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R.Cambini and L.Carosi and S.Schaible

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Via Cosimo Ridolfi, 10 – 56124 PISA – Tel. Segr. Amm. 050 945231 Segr. Stud. 050 945317 Fax 050 945375
Cod. Fisc. 80003670504 - P. IVA 00286820501 - Web http://statmat.ec.unipi.it/ - E-mail: dipstat@ec.unipi.it

DUALITY IN FRACTIONAL OPTIMIZATION PROBLEMS WITH SET CONSTRAINTS

Riccardo Cambini, Laura Carosi*
Dept. of Statistics and Applied Mathematics
University of Pisa, ITALY

Siegfried Schaible
A. Gary Anderson Graduate School of Management
University of California, U.S.A.

Abstract The dual program of a minimization problem with a set constraint is defined. Under suitable generalized convexity assumptions, weak, strong and strict converse duality theorems are stated. By means of a suitable transformation the obtained results can be applied to a class of fractional program where the objective function is the ratio between a convex function and an affine one. In this case, it is proved that the objective function of the dual problem is linear.

Keywords: Duality, Fractional Programming, Set Constraints.

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email: cambric@ec.unipi.it, lcarosi@ec.unipi.it, siegfried.schaible@ucr.edu
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1. Introduction

In the recent years there has been an extensive interest in duality, many different pairs of dual problems have been introduced for both scalar and vector optimization, and many solution algorithms based on duality properties have been proposed.

On the other hand, at the best of our knowledge, the most of the existing results deal with problems whose feasible region is compact or defined by equalities and inequalities. Duality results for problems with set constraints and inequality constraints can be found in Giorgi and Guerraggio (1994).

In this paper, we aim to study duality results for minimization problems with a set constraint defined by a convex set X which does not need to be closed or open. A necessary optimality condition of the minimum-principle type holds for this class of problems (see for all Mangasarian (1969)) and this allows us to suggest a Wolfe-type dual problem and to prove Weak, Strong and Strict converse duality results.

Furthermore we consider fractional problems, having a set constraint, where the numerator is a convex function and the denominator is an affine one. Even in this particular case, there are a variety of approaches to find a corresponding dual problem when the primal one has no set constraints (see for example Barros et al (1996); Bector (1973); Bector et al (1977); Craven (1981); Jagannathan (1973); Liang et al (2001); Liu (1996); Mahajan and Vartak (1977); Schaible (1976a); Schaible (1976b); Scott and Jefferson (1996)).

By performing a proper transformation, we can apply our duality results to the fractional problem. In this case, we prove that the objective function of the dual problem is linear and hence our results can be seen as a generalization of the known ones related to fractional problems where the feasible region is not defined by any set constraints (see Mahajan and Vartak (1977); Schaible (1974); Schaible (1976a); Schaible (1976b)).

2. Duality results

Definition 2.1 (Primal Problem) *The following problem is assumed to be the Primal Problem:*

$$P : \begin{cases} \min \ f(x) \\ x \in S_P \end{cases} \equiv \begin{cases} \min \ f(x) & \\ g(x) \in -V & \textit{inequality constraints} \\ h(x) = 0 & \textit{equality constraints} \\ x \in X & \textit{set constraint} \end{cases}$$

where

- $S_P = \{x \in A : \ g(x) \in V, \ h(x) = 0, \ x \in X\}$,

- *$A \subseteq \mathbb{R}^n$ is an open convex set,*
- *$f : A \to \mathbb{R}$, $g : A \to \mathbb{R}^m$ and $h : A \to \mathbb{R}^p$ are differentiable functions,*
- *$V \subset \mathbb{R}^m$ is a closed convex pointed cone with nonempty interior (that is to say a convex pointed solid cone),*
- *$X \subseteq A$ is a convex set with nonempty interior which is not required to be open or closed.*

The following necessary optimality condition, known as minimum principle condition (see for all Mangasarian (1969)), holds for problem P. Recall that V^+ denotes the positive polar cone of V while $\mathbb{R}_+$ denotes the nonnegative numbers.

