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DUALITY IN MULTIOBJECTIVE OPTIMIZATION PROBLEMS WITH SET CONSTRAINTS

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Abstract We propose four different duality problems for a vector optimization program with a set constraint, equality and inequality constraints. For all dual problems we state weak and strong duality theorems based on different generalized concavity assumptions. The proposed dual problems provide a unified framework generalizing Wolfe and Mond-Weir results.

Keywords: Vector Optimization, Duality, Maximum Principle Conditions, Generalized Convexity, Set Constraints.

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1. Introduction

Vector optimization programs are extremely useful in order to model real life problems where several objectives conflict with one another, and so the interest of this topics crosses many different fields such as operation research, economic theory, location theory and management science. During the last decades the analysis of duality in multiobjective

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theory has been a focal issue. We can find papers dealing with duality under smooth and non smooth assumptions for both the objective and constraint functions, some other papers consider particular objective functions such as vector fractional ones (see for example the recent contributions by Bathia and Pankaj (1998); Patel (2000); Zalmai (1997)). Moreover many different kinds of generalized convexity properties have been investigated in order to get the usual duality results. Despite of a very large number of papers on duality the most part of the recent literature deals with vector optimization problems where the feasible region is defined by equality and inequality constraint or by a compact set (for this latter case the reader can see for example the leading article by Tanino and Sawaragy (1979)).

In this paper we aim to deal with a vector optimization problem where the feasible region is defined by equality constraint, inequality and set constraint and we do not require any topological properties on the set constraint. Since our duality results are related to the concepts of C-maximal and weakly C-maximal point we first recall these definitions and then we propose some necessary optimality conditions which can be classified as a maximum principle conditions. These suggest the introduction of the first dual D_1 which is a generalization of the Wolfe-dual problem ([1]). Then we propose three further dual programs which are called D_2, D_3 and D_4. While problem D_4 can be classified as a generalization of the Mond-Weir dual problem (see Mond and Weir (1981); Weir et al (1986)), D_2, D_3 are a sort of mixed duals. In the recent literature (see for example Aghezzaf and Hachimi (2001); Mishra (1996)) similar mixed dual have been proposed, but they refer to a primal problem with feasible region defined only by equality and inequality constraints. For all our dual programs, duality theorems are stated and for each one, different generalized convexity properties are assumed. For a feasible region without set constraint, there are many duality results dealing with several kind of generalized convexity properties such as invexity, generalized invexity (see for all Bector et al (1993); Bector et al (1994); Bector (1996); Giorgi and Guerraggio (1998); Hanson and Mond (1987); Kaul et al (1994); Rueda et al (1995)), or (F, ρ)-convexity (see for example Aghezzaf and Hachimi (2001); Bhatia and Jain (1994); Bathia and Pankaj (1998); Gulati and Islam (1994); Mishra (1996); Preda (1992)). In our case the objective function f is C-concave or $(\mathrm{Int}(C), \mathrm{Int}(C))$-pseudoconcave while the inequality constraint function g is assumed to be V-concave or polarly V-quasiconcave and the equality constraint function h is affine or polarly quasiaffine.

Finally, we compare the four dual programs in order to analyze them in a unified framework and to appreciate the differences among them.

2. Definitions and preliminary results

We consider the following multiobjective nonlinear programming P.

Definition 2.1 (Primal Problem)

$$P : \begin{cases} C\_\max \ f(x) \\ x \in S_P \end{cases} \equiv \begin{cases} C\_\max \ f(x) & \\ g(x) \in V & \textit{inequality constraints} \\ h(x) = 0 & \textit{equality constraints} \\ x \in X & \textit{set constraint} \end{cases}$$

where

$$S_P = \{x \in A : \ g(x) \in V, \ h(x) = 0, \ x \in X\},$$

$A \subseteq \mathbb{R}^n$ is an open convex set, $f : A \to \mathbb{R}^s$ and $g : A \to \mathbb{R}^m$ are Gâteaux differentiable functions, $h : A \to \mathbb{R}^p$ is a Fréchet differentiable function with a continuous Jacobian matrix $J_h(x)$. Moreover $C \subset \mathbb{R}^s$ and $V \subset \mathbb{R}^m$ are closed convex pointed cones with nonempty interior (that is to say convex pointed solid cones), and $X \subseteq A$ is a set verifying no particular topological properties. In other words, X is not required to be open or convex or with nonempty interior. Throughout the paper we will denote with C^+ and V^+ the positive polar cones of C and V, respectively.

For a better understanding of the paper, we recall some useful definitions and notations.

