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Optimality Conditions and Duality in Fractional Programming Involving Semilocally Preinvex and Related Functions

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Abstract

In this paper, a nonlinear fractional programming problem is considered, where the functions involved are η-semidifferentiable.Necessary and sufficient optimality conditions are obtained. A dual is formulated and duality results are proved using concepts of semilocally preinvex, semilocally quasi-preinvex and semilocally pseudo-preinvex.Our results generalize the results obtained by Lyall, Suneja and Aggarwal [8].

Keywords : Generalized convexity; fractional programming; optimality conditions; duality

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1. Introduction

The importance of convex functions is well known in optimization theory. But for many mathematical models used in decision sciences, economics, management science, stochastics, applied mathematics and engineering, the notion of convexity does no longer suffice. Various generalizations of convex functions have been introduced in literature. Many of such functions preserve one or more properties of convex functions and give rise to models which are more adaptable to real-world situations than convex models. Between these we recall the class of semilocally convex, semilocally preinvex and related functions. Ewing [2] defined semilocally convex functions which he applied it to derive sufficient optimality conditions for variational and control problems. Such functions have certain important convex type properties, e.g., local minima of semilocally convex functions defined on locally starshaped sets are also global minima, and nonnegative linear combinations of semilocally convex functions are also semilocally convex. Some generalizations of semilocally convex functions and their properties were investigated by Kaur [7], Kaul and Kaur [3],[4],[5], Suneja and Gupta [17], Weir [18], Preda [12], Preda, Stancu-Minasian and Bătătorescu [14], Mukherjee and Mishra [11], Weir and Jeyakumar [19] and Mohan and Neogy [9]. Kaur [7] and Kaul and Kaur [3] defined semilocally quasiconvex and semilocally pseudoconvex functions. Kaul and Kaur [4] derived sufficient optimality criteria for a class of nonlinear programming problems by using generalized semilocally functions. Optimality conditions and duality results were given by Kaul and Kaur [5] for a nonlinear programming problem where the functions involved are semidifferentiable and generalized semilocally. Preda, Stancu-Minasian and Bătătorescu [14] (see also Preda and Bătătorescu [13]) obtained results related to optimality and duality in nonlinear programming involving semilocally preinvex and related functions. These results are extended to the multiple objective programming by Preda and Stancu-Minasian [15]. Optimality conditions and duality results were given by Lyall et al. [8] for a fractional programming problem involving semilocally convex and related functions.

In this paper, a nonlinear fractional programming problem is considered, where the functions involved are η-semidifferentiable.Necessary and sufficient optimality conditions are obtained. A dual is formulated and duality results are proved using concepts of semilocally preinvex, semilocally quasi-preinvex and semilocally pseudo-preinvex.

Due to the fact the class of semilocally preinvex functions is larger than the class of semilocally convex functions it results that this paper generalizes the work of Lyall, Suneja and Aggarwal [8]

The organization of the remainder of this paper is as follows. In Section 2, we shall introduce some notations and definitions which are used throughout the paper. In Section 3, we shall give necessary optimality criteria for a nonlinear fractional programming problem. In Section 4, we shall give sufficient optimality criteria. In Section 5, a dual is formulated and duality results of weak and strong duality for the pair of primal and dual programs are proved.

2. Definitions and Preliminaries

In this section, we shall introduce some notations and definitions which are used throughout the paper.

Let $\mathbf{R}^n$ be the n-dimensional Euclidean space and $\mathbf{R}^n_+$ be its positive orthant, i.e., $\mathbf{R}^n_+ = \{x \in \mathbf{R}^n, x_j \geqq 0,\ j = 1, \ldots, n\}$. Throughout this paper, the following conventions for vectors in $\mathbf{R}^n$ will be followed :

$x > y$ if and only if $x_i > y_i$ $(i = 1, \ldots, n)$,

$x \geqq y$ if and only if $x_i \geqq y_i$ $(i = 1, \ldots, n)$,

$x \geq y$ if and only if $x_i \geqq y_i$ $(i = 1, \ldots, n)$, but $x \neq y$.

Throughout this paper, all definitions, theorems, lemmas, corollaries, remarks are numbered consecutively in a single numerotation system in each section.

Let $X^0 \subseteq \mathbf{R}^n$ be a set and $\eta : X^0 \times X^0 \longrightarrow \mathbf{R}^n$ be a vectorial application.

Definition 2.1. *We say that the set X^0 is η-vex at $\bar{x} \in X^0$ if $\bar{x} + \lambda\eta(x, \bar{x}) \in X^0$ for any $x \in X^0$ and any $\lambda \in [0, 1]$.*

We say that the set X^0 is η-vex if X^0 is η-vex at any $x \in X^0$.

We remark that if $\eta(x, \bar{x}) = x - \bar{x}$ for any $x \in X^0$ then X^0 is η-vex at $\bar{x}$ iff X^0 is a convex set at $\bar{x}$.

