_{\vartheta(s)}(\eta(t_0)).$$

The rolling-damping is defined as

$$\tilde{Y}_{s_0} = \exp\left(\int_{t_0}^{s_0} \lambda(s, Y_s)\,ds\right) Y_{s_0}. \tag{3.27}$$

Consider an European contingent claim F with maturity $T > t_0$ then we have the Greeks formula

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} E_{x_W(t_0)+\varepsilon Y_{t_0}}\left[F(x_W(T))\right] = E\left[<\tilde{Y}_T, dF>_{x_W(T)}\right] = E[F\Gamma] \tag{3.28}$$

where the weight Γ satisfies the inequality

$$E(\Gamma^2) \le \frac{\|\eta(t_0)\|^2}{(T - t_0)^2} E\left[\int_{t_0}^{T} \exp\left(2\int_{t_0}^{s} \lambda(t, Y_t)\,dt\right) ds\right]. \tag{3.29}$$

Proof. Cruzeiro and Malliavin (1998). □

It results from (3.29) that the market is stable when the two eigenvalues of the elasticity matrix are negative.

4 Numerical implementation

In this section we come back to the one-dimensional setting of Section 2. The functions A, B, C can be computed through the Fourier method developed in Malliavin and Mancino (2002).

4.1 Computation of the feedback rate

We assume that x is measured on the time interval $[0, 2\pi]$. The Fourier coefficients of dx are given by

$$\begin{aligned}\alpha_0 &:= a_0(dx) = \frac{1}{2\pi}\int_0^{2\pi} dx(t),\\ \alpha_k &:= a_k(dx) = \frac{1}{\pi}\int_0^{2\pi} \cos(kt)\, dx(t),\\ \beta_k &:= b_k(dx) = \frac{1}{\pi}\int_0^{2\pi} \sin(kt)\, dx(t).\end{aligned}$$

By Malliavin and Mancino (2002) we have

$$\begin{aligned}a_0(A) &= \lim_N \frac{\pi}{N+1-n_0}\sum_{s=n_0}^{N} \frac{1}{2}(\alpha_s^2 + \beta_s^2)\\ a_k(A) &= \lim_N \frac{2\pi}{N+1-n_0}\sum_{s=n_0}^{N} \alpha_s \alpha_{s+k}\\ b_k(A) &= \lim_N \frac{2\pi}{N+1-n_0}\sum_{s=n_0}^{N} \alpha_s \beta_{s+k}.\end{aligned}$$

The reconstruction of A from its Fourier coefficients is achieved by the classical Fourier-Féjer inversion formula:

$$A(t) = \lim_N \sum_{k=0}^{N}\Big(1 - \frac{k}{N}\Big)(a_k(A)\cos(kt) + b_k(A)\sin(kt)).$$

The coefficients of $a_0(dA)$, $a_k(dA)$, $b_0(dA)$ of dA are then computed using integration by parts, e.g.,

$$\begin{aligned}a_k(dA) &= \frac{1}{\pi}\int_0^{2\pi} \cos(kt)\, dA(t)\\ &= \frac{1}{\pi}(A(2\pi) - A(0)) + \frac{1}{\pi}k\int_0^{2\pi} \sin(kt)A(t)dt\\ &= \frac{1}{\pi}(A(2\pi) - A(0)) + k\ b_k(A), \quad k \geq 1.\end{aligned}$$