Definition 2.2 *Let $f : A \to \mathbb{R}^s$, $A \subseteq \mathbb{R}^n$, let $C \subset \mathbb{R}^s$ be a closed convex pointed cones with nonempty interior and let $S \subset \mathbb{R}^n$ be a set. Consider the following multiobjective problem:*

$$P : \begin{cases} C\_\max / C\_\min \ \ f(x) \\ x \in S \end{cases}$$

Using the notation $C^0 = C \setminus \{0\}$, a feasible point $x_0 \in S$ is said to be:

- a C-maximal [C-minimal] point for P *if:*

  $$\nexists y \in S \textit{ such that } f(y) \in f(x_0) + C^0 \quad [f(y) \in f(x_0) - C^0]$$

  in this case we will say that

  $$x_0 \in C^0\_\arg\max(P) \quad [x_0 \in C^0\_\arg\min(P)],$$

- a weak C-maximal [weak C-minimal] point for P *if:*

  $$\nexists y \in S \textit{ such that } f(y) \in f(x_0) + \mathrm{Int}(C) \quad [f(y) \in f(x_0) - \mathrm{Int}(C)]$$

in this case we will say that

$$x_0 \in \mathrm{Int}(C)\_\arg\max(P) \quad [x_0 \in \mathrm{Int}(C)\_\arg\min(P)].$$

The following necessary optimality condition of the maximum principle type holds for problem P (see Cambini (2001)) ([2]).

Theorem 2.1 *Consider problem P and let $x_0 \in X$ be a local C-maximal point. Suppose also that X is convex with* $\mathrm{Int}(X) \neq \emptyset$.

Then $\exists \alpha_f \in C^+$, $\exists \alpha_g \in V^+$, $\exists \alpha_h \in \mathbb{R}^p$, $(\alpha_f, \alpha_g, \alpha_h) \neq 0$, such that:

$$\alpha_g^T g(x_0) = 0 \quad \textit{and} \quad [\alpha_f^T J_f(x_0) + \alpha_g^T J_g(x_0) + \alpha_h^T J_h(x_0)](x - x_0) \leq 0 \ \forall x \in \mathrm{Cl}(X).$$

If in addiction a constraint qualification holds then $\alpha_f \neq 0$.

As it is well known, a constraint qualification is any condition guaranteeing that $\alpha_f \neq 0$ ([3]). The following proposition presents a constraint qualification condition for problem P.

Proposition 2.1 *Consider problem P and let $x_0 \in X$ be a feasible local C-maximal point. Suppose also that X is convex with* $\mathrm{Int}(X) \neq \emptyset$.

The condition $\mathrm{Co}[\mathcal{L}_{x_0}] = \mathbb{R}^{m+p}$, *where* ([4])*:*

$$\mathcal{L}_{x_0} = \{(t_g, t_h) \in \mathbb{R}^{m+p} : \ (t_g, t_h) = [J_g(x_0), J_h(x_0)](x - x_0), \ x \in \mathrm{Cl}(X)\}$$

is a constraint qualification.

Proof. For the first part of Theorem 2.1 $\exists \alpha_f \in C^+$, $\exists \alpha_g \in V^+$, $\exists \alpha_h \in \mathbb{R}^p$, $(\alpha_f, \alpha_g, \alpha_h) \neq 0$, such that:

$$\alpha_g^T g(x_0) = 0 \quad \text{and} \quad [\alpha_f^T J_f(x_0) + \alpha_g^T J_g(x_0) + \alpha_h^T J_h(x_0)](x - x_0) \leq 0 \ \forall x \in \mathrm{Cl}(X).$$

Suppose now by contradiction that $\alpha_f = 0$, then $(\alpha_g, \alpha_h) \neq 0$ and:

$$(\alpha_g^T t_g + \alpha_h^T t_h) \leq 0 \ \forall (t_g, t_h) \in \mathcal{L}_{x_0}$$

This implies also that:

$$(\alpha_g^T t_g + \alpha_h^T t_h) \leq 0 \ \forall (t_g, t_h) \in \mathrm{Co}[\mathcal{L}_{x_0}] = \mathbb{R}^{m+p}$$

and hence $(\alpha_g, \alpha_h) = 0$, which is a contradiction. □

The maximum principle condition of Theorem 2.1 will suggest the definition of some dual problems for P.

3. Duality

In this section we aim to provide different kinds of dual problems for P and to study them in a unified framework. Starting from the necessary optimality condition of Theorem 2.1 we are able to define four dual problems D_1, D_2, D_3 and D_4. As the reader will see, D_1 is a Wolfe-type dual problem, D_4 is a Mond-Weir-type dual while D_2 and D_3 can be classified as a sort of mixed dual problems.

3.1 Dual problems

Definition 3.1 (1^{st} Dual Problem) *Consider problem P and let $c \in \mathrm{Int}(C)$. The following Dual problem can be introduced:*

$$D_1 : \left\{ \begin{array}{c} C\_\min \;\; L_1(x,\alpha_f,\alpha_g,\alpha_h) = f(x) + \frac{c}{\alpha_f^T c}[\alpha_g^T g(x) + \alpha_h^T h(x)] \\ (x,\alpha_f,\alpha_g,\alpha_h) \in S_{D_1} \end{array} \right. ,$$

where

$$S_{D_1} = \left\{ \begin{array}{c} (x,\alpha_f,\alpha_g,\alpha_h) \in (A \times C^+ \times V^+ \times \mathbb{R}^p), \;\; \alpha_f \neq 0, \\ \left[\alpha_f^T J_f(x) + \alpha_g^T J_g(x) + \alpha_h^T J_h(x)\right](y-x) \leq 0 \;\; \forall y \in \mathrm{Cl}(X) \end{array} \right\}$$

Some other different duals can be proposed, with different objective functions, different feasible regions and different generalized concavity properties of the functions.