Theorem 2.1 *Consider problem P and suppose $\bar{x} \in S_P$. If $\bar{x} \in \arg\min_{x \in S_P} f(x)$ then $\exists(\alpha_f, \alpha_g, \alpha_h) \in (\mathbb{R}_+ \times V^+ \times \mathbb{R}^p)$, $(\alpha_f, \alpha_g, \alpha_h) \neq 0$, such that:*

$$\alpha_g^T g(\bar{x}) = 0 \quad \text{and} \quad [\alpha_f \nabla f(\bar{x})^T + \alpha_g^T J_g(\bar{x}) + \alpha_h^T J_h(\bar{x})](x - \bar{x}) \geq 0 \ \forall x \in \mathrm{Cl}(X)$$

If in addiction, a constraint qualification holds then $\exists(\alpha_g, \alpha_h) \in (V^+ \times \mathbb{R}^p)$ such that:

$$\alpha_g^T g(\bar{x}) = 0 \quad \text{and} \quad [\nabla f(\bar{x})^T + \alpha_g^T J_g(\bar{x}) + \alpha_h^T J_h(\bar{x})](x - \bar{x}) \geq 0 \ \forall x \in \mathrm{Cl}(X)$$

Remark 2.1 *It can be easily proved that $\mathrm{Co}[W] = \mathbb{R}^{m+p}$, where (1):*

$$W = \{(t_g, t_h) \in \mathbb{R}^{m+p} : \ (t_g, t_h) = [J_g(x_0), J_h(x_0)](x - x_0), \ x \in \mathrm{Cl}(X)\},$$

is a constraint qualification condition for Problem P (see Cambini and Carosi (2002)) A complete study of constraint qualification conditions for scalar problems with set constraints can be seen for example in Giorgi and Guerraggio (1994).

The above optimality condition suggests us the introduction of the following Wolfe-type dual problem.

Definition 2.2 (Dual Problem) *The following problem is assumed to be the Dual Problem of P:*

$$D : \left\{ \begin{array}{c} \max \ L(x, \alpha_g, \alpha_h) \\ (x, \alpha_g, \alpha_h) \in S_D \end{array} \right. \quad \equiv \quad \left\{ \begin{array}{c} \max \ L(x, \alpha_g, \alpha_h) \\ \nabla_x L(x, \alpha_g, \alpha_h)(y - x) \geq 0 \ \forall y \in \mathrm{Cl}(X) \\ x \in A, \ \alpha_g \in V^+, \ \alpha_h \in \mathbb{R}^p \end{array} \right.$$

where

- $L(x, \alpha_g, \alpha_h) = f(x) + \alpha_g^T g(x) + \alpha_h^T h(x)$,
- $\nabla_x L(x, \alpha_g, \alpha_h) = \nabla f(x)^T + \alpha_g^T J_g(x) + \alpha_h^T J_h(x)$,

and

$$S_D = \{(x, \alpha_g, \alpha_h) \in (A \times V^+ \times \mathbb{R}^p) : \ \nabla_x L(x, \alpha_g, \alpha_h)(y - x) \geq 0, \ y \in \mathrm{Cl}(X)\}.$$

Remark 2.2 *Note that if X is open the dual problem D coincides with the one proposed in Mahajan and Vartak (1977). Moreover if $X = \mathbb{R}^n$ then D can be rewritten as*

$$\begin{cases} \max \ L(x, \alpha_g, \alpha_h) \\ \nabla_x L(x, \alpha_g, \alpha_h) = 0 \\ x \in A, \ \ \alpha_g \in V^+, \ \ \alpha_h \in \mathbb{R}^p \end{cases}$$

that is the well known Wolfe dual problem (see for example Mangasarian (1969)).

The pseudoconvexity of function L, with respect to the variable x, allows us to prove the following duality results. As the reader can expect, the Weak Duality result is obtained just assuming the pseudoconvexity of L at a given point x_2. On the other hand, the Strong Duality theorem requires a constraint qualification condition, and the pseudoconvexity of L, with respect to x, on the whole set A. Finally, under the strict pseudoconvexity assumption on the dual objective function L we are going to prove the Strict Converse Duality theorem. Note that, using the same assumptions, Mahajan and Vartak (1977) prove the duality results for a problem P whose feasible region is defined by only equality and inequality constraints. On the other hand, under pseudoinvexity properties, Giorgi and Guerraggio (1994) state duality theorems for problems with only a set constraint and inequality ones.