Next we compute the coefficients $a_0(B)$, $a_k(B)$, $b_k(B)$ of B. Because of $B\,dt = dA * dx$ we use results in Malliavin and Mancino (2002) for cross-volatilities, e.g.,

$$\begin{aligned}a_k(B) &= \lim_N \frac{2\pi}{N+1-n_0}\sum_{s=n_0}^{N} \frac{1}{2}(a_s(dA)\, a_{s+k}(dx) + a_s(dx)\, a_{s+k}(dA))\\ &= \lim_N \frac{2\pi}{N+1-n_0}\sum_{s=n_0}^{N} \frac{1}{2}(s\, b_s(A)\alpha_{s+k} + (s+k)\alpha_s b_{s+k}(A))\\ &= \lim_N \lim_M \frac{2\pi}{N+1-n_0}\frac{2\pi}{M+1-m_0}\sum_{s=n_0}^{N}\sum_{r=m_0}^{M} \frac{1}{2}\alpha_r(s\ \beta_{r+s}\alpha_{s+k} + (s+k)\alpha_s\beta_{r+s+k}).\end{aligned}$$

The coefficients of dB are then obtained by integration by parts. Finally, the computation of the coefficients of C is done in an analogous way. Since $C\,dt = dB * dx$, we use again the results in Malliavin and Mancino (2002) for cross-volatilities, e.g.,

$$\begin{aligned}
a_k(C) &= \lim_N \frac{2\pi}{N+1-n_0} \sum_{s=n_0}^{N} \frac{1}{2}(a_s(dB)\, a_{s+k}(dx) + a_s(dx)\, a_{s+k}(dB)) \\
&= \lim_N \frac{2\pi}{N+1-n_0} \sum_{s=n_0}^{N} \frac{1}{2}(s b_s(B)\alpha_{s+k} + (s+k)\alpha_s b_{s+k}(B)) \\
&= \lim_{N,M,L} \frac{2\pi}{N+1-n_0} \frac{2\pi}{M+1-m_0} \frac{\pi}{L+1-l_0} \\
&\qquad \sum_{s=n_0}^{N} \sum_{r=m_0}^{M} \sum_{t=l_0}^{L} \frac{1}{2}\Big\{ s\alpha_{s+k}[r\alpha_t\beta_{r+t}\beta_{r+s} - (r+s)\,\alpha_r\alpha_t\alpha_{r+t+s}]\Big\} \\
&\qquad + \Big\{(s+k)\alpha_s[r\alpha_t\beta_{r+t}\beta_{r+s+k} - (r+s+k)\,\alpha_r\alpha_t\alpha_{r+t+s+k}]\Big\}.
\end{aligned}$$

4.2 Estimation of λ on market data

Estimating λ on market data can be of great interest, since the theory suggests that the sign of λ is associated with stability of the market. A negative λ would witness a period of stability, while a positive λ would signal instability.

An estimate of λ can be achieved via the Fourier expansions explained in Section 4.1. Implementing these formulae, two practical problems are in order: first, the Fourier coefficients are given in the form of limits, so that we have to replace them with finite sums; secondly, the price process is not observed continuously. To recover the continuous-time price from discrete observations, several interpolation schemes have been adopted in the literature, see Barucci and Renò (2000). We will adopt the previous-tick interpolation scheme (stepwise function), which has been shown, in a Monte Carlo setting, to perform better than the linear interpolation scheme.

Precise estimation of quadratic and higher order variations asks for a huge quantity of data; then, high-frequency data are a natural candidate for this purpose. We used three data-sets; a data set containing quotes of the JPY-USD and DEM-USD foreign exchange rates, from October 1992 to September 1993, and a data set of IBM quotes, from January to December 1999[1].

The estimation of $A(t), B(t), C(t)$ can be accomplished using the formulas in Section 4.1, after truncating them to an empirically chosen order. We have two expansions to truncate: the first one is that relating the coefficients of A, B, C to the coefficients of dx; the second one is the inversion formula that provides A, B, C from their coefficients. The computation of the coefficients of dx must be stopped to an empirically chosen frequency, see Barucci and Renò (2000) for a discussion of this point. For instance, computing them for frequencies larger than the inverse of the smallest distance between adjacent observations induces aliasing effects.