Definition 2.2. [1].*Let $X^0 \subseteq \mathbf{R}^n$ be a non-empty set. The function $f : X^0 \to \mathbf{R}$ is preinvex on X^0(with respect to η) (briefly, f is η-vex) if there exists an n-dimensional vector function $\eta : X^0 \times X^0 \to \mathbf{R}^n$ such that, for all $x, u \in X^0$ and $\lambda \in [0, 1]$, we have*

$$f\left(u + \lambda\eta\left(x, u\right)\right) \leqq \lambda f\left(x\right) + \left(1 - \lambda\right) f\left(u\right).$$

An m-dimensional vector-valued function $\psi : X^0 \to \mathbf{R}^m$ is preinvex on X^0 (with respect to η) if each of its components is preinvex on X^0 (with respect to η).

Definition 2.3. *We say that the set $X^0 \subseteq \mathbf{R^n}$ is an η-locally starshaped set at $\bar{x}$ ($\bar{x} \in X^0$) if for any $x \in X^0$ there exists $0 < a_\eta(x,\bar{x}) \leqq 1$ such that $\bar{x} + \lambda\eta(x,\bar{x}) \in X^0$ for any $\lambda \in [0, a_\eta(x,\bar{x})]$.*

We say that the set X^0 is η-locally starshaped if X^0 is η-locally starshaped at any $x \in X^0$.

Definition 2.4.[14]. *Let $f : X^0 \longrightarrow \mathbf{R}$ be a function, where $X^0 \subseteq \mathbf{R^n}$ is an η-locally starshaped set at $\bar{x} \in X^0$. We say that f is:*

(i_1) semilocally preinvex (slpi) at $\bar{x}$ if corresponding to $\bar{x}$ and each $x \in X^0$, there exists a positive number $d_\eta(x,\bar{x}) \leqq a_\eta(x,\bar{x})$ such that

$$f(\bar{x} + \lambda\eta(x,\bar{x})) \leqq \lambda f(x) + (1-\lambda)f(\bar{x}), \quad 0 < \lambda < d_\eta(x,\bar{x}). \tag{2.1}$$

The function f is strictly semilocally preinvex (sslpi) at $\bar{x} \in X^0$ if for each $x \in X^0, x \neq x^0$ the inequality (2.1) is strict.

If f is slpi (sslpi) at each $\bar{x} \in X^0$ then f is said to be slpi (sslpi) on X^0.

(i_2) semilocally quasi-preinvex (slqpi) at $\bar{x}$ if corresponding to $\bar{x}$ and each $x \in X^0$, there exists a positive number $d_\eta(x,\bar{x}) \leqq a_\eta(x,\bar{x})$ such that $f(x) \leqq f(\bar{x})$, $0 < \lambda < d_\eta(x,\bar{x})$ implies $f[\bar{x} + \lambda\eta(x,\bar{x})] \leqq f(\bar{x})$.

If f is slqpi at each $\bar{x} \in X^0$ then f is said to be slqpi on X^0.

Definition 2.5.[14, 15]. *Let $f : X^0 \longrightarrow \mathbf{R}$ be a function, where $X^0 \subseteq \mathbf{R^n}$ is an η-locally starshaped set at $\bar{x} \in X^0$. We say that f is η-semidifferentiable at $\bar{x}$ if $(df)^+(\bar{x},\eta(x,\bar{x}))$ exists for each $x \in X^0$, where*

$$(df)^+(\bar{x},\eta(x,\bar{x})) = \lim_{\lambda\to 0^+} \frac{1}{\lambda}[f(\bar{x} + \lambda\eta(x,\bar{x})) - f(\bar{x})]$$

(the right derivative at $\bar{x}$ along the direction $\eta(x,\bar{x})$).
If f is η-semidifferentiable at any $\bar{x} \in X^0$, then f is said to be η-semidifferentiable on X^0.

It may be noted that the semidifferentiable functions correspond to $\eta(x,\overline{x}) = x - \overline{x}$. Some properties possesed by the semidifferentiable functions are given by Kaul and Lyall [6]

Definition 2.6. *Let $f : X^0 \longrightarrow \mathbf{R}$ be an η-semidifferentiable function on $X^0 \subseteq \mathbf{R}^n$. We say that f is semilocally pseudo-preinvex (slppi) at $\bar{x} \in X^0$ if*

$$(df)^+(\bar{x},\eta(x,\bar{x})) \geqq 0 \Rightarrow f(x) \geqq f(\overline{x})$$

If f is slppi at each $\bar{x} \in X^0$ then f is said to be slppi on X^0.

We define a function f to be semilocally preincave, semilocally quasi-preincave, semilocally pseudo-preincave, according as $-f$ is semilocally preinvex, semilocally quasi-preinvex, semilocally pseudo-preinvex, respectively.

Theorem 2.7. *Let $f : X^0 \longrightarrow \mathbf{R}$ be an η-semidifferentiable function on X^0 an η-locally starshaped set. The function f is slpi at $\bar{x} \in X^0$ if and only if $(df)^+(\bar{x}, \eta(x, \bar{x}))$ exists and*

$$f(x) - f(\overline{x}) \geqq (df)^+(\bar{x}, \eta(x, \bar{x})) \tag{2.2}$$

and if f is strictly semilocally preinvex at $\overline{x} \in X^0$, then

$$f(x) - f(\overline{x}) > (df)^+(\bar{x}, \eta(x, \bar{x})),\ \forall\, x \neq \bar{x}.$$

Proof: Suppose f is slpi at $\bar{x} \in X^0$. Therefore, for each $x \in X^0$, there exists a positive number $d_\eta(x, \bar{x}) \leqq a_\eta(x, \bar{x})$ such that $f(\bar{x}+\lambda\eta(x, \bar{x})) \leqq \lambda f(x)+(1-\lambda)f(\bar{x})$ for $0 < \lambda < d_\eta(x, \bar{x})$, i.e.:

$$\frac{f(\bar{x} + \lambda\eta(x, \bar{x})) - f(\overline{x})}{\lambda} \leqq [f(x) - f(\overline{x})].$$

Taking the limit as $\lambda \to 0^+$, we have the relation (2.2). The converse part is evidently.