Theorem 2.2 (Weak Duality) *Let us consider problems P and D and let $x_1 \in S_P$, $(x_2, \alpha_g, \alpha_h) \in S_D$. If for each fixed $\alpha_g \in V^+$ and $\alpha_h \in \mathbb{R}^p$ the function L is pseudoconvex in the variable x, at the point x_2, then*

$$f(x_1) \geq L(x_2, \alpha_g, \alpha_h)$$

Proof. Since $(x_2, \alpha_g, \alpha_h) \in S_D$ and $S_P \subset \mathrm{Cl}(X)$,

$$\nabla_x L(x_2, \alpha_g, \alpha_h)(x_1 - x_2) \geq 0$$

and from the pseudoconvexity of L it is

$$f(x_1) + \alpha_g^T g(x_1) + \alpha_h^T h(x_1) \geq f(x_2) + \alpha_g^T g(x_2) + \alpha_h^T h(x_2) \qquad (1.1)$$

Since $\alpha_g \in V^+$, $\alpha_g^T g(x_1) \leq 0$ and being $h(x_1) = 0$ it follows

$$f(x_1) \geq f(x_1) + \alpha_g^T g(x_1) + \alpha_h^T h(x_1) \geq f(x_2) + \alpha_g^T g(x_2) + \alpha_h^T h(x_2)$$

□

Remark 2.3 *Observe that in the case f is convex at x_2, g is V-convex at x_2 ([2]) and h is affine, the dual objective function L is pseudoconvex at x_2, for each fixed $\alpha_g \in V^+$ and $\alpha_h \in \mathbb{R}^p$, and hence the assumptions of the weak duality theorem are verified.*

Theorem 2.3 (Strong Duality) *Let us consider problems P and D. If for each fixed $\alpha_g \in V^+$ and $\alpha_h \in \mathbb{R}^p$ the function L is pseudoconvex in the variable x, on the set A, and a constraint qualification holds then $\forall \overline{x} \in \arg\min_{x \in S_P} f(x)$ $\exists \overline{\alpha}_g \in V^+$, $\exists \overline{\alpha}_h \in \mathbb{R}^p$ such that $(\overline{x}, \overline{\alpha}_g, \overline{\alpha}_h) \in S_D$ and*

$$f(\overline{x}) = L(\overline{x}, \overline{\alpha}_g, \overline{\alpha}_h) = \max_{(x,\alpha_g,\alpha_h) \in S_D} L(x, \alpha_g, \alpha_h)$$

so that $(\overline{x}, \overline{\alpha}_g, \overline{\alpha}_h) \in \arg\max_{(x,\alpha_g,\alpha_h) \in S_D} L(x, \alpha_g, \alpha_h)$.

Proof. Since $\overline{x} \in \arg\min_{x \in S_P} f(x)$, for Theorem 2.1 $\exists \overline{\alpha}_g \in V^+$, $\exists \overline{\alpha}_h \in \mathbb{R}^p$ such that

$$\overline{\alpha}_g^T g(\overline{x}) = 0 \quad \text{and} \quad [\nabla f(\overline{x})^T + \overline{\alpha}_g^T J_g(\overline{x}) + \overline{\alpha}_h^T J_h(\overline{x})](x - \overline{x}) \geq 0 \ \forall x \in \mathrm{Cl}(X)$$

and this implies that $(\overline{x}, \overline{\alpha}_g, \overline{\alpha}_h) \in S_D$. Being $\overline{x} \in S_P$ it is $h(\overline{x}) = 0$ so that for the weak duality theorem:

$$\begin{aligned} L(\overline{x}, \overline{\alpha}_g, \overline{\alpha}_h) &= f(\overline{x}) + \overline{\alpha}_g^T g(\overline{x}) + \overline{\alpha}_h^T h(\overline{x}) = \\ &= f(\overline{x}) \geq L(x_2, \alpha_g, \alpha_h) \quad \forall (x_2, \alpha_g, \alpha_h) \in S_D \end{aligned}$$

and hence the result is proved. □

Theorem 2.3 allows us to prove the following results.

Corollary 2.1 *Let us consider problems P and D; assume that for each fixed $\alpha_g \in V^+$ and $\alpha_h \in \mathbb{R}^p$ the function L is pseudoconvex in the variable x, on the set A, and a constraint qualification holds. If $S_D = \emptyset$ then $\arg\min_{x \in S_P} f(x) = \emptyset$.*