In the case of the foreign exchange rates, estimates of $A(t), B(t), C(t)$ have been computed for a one day time window (24 hours). We used 500 coefficients of dx for the DEM-USD exchange rate and 160 coefficients for the JPY-USD exchange rate; in the inversion formula only 20 coefficients of A, 10 of B and 5 of C are used (leading to relatively smooth curves).

We summarize the computations by averaging over the full sample for illustrative purposes. Figures 1, 2 report the average estimate of $A(t), B(t), C(t)$ obtained with this technique, plus

[1]We acknowledge Olsen & Associates for the provision of the data set

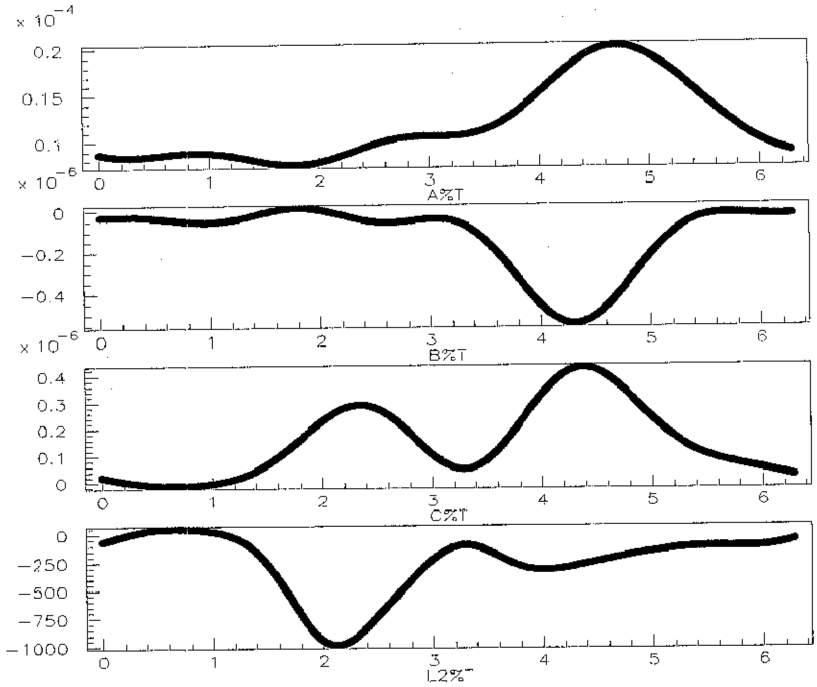

Figure 1: Average estimate of daily $A(t), B(t), C(t)$ on DEM-USD data, and corresponding value of $\lambda(t)$. On the x axis, the time window $[0, 2\pi]$ corresponds to one trading day (24 hours, starting at 21:00 GMT). $A(t)$ displays a three peak structure, each peak correspond to the opening of major markets (Asian, European, North-American).