Definition 3.2 (2^{nd} Dual Problem) *Consider problem P and let $c \in \mathrm{Int}(C)$. The following Dual problem can be introduced:*

$$D_2 : \left\{ \begin{array}{c} C\_\min \;\; L_2(x,\alpha_f,\alpha_g,\alpha_h) = f(x) + \frac{c}{\alpha_f^T c}[\alpha_h^T h(x)] \\ (x,\alpha_f,\alpha_g,\alpha_h) \in S_{D_2} \end{array} \right. ,$$

where $S_{D_2} = \{(x,\alpha_f,\alpha_g,\alpha_h) \in S_{D_1} : \; \alpha_g^T g(x) \leq 0\}$

Definition 3.3 (3^{rd} Dual Problem) *Consider problem P and let $c \in \mathrm{Int}(C)$. The following Dual problem can be introduced:*

$$D_3 : \left\{ \begin{array}{c} C\_\min \;\; L_3(x,\alpha_f,\alpha_g,\alpha_h) = f(x) + \frac{c}{\alpha_f^T c}[\alpha_g^T g(x)] \\ (x,\alpha_f,\alpha_g,\alpha_h) \in S_{D_3} \end{array} \right. ,$$

where $S_{D_3} = \{(x,\alpha_f,\alpha_g,\alpha_h) \in S_{D_1} : \; \alpha_h^T h(x) = 0\}$

Definition 3.4 (4^{th} Dual Problem) *Consider problem P. The following Dual problem can be introduced:*

$$D_4 : \left\{ \begin{array}{c} C\_\min \;\; L_4(x,\alpha_f,\alpha_g,\alpha_h) = f(x) \\ (x,\alpha_f,\alpha_g,\alpha_h) \in S_{D_4} \end{array} \right. ,$$

where $S_{D_4} = \{(x, \alpha_f, \alpha_g, \alpha_h) \in S_{D_1} : \ \alpha_g^T g(x) \leq 0, \ \alpha_h^T h(x) = 0\}$

In order to prove weak and strong duality results for the introduced pairs of primal-dual problems some generalized convexity properties are needed.

Definition 3.5 *Consider the primal problem P and the dual problems D_j, $j \in \{1,2,3,4\}$. We say that functions f, g and h verify the generalized convexity properties (GC_j) if:*

- *in the case $j = 1$, f is C-concave in A, g is V-concave in A and h is affine in A,*
- *in the case $j = 2$, f is C-concave in A, g is polarly V-quasiconcave in A and h is affine in A,*
- *in the case $j = 3$, f is C-concave in A, g is V-concave in A and h is polarly quasiaffine in A,*
- *in the case $j = 4$, f is* $(\mathrm{Int}(C), \mathrm{Int}(C))$*-pseudoconcave in A, g is polarly V-quasiconcave in A and h is polarly quasiaffine in A.*

3.2 Weak Duality

Let us now prove weak duality results for the pairs of duals problems introduced so far. With this aim, it is worth noticing that we do not need to assume the convexity of the set X.

Theorem 3.1 *Let us consider the primal problem P and the dual problems D_j, $j \in \{1,2,3,4\}$. If (GC_j) property holds for $j \in \{1,2,3,4\}$ then:*

$$f(x_1) \notin L_j(x_2, \alpha_f, \alpha_g, \alpha_h) + \mathrm{Int}(C)$$

$\forall x_1 \in S_P$ *and* $\forall (x_2, \alpha_f, \alpha_g, \alpha_h) \in S_{D_j}$.

Proof. *Case $j = 1$)* Suppose by contradiction that

$$f(x_1) \in f(x_2) + \frac{1}{\alpha_f^T c} c[\alpha_g^T g(x_2) + \alpha_h^T h(x_2)] + \mathrm{Int}(C)$$

so that, being $\alpha_f \in C^+$, $\alpha_f \neq 0$, it is

$$\alpha_f^T f(x_1) > \alpha_f^T f(x_2) + \alpha_g^T g(x_2) + \alpha_h^T h(x_2). \qquad (1.1)$$

Since f C-concave it is:

$$f(x_1) \in f(x_2) + J_f(x_2)(x_1 - x_2) - C,$$

so that, since $\alpha_f \in C^+$,

$$\alpha_f^T f(x_1) \leq \alpha_f^T f(x_2) + \alpha_f^T J_f(x_2)(x_1 - x_2); \tag{1.2}$$

from the V-concavity of g it is:

$$g(x_1) \in g(x_2) + J_g(x_2)(x_1 - x_2) - V,$$

so that, since $\alpha_g \in V^+$ and $g(x_1) \in V$,

$$0 \leq \alpha_g^T g(x_1) \leq \alpha_g^T g(x_2) + \alpha_g^T J_g(x_2)(x_1 - x_2); \tag{1.3}$$

finally, being h affine it is:

$$h(x_1) = h(x_2) + J_h(x_2)(x_1 - x_2),$$

so that, $h(x_1) = 0$ implies

$$0 = \alpha_h^T h(x_1) = \alpha_h^T h(x_2) + \alpha_h^T J_h(x_2)(x_1 - x_2). \tag{1.4}$$

Adding the leftmost and rightmost components of inequalities (1.2), (1.3) and (1.4) we then have, for the definition of S_{D_1} and since $x_1 \in X$:

$$\begin{aligned}\alpha_f^T f(x_1) &\leq \alpha_f^T f(x_2) + \alpha_g^T g(x_2) + \alpha_h^T h(x_2) + \\ &\quad + [\alpha_f^T J_f(x_2) + \alpha_g^T J_g(x_2) + \alpha_h^T J_h(x_2)](x_1 - x_2) \\ &\leq \alpha_f^T f(x_2) + \alpha_g^T g(x_2) + \alpha_h^T h(x_2)\end{aligned}$$

which contradicts condition (1.1).