Corollary 2.2 *Let us consider problems P and D. If for each fixed $\alpha_g \in V^+$ and $\alpha_h \in \mathbb{R}^p$ the function L is pseudoconvex in the variable x, on the set A, and a constraint qualification holds then $\forall \overline{x} \in \arg\min_{x \in S_P} f(x)$ and $\forall (\widehat{x}, \widehat{\alpha}_g, \widehat{\alpha}_h) \in \arg\max_{(x,\alpha_g,\alpha_h) \in S_D} L(x, \alpha_g, \alpha_h)$ it is:*

$$f(\overline{x}) = L(\widehat{x}, \widehat{\alpha}_g, \widehat{\alpha}_h)$$

Proof. Let $(\widehat{x}, \widehat{\alpha}_g, \widehat{\alpha}_h) \in \arg\max_{(x,\alpha_g,\alpha_h)\in S_D} L(x, \alpha_g, \alpha_h)$ and let $\overline{x} \in \arg\min_{x\in S_P} f(x)$; for the Strong Duality

$$\exists(\overline{x}, \overline{\alpha}_g, \overline{\alpha}_h) \in \arg \max_{(x,\alpha_g,\alpha_h)\in S_D} L(x, \alpha_g, \alpha_h)$$

so that $f(\overline{x}) = L(\overline{x}, \overline{\alpha}_g, \overline{\alpha}_h) = L(\widehat{x}, \widehat{\alpha}_g, \widehat{\alpha}_h)$. □

Theorem 2.4 (Strict Converse Duality) *Let us consider problems P and D with $(\widehat{x}, \widehat{\alpha}_g, \widehat{\alpha}_h) \in \arg\max_{(x,\alpha_g,\alpha_h)\in S_D} L(x, \alpha_g, \alpha_h)$ and $\overline{x} \in \arg\min_{x\in S_P} f(x)$. Assume that for each fixed $\alpha_g \in V^+$ and $\alpha_h \in \mathbb{R}^p$ the function $L(x, \alpha_g, \alpha_h)$ is pseudoconvex in the variable x, on the set A, and strictly pseudoconvex at $\widehat{x}$. If a constraint qualification holds then $\overline{x} = \widehat{x}$ and trivially $\widehat{x} \in \arg\min_{x\in S_P} f(x)$.*

Proof. Being $\overline{x} \in S_P$ and $(\widehat{x}, \widehat{\alpha}_g, \widehat{\alpha}_h) \in S_D$ it is $h(\overline{x}) = 0$, $g(\overline{x}) \in -V$, $\widehat{\alpha}_g \in V^+$, so that $\widehat{\alpha}_g^T g(\overline{x}) \leq 0$ and for the previous corollary:

$$L(\widehat{x}, \widehat{\alpha}_g, \widehat{\alpha}_h) = f(\overline{x}) \geq f(\overline{x}) + \widehat{\alpha}_g^T g(\overline{x}) + \widehat{\alpha}_h^T h(\overline{x}) = L(\overline{x}, \widehat{\alpha}_g, \widehat{\alpha}_h)$$

Suppose now by contradiction that $\overline{x} \neq \hat{x}$; because of the strict pseudoconvexity of L at $\hat{x}$ with respect to the variable x, condition $L(\widehat{x}, \widehat{\alpha}_g, \widehat{\alpha}_h) \geq L(\overline{x}, \widehat{\alpha}_g, \widehat{\alpha}_h)$ implies

$$\nabla_x L(\widehat{x}, \widehat{\alpha}_g, \widehat{\alpha}_h)^T(\overline{x} - \widehat{x}) = [\nabla f(\widehat{x})^T + \widehat{\alpha}_g^T J_g(\widehat{x}) + \widehat{\alpha}_h^T J_h(\widehat{x})](\overline{x} - \widehat{x}) < 0 \tag{1.2}$$

Since $\overline{x} \in X$, (1.2) implies that $(\widehat{x}, \widehat{\alpha}_g, \widehat{\alpha}_h) \notin S_D$ which is a contradiction. □

3. The fractional case

In this section we consider a fractional problem where the objective function f is the ratio between a convex function and an affine one and the feasible region is defined as in Problem P, that is

$$P_F : \begin{cases} \min\ f(x) = \frac{n(x)}{d(x)} & \\ g(x) \in -V & \textit{inequality constraints} \\ h(x) = 0 & \textit{equality constraints} \\ x \in X & \textit{set constraint} \end{cases} \tag{1.3}$$

where

- $n : A \to \mathbb{R}$ and $g : A \to \mathbb{R}^m$ are differentiable functions,

- $A \subseteq \mathbb{R}^n$ is an open convex set,
- $V \subset \mathbb{R}^m$ is a closed convex pointed cone with nonempty interior,
- $d(x) = d^T x + d_0$ with $d \in \mathbb{R}^n$, $d_0 \in \mathbb{R}$ and $d^T x + d_0 > 0 \ \forall x \in A$,
- $h(x) = Bx + b_0$ with $B \in \mathbb{R}^{p \times n}$, $b_0 \in \mathbb{R}^p$,
- n is convex in A and g is V-convex in A,
- $X \subseteq A$ is a convex set with nonempty interior which is not required to be open or closed.