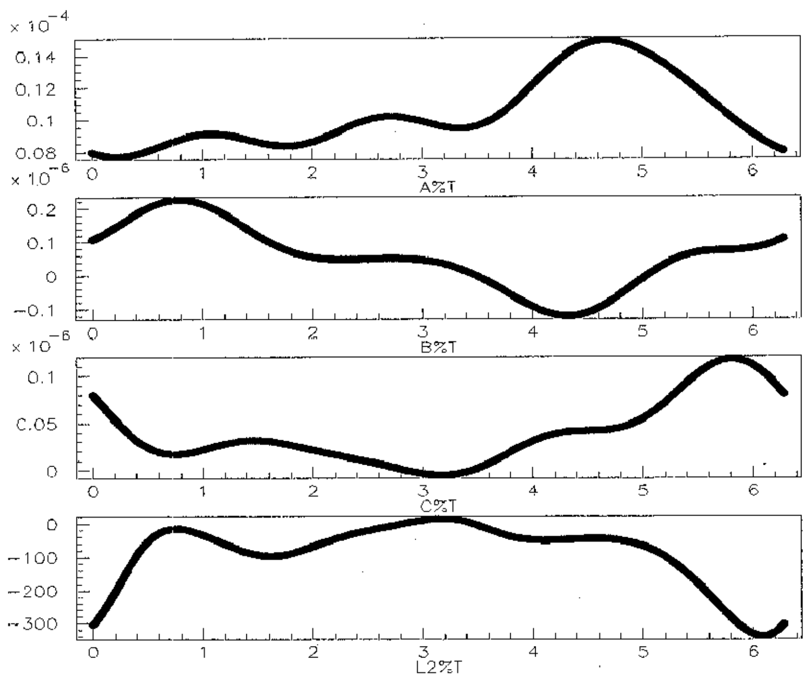

Figure 2: Average estimate of daily $A(t), B(t), C(t)$ on JPY-USD data, and corresponding value of $\lambda(t)$.

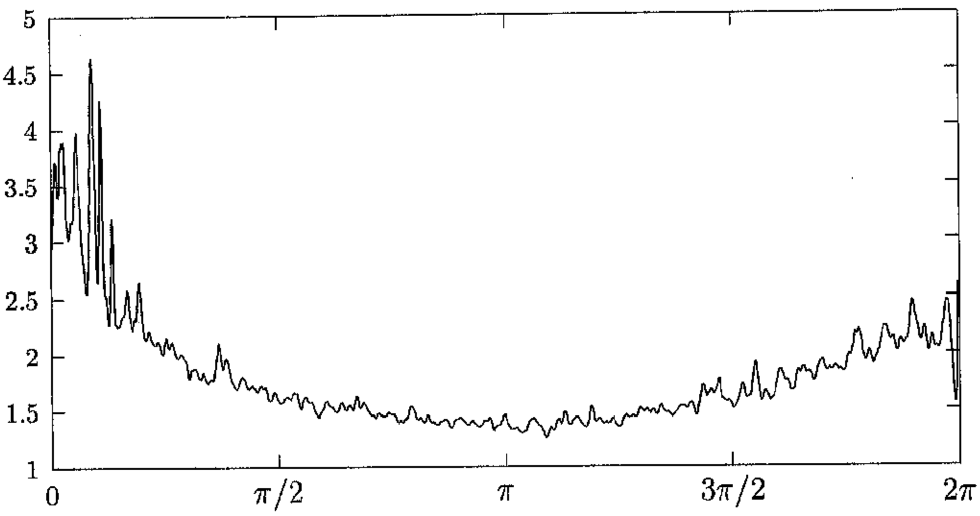

Figure 3: Average estimate of daily volatility $A(t)$ on IBM data. On the x axis, the time window $[0, 2\pi]$ corresponds to one trading day (6.5 hours). $A(t)$ displays the typical U-shape of volatility in stock markets.

the corresponding value of $\lambda(t)$ obtained by Eq. (2.1). $A(t)$ is linked to the intra-day volatility, which has been widely documented to display deterministic patterns. Looking at Figures 1, 2 we observe a three-peak structure in $A(t)$; the three peaks correspond, respectively, to the opening of Asian markets, European markets and North-American markets; the latter peak is the larger, as expected.

In the case of the IBM stock price, estimates of $A(t), B(t), C(t)$ have been computed for a one day time window (6.5 hours); the larger number of Fourier coefficients produces a higher resolution of the plots. Figure 3 gives the volatility $A(t)$ averaged over the full year; here we recognize the U-shape pattern which is typical of stock market intra-day volatility.

Short-horizon estimates of λ are the most important for traders. We present in Figure 4 as typical sample of daily (non-averaged) estimates the values for $A(t), B(t), C(t)$ and $\lambda(t)$ for two days in 1999.

It is noteworthy that taking the logarithm of the stock price mainly changes the scales of $A(t), B(t)$ and $C(t)$, but lets the shape of the curves more or less invariant. For this reason, in Figures 3, 4 the scales have been chosen according to the stock price (without taking logarithms).