Case $j = 4$*)* Suppose by contradiction that $f(x_1) \in f(x_2) + \text{Int}(C)$; for the $(\text{Int}(C), \text{Int}(C))$-pseudoconcavity of f it follows that $J_f(x_2)(x_1 - x_2) \in \text{Int}(C)$; being $\alpha_f \in C^+$, $\alpha_f \neq 0$, it then results:

$$\alpha_f^T J_f(x_2)(x_1 - x_2) > 0. \tag{1.5}$$

For the hypotheses we have $g(x_1) \in V$, $\alpha_g \in V^+$, $\alpha_g^T g(x_2) \leq 0$, so that $\alpha_g^T g(x_2) \leq 0 \leq \alpha_g^T g(x_1)$; if $\alpha_g \neq 0$ then the polar V-quasiconcavity of g implies that

$$\alpha_g^T J_g(x_2)(x_1 - x_2) \geq 0, \tag{1.6}$$

while if $\alpha_g = 0$ then (1.6) holds trivially. For the hypotheses we have $h(x_1) = 0$ and $\alpha_h^T h(x_2) = 0$, so that $\alpha_h^T h(x_1) = 0 = \alpha_h^T h(x_2)$; if $\alpha_h \neq 0$ then the polar quasiaffinity of h implies that

$$\alpha_h^T J_h(x_2)(x_1 - x_2) = 0, \tag{1.7}$$

while if $\alpha_h = 0$ then (1.7) holds trivially. Adding the leftmost and rightmost components of inequalities (1.5), (1.6) and (1.7) we then have:

$$[\alpha_f^T J_f(x_2)^T + \alpha_g^T J_g(x_2) + \alpha_h^T J_h(x_2)](x_1 - x_2) > 0$$

so that, since $x_1 \in X$, it is $(x_2, \alpha_f, \alpha_g, \alpha_h) \notin S_{D_4}$ which is a contradiction.

Case $j = 2, 3$*)* The proofs are analogous to those of cases $j = 1, 4$.
□

In the same way, the following stronger version of the weak duality theorem can be proved just changing the generalized convexity assumptions of function f.

Theorem 3.2 *Let us consider the primal problem P and the dual problems D_j, $j \in \{1,2,3,4\}$. The following statements hold:*

i) in the case of $j \in \{1,2,3\}$, if (GC_j) property holds and f is Int(C)*-concave then:*

$$f(x_1) \notin L_j(x_2, \alpha_f, \alpha_g, \alpha_h) + C$$

$\forall x_1 \in S_P$ *and* $\forall (x_2, \alpha_f, \alpha_g, \alpha_h) \in S_{D_j}$ *such that* $x_1 \neq x_2$*,*

ii) in the case of $j = 4$, if (GC_4) property holds and f is $(C, \text{Int}(C))$-pseudoconcave then:

$$f(x_1) \notin L_4(x_2, \alpha_f, \alpha_g, \alpha_h) + C$$

$\forall x_1 \in S_P$ *and* $\forall (x_2, \alpha_f, \alpha_g, \alpha_h) \in S_{D_4}$ *such that* $x_1 \neq x_2$*,*

iii) in the case of $j = 4$, if (GC_4) property holds and f is $(C^0, \text{Int}(C))$-pseudoconcave then:

$$f(x_1) \notin L_4(x_2, \alpha_f, \alpha_g, \alpha_h) + C^0$$

$\forall x_1 \in S_P$ *and* $\forall (x_2, \alpha_f, \alpha_g, \alpha_h) \in S_{D_4}$ *such that* $x_1 \neq x_2$*.*

3.3 Strong Duality

We are now ready to prove the following results related to strong duality. With this aim, from now on we will assume the set X to be convex and with nonempty interior.