The proof in the strict case is similarly.

Theorem 2.8. *Let $f : X^0 \longrightarrow \mathbf{R}$ be an η-semidifferentiable function on X^0 an η-locally starshaped set. If f is slqpi then:*

$$f(x) \leqq f(\overline{x}) \Rightarrow (df)^+(\bar{x}, \eta(x, \bar{x})) \leqq 0$$

Proof: The function f is η-semidifferentiable, therefore $(\mathrm{d}f)^+(\bar{x}, \eta(x, \bar{x}))$ exists. Also, f is slqpi, therefore there exists a positive number $d_\eta(x, \bar{x}) \leqq a_\eta(x, \bar{x})$ such that

$$\left.\begin{array}{l} f(x) \leqq f(\overline{x}) \\ 0 < \lambda < \mathrm{d}_\eta(x, \bar{x}) \end{array}\right\} \Rightarrow f[\bar{x} + \lambda\eta(x, \bar{x})] \leqq f(\overline{x})$$

i.e.

$$f[\bar{x} + \lambda\eta(x, \bar{x})] - f(\overline{x}) \leqq 0.$$

Dividing by $\lambda > 0$ and taking the limit as $\lambda \longrightarrow 0^+$, we get

$$(\mathrm{d}f)^+(\bar{x}, \eta(x, \bar{x})) \leqq 0.$$

This completes the proof.

A number of properties of semilocally preinvex, quasi-preinvex, pseudo-preinvex, η-semidifferentiable functions are given by Preda, Stancu-Minasian and Bătătorescu [14]. The main properties are summarized in the following

Theorem 2.9. [14].*Let functions* $f, g : X^0 \to \mathbf{R}_+$.

a) If f and $-g$ are slpi on X^0 and g is strictly positive and finite on X^0, then $q = f/g$ is a slqpi function on X^0;

b) If $-f$ is slpi and non-negative function on X^0 and g is slpi, strictly positive and finite function on X^0, then $-q$ is slqpi on X^0;

c) If g is strictly positive and finite function on X^0, then $1/g$ is slqpi on X^0 if and only if $-g$ is slqpi on X^0;

d) If f is a slpi and non-negative function on X^0, $-g$ is a slpi, strictly negative and finite function on X^0, then f^2/g is slpi on X^0;

e) If f and $-g$ are slpi and non-negative functions on X^0, then $-fg$ is slqpi on X^0.

3. Necessary Optimality Criteria

Consider the following nonlinear fractional programming problem (P) :

$$\text{Max } q(x) = \frac{f(x)}{g(x)}$$

(P) subject to

$$h(x) \leqq 0$$
$$x \in X^0$$

where

i) $X^0 \subseteq \mathbf{R}^n$ is a non-empty η-locally starshaped set,

ii) $f : X^0 \to \mathbf{R}$, is η-semidifferentiable function and $f(x) \geqq 0, \forall x \in X^0$,

iii) $g : X^0 \to \mathbf{R}$ is η-semidifferentiable function and $g(x) > 0, \forall x \in X^0$,

iv) $h : X^0 \to \mathbf{R}^m$ is η-semidifferentiable function.

Let $X = \{x \in X^0 \mid h(x) \leqq 0\}$ be the set of all feasible solutions for (P).

Let $N_\varepsilon(\overline{x})$ denote the neighbourhood of $\bar{x} \in \mathbf{R}^n$, i.e.,

$$N_\varepsilon(\overline{x}) = \{x \in \mathbf{R}^n \mid \|x - \overline{x}\| < \varepsilon\}.$$

Definition 3.1. *$\overline{x}$ is a local maximum solution of the problem (P) if $\bar{x} \in X$ and there exists $\varepsilon > 0$ such that*

$$x \in N_\varepsilon(\overline{x}) \cap X \Rightarrow f(\bar{x}) \geqq f(x).$$

For $\bar{x} \in X$ we denote

$$I = \{i | h_i(\bar{x}) = 0\},$$
$$J = \{i | h_i(\bar{x}) < 0\},$$

and

$$h_I = (h_i)_{i \in I}.$$

Obviously $I \cup J = \{1, 2, \ldots, m\}$.

Definition 3.2. *We say that the function h satisfies the generalized Slater's constraint qualification (GSQ) at $\bar{x} \in X$, if h_i is semilocally pseudo-preinvex at $\bar{x}$ and there exists an $\hat{x} \in X$ such that $h_i(\hat{x}) < 0$ for $i \in I$.*

In what follows we need the following theorem of the alternatives stated by Weir and Mond [20].