On January 4, 1999, the beginning of the trading day reveals positivity of λ which detects instability of the market and which is revealed by subsequent large picks of A. Over the whole day the positive values of λ dominate and indicate an instable trading day.

On April 9, 1999, the beginning of the day reveals negativity of λ which detects stability of the market and which is revealed by a subsequent progressive damping of A. In contrast to January 4, over the whole trading day, small and mainly negative values of λ dominate and indicate stability.

As argued in this paper, precise estimation of λ could, in principle, result in important consequences for trading strategies. In this context, it can be of some help to analyze the sign of λ and λ_1 week-by-week or even day-by-day; our results show that, by using high-frequency data, an estimate of these effect can be readily accomplished.

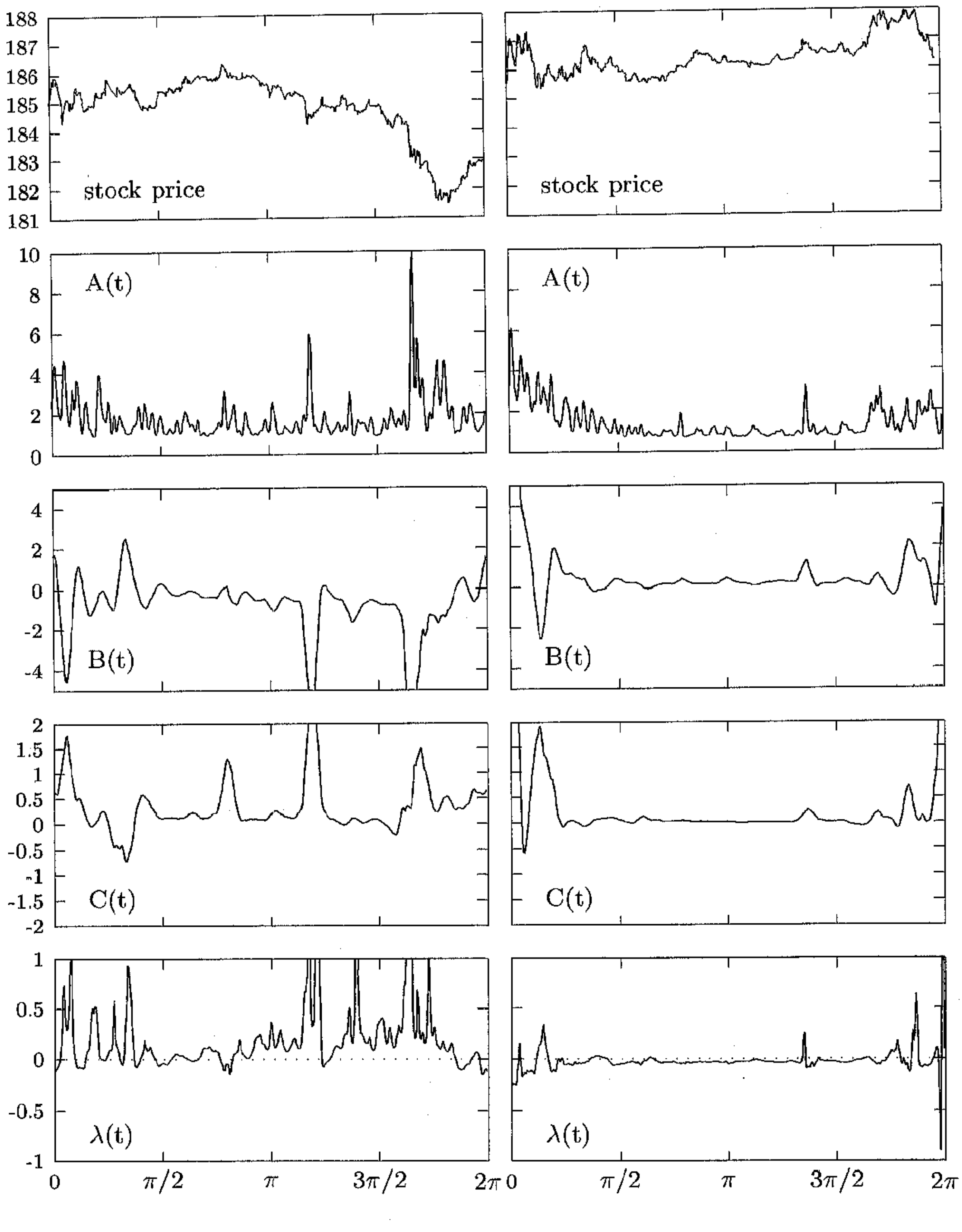

Figure 4: Daily values of $A(t), B(t), C(t), \lambda(t)$ on IBM data. The time windows $[0, 2\pi]$ correspond to two typical trading days in 1999 (6.5 hours each). Jan 4 (left-hand side) displays positivity of λ with large picks of A (instable market); April 9 (right-hand side) displays small and mainly negative values of λ along with a progressive damping of A (stable market).

References

Andersen, T. and Bollerslev, T. (1998) Answering the Skeptics: Yes, Standard Volatility Models do Provide Accurate Forecasts. *International Economic Review*, 39, pp. 885–905.

Barucci, E. and Renò, R. (2000) On measuring volatility and the GARCH forecasting performance. Forthcoming *Journal of International Financial Markets, Institutions and Money.*

Barucci, E. and Renò, R. (2002). On measuring volatility of diffusion processes with high frequency data. *Economics Letters*, 74, pp. 371–378.

Bates, D. (1995) Testing option pricing models. *NBER working paper*, No. 5129.