Theorem 3.3 *Let us consider the primal problem P and the dual problems D_j, $j \in \{1,2,3,4\}$. Suppose that X is convex with nonempty interior and a constraint qualification holds for problem P. If (GC_j) property holds for $j \in \{1,2,3,4\}$ then $\forall x \in C^0\text{\_}\arg\max(P)$ $\exists \alpha_f \in C^+ \setminus \{0\}$, $\exists \alpha_g \in V^+$, $\exists \alpha_h \in \mathbb{R}^p$ such that:*

$$(x, \alpha_f, \alpha_g, \alpha_h) \in \text{Int}(C)\text{\_}\arg\min(D_j) \quad \textit{and} \quad f(x) = L_j(x, \alpha_f, \alpha_g, \alpha_h)$$

Proof. Let $x \in C^0\text{\_}\arg\max(P)$; by means of Theorem 2.1 $\exists \alpha_f \in C^+ \setminus \{0\}$, $\exists \alpha_g \in V^+$, $\exists \alpha_h \in \mathbb{R}^p$ such that $\alpha_g^T g(x) = 0$ and

$$[\alpha_f^T J_f(x)^T + \alpha_g^T J_g(x) + \alpha_h^T J_h(x)](y - x) \leq 0 \; \forall y \in \mathrm{Cl}(X).$$

Since $h(x) = 0$ and $\alpha_g^T g(x) = 0$ it results $f(x) = L_j(x, \alpha_f, \alpha_g, \alpha_h)$ for all $j \in \{1,2,3,4\}$. It results also that $(x, \alpha_f, \alpha_g, \alpha_h) \in S_{D_4}$ and hence $(x, \alpha_f, \alpha_g, \alpha_h) \in S_{D_j}$ for all $j \in \{1,2,3,4\}$ since $S_{D_4} \subseteq S_{D_2} \subseteq S_{D_1}$ and $S_{D_4} \subseteq S_{D_3} \subseteq S_{D_1}$. Let $j \in \{1,2,3,4\}$, for the weak duality theorem $\nexists(\hat{x}, \hat{\alpha_f}, \hat{\alpha_g}, \hat{\alpha_h}) \in S_{D_j}$ such that

$$L_j(x, \alpha_f, \alpha_g, \alpha_h) = f(x) \in L_j(\hat{x}, \hat{\alpha_f}, \hat{\alpha_g}, \hat{\alpha_h}) + \mathrm{Int}(C)$$

In other words, $\nexists(\hat{x}, \hat{\alpha_f}, \hat{\alpha_g}, \hat{\alpha_h}) \in S_{D_j}$ such that

$$L_j(\hat{x}, \hat{\alpha_f}, \hat{\alpha_g}, \hat{\alpha_h}) \in L_j(x, \alpha_f, \alpha_g, \alpha_h) - \mathrm{Int}(C)$$

and hence $(x, \alpha_f, \alpha_g, \alpha_h) \in \mathrm{Int}(C)\text{\_}\arg\min(D_j)$. □

The following result follows directly from Theorem 3.3.

Corollary 3.1 *Let us consider the primal problem P and the dual problems D_j, $j \in \{1,2,3,4\}$. Suppose that X is convex with nonempty interior and a constraint qualification holds for problem P. If there exists an index $j \in \{1,2,3,4\}$ such that (GC_j) property holds and* $\mathrm{Int}(C)\text{\_}\arg\min(D_j) = \emptyset$ *then* $C^0\text{\_}\arg\max(P) = \emptyset$.

The following further duality result follows from the weak and the strong duality theorems.

Corollary 3.2 *Let us consider the primal problem P and the dual problems D_j, $j \in \{1,2,3,4\}$. Suppose that X is convex with nonempty interior and a constraint qualification holds for problem P. If (GC_j) property holds for $j \in \{1,2,3,4\}$ then*

$$f(x_1) - L_j(x_2, \alpha_f, \alpha_g, \alpha_h) \notin (\mathrm{Int}(C) \cup \mathrm{Int}(-C))$$

$\forall x_1 \in C^0\text{\_}\arg\max(P)$ *and* $\forall (x_2, \alpha_f, \alpha_g, \alpha_h) \in \mathrm{Int}(C)\text{\_}\arg\min(D_j)$.

Proof. Let $j \in \{1,2,3,4\}$, $x_1 \in C^0\text{\_}\arg\max(P)$ and $(x_2, \alpha_f, \alpha_g, \alpha_h) \in \mathrm{Int}(C)\text{\_}\arg\min(D_j)$; for the weak duality theorem it is

$$f(x_1) - L_j(x_2, \alpha_f, \alpha_g, \alpha_h) \notin \mathrm{Int}(C)$$

For the strong duality theorem $\exists \alpha_f \in C^+ \setminus \{0\}$, $\exists \alpha_g \in V^+$, $\exists \alpha_h \in \mathbb{R}^p$ such that $(x_1, \alpha_f, \alpha_g, \alpha_h) \in \mathrm{Int}(C)\text{\_}\arg\min(D_j)$ and $f(x_1) = L_j(x_1, \alpha_f, \alpha_g, \alpha_h)$. Hence, condition $(x_2, \alpha_f, \alpha_g, \alpha_h) \in \mathrm{Int}(C)\text{\_}\arg\min(D_j)$ implies

$$L_j(x_1, \alpha_f, \alpha_g, \alpha_h) \notin L_j(x_2, \alpha_f, \alpha_g, \alpha_h) - \mathrm{Int}(C)$$

so that, for the equality $f(x_1) = L_j(x_1, \alpha_f, \alpha_g, \alpha_h)$, we have