Theorem 3.3. ([20], Theorem 2.1). *Let X^0 be a non-empty set in $\mathbf{R}^n$ and let $f : X^0 \to \mathbf{R}^k$, be a preinvex function on X^0(with respect to η). Then either*

$f(x) < 0$ *has a solution* $x \in X^0$

or

$\lambda f(x) \geqq 0$ *for all* $x \in X^0$, *for some* $\lambda \in \mathbf{R}^k, \lambda \geqq 0$,

but both alternatives are never true.

Lemma 3.4. *Let $\bar{x} \in X$ be a (local) maximum solution for (P). We assume that h_i is continuous at $\bar{x}$ for any $i \in J$, and that f, g and h_I are η-semidifferentiable at $\bar{x}$. Then the system*

$$(df)^+(\bar{x}, \eta(x, \bar{x})) > 0 \tag{3.1}$$

$$(dg)^+(\bar{x}, \eta(x, \bar{x})) < 0 \tag{3.2}$$

$$(dh_I)^+(\bar{x}, \eta(x, \bar{x})) < 0 \tag{3.3}$$

has no solution $x \in X^0$.

Proof. Let $\bar{x} \in X$ be a (local) maximum solution for (P). We assume *ad absurdum* that the system (3.1)-(3.3) has a solution $x^0 \in X^0$ i.e.

$$(df)^+(\bar{x}, \eta(x^0, \bar{x})) > 0 \tag{3.4}$$

$$(dg)^+(\bar{x}, \eta(x^0, \bar{x})) < 0 \tag{3.5}$$

$$(dh_I)^+(\bar{x}, \eta(x^0, \bar{x})) < 0 \tag{3.6}$$

Consider the function

$$\Phi_1\left(\bar{x}, \eta\left(x^0, \bar{x}\right), \lambda\right) = f\left(\bar{x} + \lambda\eta\left(x^0, \bar{x}\right)\right) - f\left(\bar{x}\right)$$

which vanishes at $\lambda = 0$.

The right differential of $\Phi_1\left(\bar{x}, \eta\left(x^0, \bar{x}\right), \lambda\right)$with respect to λ at $\lambda = 0$ is given by

$$\lim_{\lambda \to 0+} \frac{\Phi_1\left(\bar{x}, \eta\left(x^0, \bar{x}\right), \lambda\right) - \Phi_1\left(\bar{x}, \eta\left(x^0, \bar{x}\right), 0\right)}{\lambda} =$$

$$= \lim_{\lambda \to 0+} \frac{f\left(\bar{x} + \lambda\eta\left(x^0, \bar{x}\right)\right) - f\left(\bar{x}\right)}{\lambda} = (df)^+(\bar{x}, \eta(x^0, \bar{x})) > 0 \text{ (using (3.4))}.$$

Therefore there exists δ_1 such that

$$\Phi_1\left(\bar{x}, \eta\left(x^0, \bar{x}\right), \lambda\right) > 0, \quad \lambda \in (0, \delta_1)$$

i.e.

$$f\left(\bar{x} + \lambda\eta\left(x^0, \bar{x}\right)\right) > f\left(\bar{x}\right), \quad \lambda \in (0, \delta_1) \tag{3.7}$$

Similarly, if we consider

$$\Phi_2\left(\bar{x}, \eta\left(x^0, \bar{x}\right), \lambda\right) = g\left(\bar{x}\right) - g\left(\bar{x} + \lambda\eta\left(x^0, \bar{x}\right), \lambda\right)$$

and

$$\Phi_3\left(\bar{x}, \eta\left(x^0, \bar{x}\right), \lambda\right) = h_I\left(\bar{x}\right) - h_I\left(\bar{x} + \lambda\eta\left(x^0, \bar{x}\right), \lambda\right)$$

and using (3.5) and (3.6) we have

$$g\left(\bar{x} + \lambda\eta\left(x^0, \bar{x}\right)\right) < g\left(\bar{x}\right), \; \lambda \in (0, \delta_2) \tag{3.8}$$

and

$$h_I\left(\bar{x}+\lambda\eta\left(x^0,\bar{x}\right)\right)<h_I\left(\bar{x}\right),\ \lambda\in\left(0,\delta_3\right) \tag{3.9}$$

From (3.9) and the definition of I , it results

$$h_I\left(\bar{x}+\lambda\eta\left(x^0,\bar{x}\right)\right)<0,\ \lambda\in\left(0,\delta_3\right).$$

Also, $h_i\left(\bar{x}\right)<0$ for $i\in J$ and from the continuity of h_i at $\bar{x}$, there exists $\delta_i^*>0$ such that

$$h_i\left(\bar{x}+\lambda\eta\left(x^0,\bar{x}\right)\right)<0,\ \lambda\in\left(0,\delta_i^*\right). \tag{3.10}$$

Let $\delta^*=\min\left(\delta_1,\delta_2,\delta_3,\delta_i^*(i\in J)\right)$. For $\lambda\in\left(0,\delta^*\right)$ we have

$$\bar{x}+\lambda\eta\left(x^0,\bar{x}\right)\in S\left(\bar{x},\delta^*\right)\subseteq N_{\delta^*}\left(\bar{x}\right) \tag{3.11}$$

where $S\left(\bar{x},\delta^*\right)$ is open sphere of center $\bar{x}$ and radius δ^*.