Bekaert, G. and Wu, G. (2000) Asymmetric volatility and risk in equity markets. *Review of Financial Studies*, 13, pp. 1–42.

Black, F. (1976) Studies of stock price volatility changes. *Proceedings of the 1976 Meetings of the American Statistical Association*, pp. 177–181.

Cox, J.C. (1997). The constant elasticity of variance option pricing model. *Journal of Portfolio Management*, Vol. 23, No. 3, pp. 15–17.

Cox, J.C. and Ross, S. (1976) The valuation of options for alternative stochastic processes. *Journal of Financial Economics*, 3, pp. 407–432.

Cruzeiro, A.B. and Malliavin, P. (1998) Non perturbative construction of Invariant Measure through Confinement by Curvature. *J. Math. Pures Appl.*, 77, pp. 527–537.

Derman, E. and Kani, I. (1994) Riding on the Smile. *RISK*, 7, pp. 32–39.

Dupire, B. (1994) Pricing with a smile. *RISK*, 7, pp. 18–20.

Fang, S. and Malliavin, P. (1993) Stochastic analysis on the path space of a Riemannian manifold. I: Markovian stochastic calculus. *J. Funct. Anal.*, 118, pp. 249–274.

Frey, R. and Stremme, A. (1997) Market volatility and feedback effects from dynamic hedging. *Mathematical Finance*, 7, pp. 351–374.

Friedman, M. (1953) The Case of Flexible Exchange Rates. *Essays in Positive Economics*, University of Chicago Press, Chicago.

Ghysels, E., Harvey, A. C. and Renault, E. (1996) Stochastic volatility. *Handbook of Statistics*, vol. 14, pp. 119–191, North-Holland, Amsterdam.

Hobson, D. G. and Rogers, L. C. G. (1998) Complete models with stochastic volatility. *Mathematical Finance*, 8, pp. 27–48.

Malliavin, P. and Mancino, M.E. (2002). Fourier Series method for measurement of Multivariate Volatilities. *Finance and Stochastics*, Vol. VI, No. 1.

Platen, E. and Schweizer, M. (1998) On feedback effects from hedging derivatives. *Mathematical Finance*, 8, pp. 67–84.