$$f(x_1) - L_j(x_2, \alpha_f, \alpha_g, \alpha_h) \notin \text{Int}(-C)$$

which prove the result. □

4. Final remarks

Comparing the introduced dual programs it can be easily seen that problem D_1 (the Wolfe-type dual problem) has the most "complex" objective function while problem D_4 (the Mond-Weir type) has the simplest one. Furthermore as you move from the dual program D_1 to D_4 you can require weaker generalized concavity assumptions in order to prove duality theorems. Finally, the feasible region of D_4 is the smallest, S_{D_1} is the biggest and $S_{D_4} \subseteq S_{D_2} \subseteq S_{D_1}$ and $S_{D_4} \subseteq S_{D_3} \subseteq S_{D_1}$. As the reader has already noted, whenever you get duality results by defining a simpler objective function and by requiring weaker generalized concavity properties (see Problem D_4), the feasible region of the dual problem is bigger and viceversa a smaller feasible region (see Problem D_1) is "paid" by a more complex objective function and stronger generalized concavity assumptions. The described behavior is represented in Figure 1.1.

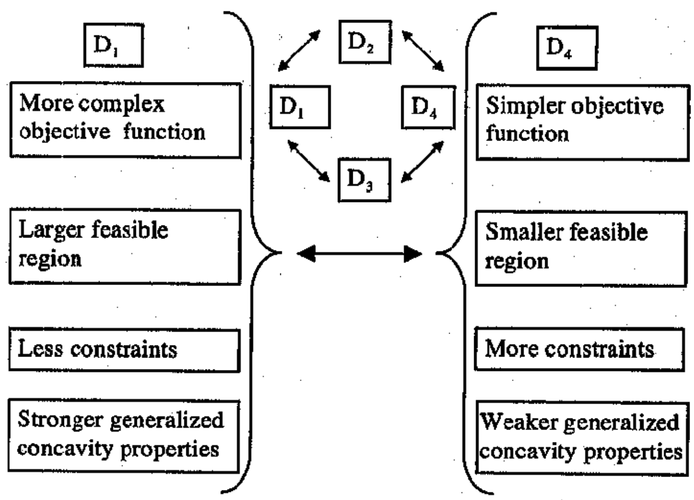

Figure 1.1.

Appendix - Generalized Concave Functions

The following classes of vector valued functions have been defined and studied in Cambini (1996); Cambini (1998); Cambini (1998).

Definition 4.1 *Let $f : A \to \mathbb{R}^m$, where $A \subset \mathbb{R}^n$ is an open convex set, be a differentiable vector valued function and let $C \subset \mathbb{R}^m$ be a closed convex cone with nonempty interior. Let also $C^0 = C \setminus \{0\}$ and C^+ the positive polar cone of C. Function f is said to be:*

- *C-concave if and only if $\forall x, y \in A$, $x \neq y$, it holds:*

$$f(y) - f(x) - J_f(x)(y - x) \in -C,$$

- *C^0-concave if and only if $\forall x, y \in A$, $x \neq y$, it holds:*

$$f(y) - f(x) - J_f(x)(y - x) \in -C^0,$$

- $\text{Int}(C)$*-concave if and only if $\forall x, y \in A$, $x \neq y$, it holds:*

$$f(y) - f(x) - J_f(x)(y - x) \in -\text{Int}(C),$$

- $(\text{Int}(C), \text{Int}(C))$*-pseudoconcave if and only if $\forall x, y \in A$, $x \neq y$, it holds:*

$$f(y) \in f(x) + \text{Int}(C) \quad \Rightarrow \quad J_f(x)(y - x) \in \text{Int}(C),$$

- $(C^0, \text{Int}(C))$*-pseudoconcave if and only if $\forall x, y \in A$, $x \neq y$, it holds:*

$$f(y) \in f(x) + C^0 \quad \Rightarrow \quad J_f(x)(y - x) \in \text{Int}(C),$$

- $(C, \text{Int}(C))$*-pseudoconcave if and only if $\forall x, y \in A$, $x \neq y$, it holds:*

$$f(y) \in f(x) + C \quad \Rightarrow \quad J_f(x)(y - x) \in \text{Int}(C).$$

See Cambini (1998); Cambini and Komlósi (1998); Cambini and Komlósi (2000) for the definition and the study of the following classes of functions.

Definition 4.2 *Let $f : A \to \mathbb{R}^m$, where $A \subset \mathbb{R}^n$ is an open convex set, be a differentiable vector valued function and let $C \subset \mathbb{R}^m$ be a closed convex cone with nonempty interior. Let also $C^0 = C \setminus \{0\}$ and C^+ the positive polar cone of C. Function f is said to be:*

- *polarly C-quasiconcave if and only if $\phi(x) = \alpha^T f(x)$ is quasiconcave $\forall \alpha \in C^+$, $\alpha \neq 0$, that is to say if and only if $\forall \alpha \in C^+$, $\alpha \neq 0$, $\forall x, y \in A$, $x \neq y$, it holds:*

$$\alpha^T f(y) \geq \alpha^T f(x) \quad \Rightarrow \quad \alpha^T J_f(x)(y - x) \geq 0,$$