By the choosing of δ^*, from (3.7), (3.8), (3.9) and (3.10), it results

$$f\left(\bar{x}+\lambda\eta\left(x^0,\bar{x}\right)\right)>f\left(\bar{x}\right),\ \lambda\in\left(0,\delta^*\right) \tag{3.12}$$

$$g\left(\bar{x}+\lambda\eta\left(x^0,\bar{x}\right)\right)<g\left(\bar{x}\right),\ \lambda\in\left(0,\delta^*\right) \tag{3.13}$$

$$h_I\left(\bar{x}+\lambda\eta\left(x^0,\bar{x}\right)\right)<0,\ \lambda\in\left(0,\delta^*\right) \tag{3.14}$$

$$h_J\left(\bar{x}+\lambda\eta\left(x^0,\bar{x}\right)\right)<0,\ \lambda\in\left(0,\delta^*\right). \tag{3.15}$$

From (3.11), (3.14) and (3.15), it follows that

$$\bar{x}+\lambda\eta\left(x^0,\bar{x}\right)\in N_{\delta^*}\left(\bar{x}\right)\cap X,\ \lambda\in\left(0,\delta^*\right)$$

and from (3.12) and (3.13), we have

$$q\left(\bar{x}+\lambda\eta\left(x^0,\bar{x}\right)\right)>q\left(\bar{x}\right)$$

which contradicts the assumption that $\bar{x}$ is a local maximum solution for (P). Therefore, there exist no $x\in X^0$ such that

$$(df)^+\left(\bar{x},\eta\left(x,\bar{x}\right)\right)>0$$

$$(dg)^+\left(\bar{x},\eta\left(x,\bar{x}\right)\right)<0$$

$$(dh_I)^+\left(\bar{x},\eta\left(x,\bar{x}\right)\right)<0.$$

The proof is complete.

Now we have the following Fritz John type necessary optimality criteria.

Theorem 3.5. *Let us suppose that h_i is continuous at $\bar{x}$ for $i \in J, -(df)^+(\bar{x}, \eta(x, \bar{x}))$, $(dg)^+(\bar{x}, \eta(x, \bar{x}))$, $(dh_I)^+(\bar{x}, \eta(x, \bar{x}))$ are η-vex functions of x on X^0- a η-vex set at $\bar{x}$. If $\bar{x}$ is a (local) maximum solution for Problem (P), then there exist, $\bar{u}_0 \in \mathbf{R}$, $\bar{u} \in \mathbf{R}^m$, $\bar{\lambda} \in \mathbf{R}$ such that $(\bar{x}, \bar{u}_0, \bar{\lambda}, \bar{u})$ satisfies the following conditions:*

$$-\bar{u}_0(df)^+(\bar{x}, \eta(x, \bar{x})) + \overline{\lambda}(dg)^+(\bar{x}, \eta(x, \bar{x})) +$$

$$+\overline{u}(dh)^+(\bar{x}, \eta(x, \bar{x})) \geqq 0, \quad \forall x \in X^0 \tag{3.16}$$

$$\bar{u} \cdot h(\bar{x}) = 0 \tag{3.17}$$

$$h(\bar{x}) \leqq 0 \tag{3.18}$$

$$(\bar{u}_0, \bar{\lambda}, \bar{u}) \geqq 0, (\bar{u}_0, \bar{\lambda}, \bar{u}) \neq 0 \tag{3.19}$$

Proof. Let $\bar{x} \in X$ be a (local) maximum solution for (P). Since the conditions of Lemma 3.4 are satisfied, we get that the system (3.1)-(3.3) has no solution $x \in X^0$. But the assumptions of Theorem 3.3 also hold and since the system (3.1)-(3.3) has no solution $x \in X^0$ we obtain that there exist $\bar{u}_0 \in \mathbf{R}, \bar{u}_i \in \mathbf{R}\,(i \in I), \bar{\lambda} \in \mathbf{R}$, such that

$$-\bar{u}_0(df)^+(\bar{x}, \eta(x, \bar{x})) + \overline{\lambda}(\mathrm{d}g)^+(\bar{x}, \eta(x, \bar{x})) +$$

$$+\overline{u_I}(dh_I)^+(\bar{x}, \eta(x, \bar{x})) \geqq 0, \quad \forall x \in X^0 \tag{3.20}$$

$$(\bar{u}_0, \bar{\lambda}, \bar{u}_I) \geqq 0, (\bar{u}_0, \bar{\lambda}, \bar{u}_I) \neq 0 \tag{3.21}$$

If we define $\bar{u}_J = 0$, by (3.20), we get (3.16). Since $h_I(\bar{x}) = 0$ then for $\bar{u} = (\bar{u}_I, \bar{u}_J)$ we have

$$\bar{u} \cdot h(\bar{x}) = 0 \tag{3.22}$$

i.e. the relation (3.17).

The relation (3.18) results from $\bar{x} \in X$. The proof is complete.

Now we consider the parametric problem

$$\text{Max } f(x) - \lambda g(x), \; \lambda \in \mathbf{R} \; (\lambda \text{ parameter})$$

(P_λ) subject to

$$h(x) \leqq 0$$
$$x \in X^0.$$

It is well known that (P_λ) is closely related to problem (P).

The following lemma is well known in fractional programming [16].

Lemma 3.6. *$\bar{x}$ is an optimal solution for the Problem (P) if and only if it is optimal solution for the Problem $(P_{\bar{\lambda}})$ with $\bar{\lambda} = f(\bar{x})/g(\bar{x})$.*

The next Theorem is a Kuhn-Tucker type necessary optimality criteria and results from Lemma 3.6 and Theorem 3.5.

Theorem 3.7. *Let us suppose that h_i is continuous at $\bar{x}$ for $i \in J, -(\mathrm{d}f)^+(\bar{x}, \eta(x, \bar{x}))$, $(\mathrm{d}g)^+(\bar{x}, \eta(x, \bar{x})), (\mathrm{d}h_I)^+(\bar{x}, \eta(x, \bar{x}))$ are η-vex functions of x on X^0- a η-vex set at $\bar{x}$ and h satisfies GSQ at $\bar{x}$. If $\bar{x}$ is a (local) maximum solution for Problem (P), then there exist $\bar{\lambda} \in \mathbf{R}, \bar{u} \in \mathbf{R}^m$ such that*

$$-(df)^+(\bar{x}, \eta(x, \bar{x})) + \bar{\lambda}(dg)^+(\bar{x}, \eta(x, \bar{x})) +$$

$$+\bar{u}(dh)^+(\bar{x}, \eta(x, \bar{x})) \geqq 0, \quad \forall x \in X^0 \tag{3.23}$$

$$f(\bar{x}) - \bar{\lambda} g(\bar{x}) = 0 \tag{3.24}$$

$$\bar{u} \cdot h(\bar{x}) = 0 \tag{3.25}$$

$$h(\bar{x}) \leqq 0 \tag{3.26}$$

$$(\bar{\lambda}, \bar{u}) \geqq 0, (\bar{\lambda}, \bar{u}) \neq 0 \tag{3.27}$$

4. Sufficient optimality criteria

Theorem 4.1. *Let $\bar{x} \in X^0 \subseteq \mathbf{R}^n, \bar{u} \in \mathbf{R}^m$ and f be semilocally preincave at $\bar{x}$ and g and h be semilocally preinvex at $\bar{x}$. We assume that at $\bar{x}, f, g$ and h are η-semidifferentiable and $(\bar{x}, \bar{u})$ satisfies the following conditions :*

$$-(df)^+(\bar{x}, \eta(x, \bar{x})) + \bar{u}(dh)^+(\bar{x}, \eta(x, \bar{x})) \geqq 0, \ \forall\, x \in X \tag{4.1}$$

$$(dg)^+(\bar{x}, \eta(x, \bar{x})) \geqq 0, \ \forall\, x \in X \tag{4.2}$$

$$\bar{u} \cdot h(\bar{x}) = 0 \tag{4.3}$$

$$h(\bar{x}) \leqq 0 \tag{4.4}$$

$$\bar{u} \geqq 0, \bar{u} \neq 0 \tag{4.5}$$

Then $\bar{x}$ is a maximum solution for the Problem (P).

Proof. Let $(\bar{x}, \bar{u})$ satisfy conditions (4.1)-(4.5). Relation (4.4) yields that $\bar{x} \in X$, hence $\bar{x}$ is a feasible solution of the Problem (P).The function f is semilocally preincave. Therefore, for any $x \in X$, Theorem 2.7 , yields.

$$f(x) - f(\bar{x}) \leqq (\mathrm{d}f)^{+}(\bar{x}, \eta(x, \bar{x}))$$

or

$$\begin{aligned} f(\bar{x}) - f(x) &\geqq -(\mathrm{d}f)^{+}(\bar{x}, \eta(x, \bar{x})) \geqq -\bar{u}\,(\mathrm{d}h)^{+}(\bar{x}, \eta(x, \bar{x}))\ (\text{by } (4.1)) \\ &\geqq -\bar{u}\,[h(x) - h(\bar{x})]\ \ (\text{since } h \text{ is slpi at } \bar{x}) \\ &= -\bar{u} \cdot h(x)\ \ (\text{by } (4.3)) \\ &\geqq 0\ (\text{by } (4.5)). \end{aligned}$$

Thus

$$f(\bar{x}) \geqq f(x) \text{ for any } x \in X \tag{4.6}$$

Since g is semilocally preinvex, by Theorem 2.7, it results that

$$g(x) - g(\bar{x}) \geqq (\mathrm{d}g)^{+}(\bar{x}, \eta(x, \bar{x})) \geqq 0\ \ (\text{by } (4.2))$$

Therefore

$$g(x) \geqq g(\bar{x})\ \forall\, x \in X \tag{4.7}$$

Thus, from (4.6) and (4.7), it follows that

$$q(x) \leq g(\bar{x}),\ \forall\, x \in X.$$

Hence, $\bar{x}$ is an optimal solution of Problem (P).