- *polarly C-pseudoconcave if and only if $\phi(x) = \alpha^T f(x)$ is pseudoconcave $\forall \alpha \in C^+$, $\alpha \neq 0$, that is to say if and only if $\forall \alpha \in C^+$, $\alpha \neq 0$, $\forall x, y \in A$, $x \neq y$, it holds:*

$$\alpha^T f(y) > \alpha^T f(x) \quad \Rightarrow \quad \alpha^T J_f(x)(y-x) > 0,$$

- *polarly C^0-pseudoconcave if and only if $\forall \alpha \in C^+$, $\alpha \neq 0$, $\forall x, y \in A$, $x \neq y$, it holds:*

$$\alpha^T f(y) \geq \alpha^T f(x) \text{ with } f(y) \neq f(x) \quad \Rightarrow \quad \alpha^T J_f(x)(y-x) > 0,$$

- *polarly $\text{Int}(C)$-pseudoconcave if and only if $\phi(x) = \alpha^T f(x)$ is strictly pseudoconcave $\forall \alpha \in C^+$, $\alpha \neq 0$, that is to say if and only if $\forall \alpha \in C^+$, $\alpha \neq 0$, $\forall x, y \in A$, $x \neq y$, it holds:*

$$\alpha^T f(y) \geq \alpha^T f(x) \quad \Rightarrow \quad \alpha^T J_f(x)(y-x) > 0,$$

- *polarly quasiaffine if and only if $\phi(x) = \alpha^T f(x)$ is both quasiconvex and quasiconcave $\forall \alpha \in \mathbb{R}^m$, $\alpha \neq 0$, that is to say if and only if $\forall \alpha \in \mathbb{R}^m$, $\alpha \neq 0$, $\forall x, y \in A$, $x \neq y$, it holds:*

$$\alpha^T f(y) = \alpha^T f(x) \quad \Rightarrow \quad \alpha^T J_f(x)(y-x) = 0.$$

Note that the characterization of polarly quasiaffine functions follows from the properties of scalar generalized concave functions and scalar generalized affine functions studied in Cambini (1995). Let us finally recall that (see Cambini and Komlósi (1998); Cambini and Komlósi (2000)):

- If f is polarly C-pseudoconcave then it is also $(\text{Int}(C), \text{Int}(C))$-pseudoconcave
- If f is polarly C^0-pseudoconcave then it is also $(C^0, \text{Int}(C))$-pseudoconcave
- If f is polarly $\text{Int}(C)$-pseudoconcave then it is also $(C, \text{Int}(C))$-pseudoconcave

Notes

1. For a different duality approach when the feasible region is a subset of an arbitrary set, the reader can see for example Jahn (1994); Luc (1984); Zalmai (1997).

2. In the case f and g are Lipschitz and h are Fréchet differentiable, another necessary optimality conditions for Problem P can be found in Jiménez and Novo (2002).

3. Among the wide literature on this subject many constraint qualification conditions have been stated with various approaches and for different kind of problems (see for example Clarke (1983); Giorgi and Guerraggio (1994); Jahn (1994); Jiménez and Novo (2002); Luc (1989)).

4. We denote with $Co(X)$ the convex hull of a set X.

References

Aghezzaf, B. and Hachimi, M. (2001), Sufficiency and Duality in Multiobjective Programming Involving Generalized (F, ρ)−convexity, *Journal of Mathematical Analysis and Applications*, Vol. 258, pp. 617-628.

Bhatia, D. and Jain, P. (1994), Generalized (F, ρ)-convexity and duality for non smooth multi-objective programs, *Optimization*, Vol. 31, pp. 153-164.

Bathia, D. and Pankaj, K. G. (1998), Duality for non-smooth nonlinear fractional multiobjective programs via (F, ρ)-convexity, Optimization, Vol. 43, pp. 185-197.

Bector, C.R., Suneja, S.K. and Lalitha, C.S. (1993), Generalized B-Vex Functions and Generalized B-Vex Programming, *Journal of Optimization Theory and Application*, Vol. 76, pp. 561-576.

Bector, C.R., Bector, M.K., Gill, A. and Singh, C. (1994), Duality for Vector Valued B-invex Programming, in *Generalized Convexity*, edited by S. Komlósi, T. Rapcsák and S. Schaible, Lecture Notes in Economics and Mathematical Systems, Vol. 405, Springer-Verlag, Berlin, pp. 358-373.

Bector, C.R. (1996), Wolfe-Type Duality involving (B, η)-invex Functions for a Minmax Programming Problem, *Journal of Mathematical Analysis and Application*, Vol. 201, pp. 114-127.

Cambini, R. (1995), Funzioni scalari affini generalizzate , *Rivista di Matematica per le Scienze Economiche e Sociali*, year 18th, Vol.2, pp. 153-163.

Cambini, R. (1996), Some new classes of generalized concave vector-valued functions, *Optimization*, Vol. 36, pp. 11-24.

Cambini, R. (1998), Composition theorems for generalized concave vector valued functions, *Journal of Information and Optimization Sciences*, Vol. 19, pp. 133-150.

Cambini, R. (1998), Generalized Concavity for Bicriteria Functions, in *Generalized Convexity, Generalized Monotonicity: Recent Results*, edited by J.-P. Crouzeix, J.-E. Martinez-Legaz and M. Volle, Nonconvex Optimization and Its Applications, Vol. 27, Kluwer Academic Publishers, Dordrecht, pp. 439-451.