Corollary 4.2. *Let $\bar{x} \in X^0 \subseteq \mathbf{R}^n, \bar{u}_0 \in \mathbf{R}, \bar{u} \in \mathbf{R}^m$ and f be semilocally preincave at $\bar{x}$ and g and h be semilocally preinvex at $\bar{x}$. We assume that at $\bar{x}, f, g$ and h are η-semidifferentiable and $(\bar{x}, \bar{u}_0, \bar{u})$ satisfies the following conditions :*

$$-\bar{u}_0(df)^{+}(\bar{x}, \eta(x, \bar{x})) + \bar{u}(dh)^{+}(\bar{x}, \eta(x, \bar{x})) \geqq 0,\ \forall\, x \in X \tag{4.8}$$

$$(dg)^{+}(\bar{x}, \eta(x, \bar{x})) \geqq 0,\ \forall\, x \in X \tag{4.9}$$

$$\bar{u} \cdot h(\bar{x}) = 0 \tag{4.10}$$

$$h(\bar{x}) \leqq 0 \tag{4.11}$$

$$\bar{u} \geqq 0, \bar{u} \neq 0 \tag{4.12}$$

$$\bar{u}_0 > 0. \tag{4.13}$$

Then $\bar{x}$ is a maximum solution for the Problem (P).

Proof. Since $\bar{u}_0 > 0$ (by (4.13)), therefore $(\bar{x}, \bar{u}/\bar{u}_0)$ satisfies conditions (4.1)-(4.5) of Theorem 4.1 and hence $\bar{x}$ is an optimal solution of Problem (P).

Remark 4.3. *In the statement of the Theorem 4.1 and Corollary 4.2 it suffices to assume only the semilocal preinvexity of h_I at $\bar{x}$ instead of h.*

Theorem 4.4. *Let $\bar{x} \in X^0, \bar{u}_0 \in \mathbf{R}, \bar{u} \in \mathbf{R}^m, f$ semilocally preincave , g semilocally preinvex and h strictly semilocally preinvex at $\bar{x}$. We assume that at $\bar{x}, f, g$ and h are η-semidifferentiable and $(\bar{x}, \bar{u}_0, \bar{u})$ satisfies conditions (4.8)-(4.13).*

Then $\bar{x}$ is a maximum solution for the Problem (P).

Proof. From the relations (4.10) and (4.11) we obtain $\bar{u}_i = 0$ for $i \in J$ and thus (4.8) may be written as

$$-\bar{u}_0 \, (\mathrm{d}f)^+ \, (\bar{x}, \eta\,(x, \bar{x})) + \bar{u}_I \, (\mathrm{d}h_I)^+ \, (\bar{x}, \eta\,(x, \bar{x})) \geqq 0, \;\; \forall\, x \in X \tag{4.14}$$

From (4.12) and (4.13), we obtain

$$(\bar{u}_0, \bar{u}_I) \geqq 0, (\bar{u}_0, \bar{u}_I) \neq 0 \tag{4.15}$$

and from (4.14) and (4.15), we obtain that the system

$$\left.\begin{array}{l} (df)^+(\bar{x}, \eta(x, \bar{x})) > 0 \\ (dh_I)^+(\bar{x}, \eta(x, \bar{x})) < 0 \end{array}\right\} \tag{4.16}$$

has no solution $x \in X$. We can infer that $f\,(x) \leqq f\,(\bar{x}) \;\forall\, x \in X$. Indeed, if there exists $x^0 \in X$ such that $f\,(x^0) > f\,(\bar{x})$, then

$$h_I\left(x^0\right) \leqq 0 \leqq h_I\,(\bar{x}) \,.$$

From the semilocally preincavity of f and strict semilocally preinvexity of h_I at $\bar{x}$ we have

$$0 < f\left(x^0\right) - f\,(\bar{x}) \leqq (df)^+(\bar{x}, \eta(x^0, \bar{x}))$$
$$0 \geqq h_I\left(x^0\right) - h_I\,(\bar{x}) > (dh_I)^+(\bar{x}, \eta(x^0, \bar{x}))$$

i.e. the system

$$(df)^+(\bar{x}, \eta(x^0, \bar{x})) > 0$$
$$(dh_I)^+(\bar{x}, \eta(x^0, \bar{x})) < 0$$

has a solution x^0, which is a contradiction to (4.16). Therefore,

$$f(x) \leqq f(\bar{x}), \forall\, x \in X. \tag{4.17}$$

Similar as in the proof of Theorem 4.1 from the semilocally preinvexity of g, it results

$$g(x) \geqq g(\bar{x}), \ \forall\, x \in X. \tag{4.18}$$

Combining (4.17) and (4.18), we conclude that

$$q(x) \leqq q(\bar{x}), \ \forall\, x \in X.$$

Hence, $\bar{x}$ is an optimal solution of (P). This completes the proof of the theorem.

Theorem 4.5. *Let $\bar{x} \in X^0, \bar{u} \in \mathbf{R}^m, f$ semilocally preincave, g semilocally preinvex and h_I be semilocally quasi-preinvex at $\bar{x}$. We assume that at $\bar{x}, f, g$ and h are η-semidifferentiable and $(\bar{x}, \bar{u}_0, \bar{u})$ satisfies conditions (4.1)-(4.5). Then $\bar{x}$ is a maximum solution for the Problem (P).*

The proof follows as in Lyall et al. ([8], Theorem 4.3).

5. Duality

For Problem (P) we consider the following dual problem (D) :

$$\min \ v(\lambda) = \lambda$$

(D) subject to

$$-(\mathrm{d}f)^{+}(y, \eta(x, y)) + \lambda(\mathrm{d}g)^{+}(y, \eta(x, y)) +$$

$$+u(\mathrm{d}h)^{+}(y, \eta(x, y)) \geqq 0, \ \forall\, x \in X \tag{5.1}$$

$$f(y) - \lambda g(y) \leqq 0 \tag{5.2}$$

$$u \cdot h(y) \geqq 0 \tag{5.3}$$

$$u \geqq 0, y \in X^0, u \in \mathbf{R}^m, \lambda \in \mathbf{R}, \lambda \geqq 0. \tag{5.4}$$

Let T denote the set of all feasible solutions of Problem (D).

Theorem 5.1. *(Weak Duality). If $x \in X$ and $(y, \lambda, u) \in T$ and f is semilocally preincave and g and h are semilocally preinvex then*

$$q(x) \leqq v(\lambda).$$

Proof. Semilocally preincavity of f and Theorem 2.7, yield

$$f(x) - f(y) \leqq (\mathrm{d}f)^{+}(y, \eta(x,y))$$

$$\leqq \lambda (\mathrm{d}g)^{+}(y, \eta(x,y)) + u(\mathrm{d}h)^{+}(y, \eta(x,y)) \text{ (using (5.1))}$$

$$\leqq \lambda \{g(x) - g(y)\} + u\{h(x) - h(y)\} \text{ (by semilocally preinvexity of } g \text{ and } h).$$

Or

$$f(x) - \lambda g(x) \leqq \{f(y) - \lambda g(y)\} + u\{h(x) - h(y)\} \leqq 0$$

using (5.2), (5.3) and (5.4) and $x \in X$. Thus

$$\frac{f(x)}{g(x)} \leqq \lambda$$

i.e.

$$q(x) \leqq v(\lambda).$$

The weak duality theorem take place in weaker conditions on f, g and h.

Theorem 5.2. *If $x \in X$ and $(y, \lambda, u) \in T$ and $-f + \lambda g + uh$ is semilocally pseudo-preinvex, then $q(x) \leqq v(\lambda)$.*

Proof. Let $x \in X$ and $(y, \lambda, u) \in T$ .The relation (5.1) can be written under the form

$$(\mathrm{d}(-f + \lambda g + uh))^{+}(y, \eta(x,y)) \geqq 0.$$

Since $-f + \lambda g + uh$ is semilocally pseudo-preinvex we have

$$(-f + \lambda g + uh)(x) \geqq (-f + \lambda g + uh)(y)$$

i.e.

$$f(x) - \lambda g(x) \leqq \{f(y) - \lambda g(y)\} + u\{h(x) - h(y)\}.$$

The proof follows as in Theorem 5.1.

Corollary 5.3. *Let $\bar{x} \in X$ and $(\bar{x}, \bar{\lambda}, \bar{u}) \in T$ such that $q(\bar{x}) = v(\bar{\lambda})$. Also the hypotheses of either theorem 5.1 or 5.2 are satisfied, then $\bar{x}$ is optimal solution of (P) and $(\bar{x}, \bar{\lambda}, \bar{u})$ is optimal solution of (D).*

Proof. According to Theorems 5.1 and 5.2 , for each $x \in X$ we have

$$q(x) \leqq v(\bar{\lambda}) = q(\bar{x})$$

and hence $\bar{x}$ is an optimal solution of problem (P). Also if $(\bar{x}, \bar{\lambda}, \bar{u}) \in T$, then according to Theorems 5.1 and 5.2, we have

$$v(\lambda) \geqq q(\bar{x}) = v(\bar{\lambda})$$

and hence $(\bar{x}, \bar{\lambda}, \bar{u})$ is an optimal solution of problem (D).

Theorem 5.4. *Let $\bar{x}$ be a (local) optimal solution for (P), $h_i, i \in J$ be continuous at $\bar{x}$ and let $-(\mathrm{d}f)^+(\bar{x}, \eta(x, \bar{x})), (\mathrm{d}g)^+(\bar{x}, \eta(x, \bar{x})), (\mathrm{d}h)^+(\bar{x}, \eta(x, \bar{x}))$ be η-vex functions of x on X^0- a η-vex set at $\bar{x}$. If h satisfies GSQ at $\bar{x}$, then there exists $(\bar{x}, \bar{\lambda}, \bar{u}) \in T$ such that $q(\bar{x}) = v(\bar{\lambda})$. Moreover, if either $-f, g, h$ are semilocally preinvex or $-f + \lambda g + uh$ is semilocally pseudo-preinvex for any $(y, \lambda, u) \in T$, then $(\bar{x}, \bar{\lambda}, \bar{u})$ is an optimal solution for (D).*

Proof. Since $\bar{x}$ satisfies the conditions of Theorem 3.7 there exist $\bar{\lambda} \in \mathbf{R}, \bar{u} \in \mathbf{R}^m$ such that $(\bar{x}, \bar{\lambda}, \bar{u})$ is feasible for (D) and $q(\bar{x}) = v(\bar{\lambda})$. Hence, by Corollary 5.3 , it follows that $(\bar{x}, \bar{\lambda}, \bar{u})$ is optimal for (D).

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