Cambini, R. and Komlósi, S. (1998), On the Scalarization of Pseudoconcavity and Pseudomonotonicity Concepts for Vector Valued Functions", in *Generalized Convexity, Generalized Monotonicity: Recent Results*, edited by J.-P. Crouzeix, J.-E. Martinez-Legaz and M. Volle, Nonconvex Optimization and Its Applications, Vol. 27, Kluwer Academic Publishers, Dordrecht, pp. 277-290.

Cambini, R. and Komlósi S. (2000), On Polar Generalized Monotonicity in Vector Optimization, *Optimization*, Vol. 47, pp. 111-121.

Cambini, R. (2001), Necessary Optimality Conditions in Vector Optimization, Report n.212, Department of Statistics and Applied Mathematics, University of Pisa.

Clarke, F.H. (1983), *Optimization and Nonsmooth Analysis*, John-Wiley & Sons, New York.

Giorgi, G. and Guerraggio, A. (1994), First order generalized optimality conditions for programming problems with a set constraint, in *Generalized Convexity*, edited by S. Komlósi, T. Rapcsák and S. Schaible, Lecture Notes in Economics and Mathematical Systems, Vol. 405, Springer-Verlag, Berlin, pp. 171-185.

Giorgi, G. and Guerraggio, A. (1998), The notion of invexity in vector optimization: smooth and nonsmooth case, in *Generalized Convexity, Generalized Monotonicity: Recent Results*, edited by J.-P. Crouzeix, J.-E. Martinez-Legaz and M. Volle, Nonconvex Optimization and Its Applications, Vol. 27, Kluwer Academic Publishers, Dordrecht, pp. 389-405.

Göpfert, A. and Tammer, C. (2002), Theory of Vector Optimization, in *Multiple Criteria Optimization*, edited by M. Ehrgott and X. Gandibleux, International Series in Operations Research and Management Science, Vol. 52, Kluwer Academic Publishers, Boston.

Gulati, T. R. and Islam, M.A. (1994), Sufficiency and Duality in Multiobjective Programming Involving Generalized (F,ρ)-convexity, *Journal of Mathematical Analysis and Applications*, Vol. 183, pp. 181-195.

Hanson, M.A. and Mond, B. (1987), Necessary and Sufficiency Conditions in Constrained Optimization, *Mathematical Programming*, Vol. 37, pp. 51-58.

Jahn, J. (1994), *Introduction to the Theory of Nonlinear Optimization*, Springer-Verlag, Berlin, 1994.

Jiménez, B. and Novo, V. (2002), A finite dimensional extension of Lyusternik theorem with applications to multiobjective optimization, *Journal of Mathematical Analysis and Applications*, Vol. 270, pp. 340-356.

Kaul R.N., Suneja S.K. and Srivastava M.K. (1994), Optimality Criteria and Duality in Multiobjective Optimization Involving Generalized Invexity, *Journal of Optimization Theory and Application*, Vol. 80, pp. 465-481.

Luc, D. T. (1984), On Duality Theory in Multiobjective Programming, *Journal of Optimization Theory and Application*, Vol. 43, pp. 557-582.

Luc, D.T. (1989), *Theory of vector optimization*, Lecture Notes in Economics and Mathematical Systems, Vol. 319, Springer-Verlag, Berlin, 1989.

Maeda, T. (1994), Constraint Qualifications in Multiobjective Optimization Problems: Differentiable Case, *Journal of Optimization Theory and Application*, Vol. 80, pp. 483-500.

Mangasarian, O.L. (1969), *Nonlinear Programming*, McGraw-Hill, New York.

Mishra, S.K. (1996), On Sufficiency and Duality for Generalized Quasiconvex Nonsmooth Programs, *Optimization*, Vol. 38, pp. 223-235.

Mond, B. and Weir, T. (1981), Generalized concavity and duality, in *Generalized Concavity and Duality in Optimization and Economics*, edited by S. Schaible and W.T. Ziemba, Academic Press, New York, pp. 263-279.

Patel, R.B. (2000), On efficiency and duality theory for a class of multiobjective fractional programming problems with invexity, *Journal of Statistics and Management Systems*, Vol. 3, pp. 29-41.

Preda, V. (1992), On Efficiency and Duality in Multiobjective Programs, *Journal of Mathematical Analysis and Applications*, Vol. 166, pp. 365-377.

Rueda, N.G., Hanson, M.A. and Singh C., (1995), Optimality and Duality with Generalized Convexity, *Journal of Optimization Theory and Application*, Vol. 86, pp. 491-500.

Tanino, T. and Sawaragy, Y. (1979), Duality Theory in Multiobjective Programming, *Journal of Optimization Theory and Application*, Vol. 27, pp. 509-529.

Weir, T., Mond, B. and Craven B.D. (1986), On duality for weakly minimized vector-valued optimization problems, *Optimization*, Vol. 17, pp. 711-721.

Zalmai, G.J. (1997), Efficiency criteria and duality models for multiobjective fractional programming problems containing locally subdifferentiable and ρ-convex functions, *Optimization*, Vol. 41, pp. 321-360.