Since we consider an arbitrary convex set X we can not apply the Wolfe-like duality results that can be found in the wide literature concerning duality in fractional programming. On the other hand, even though the objective function f is pseudoconvex (see Mangasarian (1970)), the dual objective function L is not necessarily pseudoconvex. Hence we are not able to direct apply the duality results stated in the previous section. Along the lines proposed in Schaible (1976a) and Schaible (1976b) we can transform problem P_F in the following equivalent one

$$\overline{P}: \begin{cases} \min \ \overline{n}(y,t) = t \cdot n\left(\frac{y}{t}\right) \\ \overline{g}(y,t) = t \cdot g\left(\frac{y}{t}\right) \in -V \\ \overline{h}(y,t) = t \cdot h\left(\frac{y}{t}\right) = 0 \\ \overline{d}(y,t) = t \cdot d\left(\frac{y}{t}\right) - 1 = 0 \\ -t \leq 0 \\ (y,t) \in \Pi_X \end{cases}$$

where $\Pi_X = \{(y,t) \in \mathbb{R}^n \times \mathbb{R} : \frac{y}{t} \in X, \ t > 0\}$ and the functions $\overline{n}, \overline{d}, \overline{g}$ and $\overline{h}$ are defined over the set $\Pi_A = \{(y,t) \in \mathbb{R}^n \times \mathbb{R} : \frac{y}{t} \in A, \ t > 0\}$.

Thanks to the performed transformation, the new problem $\overline{P}$ verifies the following convexity properties.

Lemma 3.1 *Let us consider problem $\overline{P}$. The following statements hold:*

1 Π_X and Π_A are convex sets with nonempty interior;

2 $\overline{g}$ is V-convex in Π_A;

3 $\overline{n}$ is convex in Π_A;

4 $\overline{h}$ and $\overline{d}$ are affine in Π_A.

Proof. 1. Consider $(y_1, t_1) \in \Pi_X$ and $(y_2, t_2) \in \Pi_X$; we want to prove that

$$(y_1, t_1) + \mu\left((y_2, t_2) - (y_1, t_1)\right) \in \Pi_X \ \forall \mu \in [0, 1],$$

that is to say

$$\frac{y_1+\mu(y_2-y_1)}{t_1+\mu(t_2-t_1)} \in X \text{ and } t_1+\mu\,(t_2-t_1)>0,\ \forall \mu \in [0,1]$$

By means of simple calculations we have that $t_1+\mu\,(t_2-t_1)>0$ and

$$\frac{y_1+\mu(y_2-y_1)}{t_1+\mu(t_2-t_1)} = \frac{y_1}{t_1}+\beta\left(\frac{y_2}{t_2}-\frac{y_1}{t_1}\right)$$

where

$$\beta = \frac{1}{1+\frac{t_1(1-\mu)}{t_2\mu}}.$$

Since X is convex and β trivially belongs to $[0,1]$, $\frac{y_1+\mu(y_2-y_1)}{t_1+\mu(t_2-t_1)} \in X$ and the result is proved. The convexity of Π_A follows with the same arguments.

2. Consider $(y_1,t_1)\in \Pi_A$ and $(y_2,t_2)\in\Pi_A$; we want to show that $\overline{g}(y_2,t_2)-\overline{g}(y_1,t_1)-J_{\overline{g}}(y_1,t_1)\begin{bmatrix} y_2-y_1 \\ t_2-t_1\end{bmatrix}\in V$.

$$\begin{aligned} J_{\overline{g}}(y_1,t_1)\begin{bmatrix} y_2-y_1 \\ t_2-t_1\end{bmatrix} &= J_g\left(\frac{y_1}{t_1}\right)(y_2-y_1)+g\left(\frac{y_1}{t_1}\right)(t_2-t_1)-\frac{1}{t_1}J_g\left(\frac{y_1}{t_1}\right)y_1\,(t_2-t_1) \\ &= J_g\left(\frac{y_1}{t_1}\right)\left(y_2-y_1-\frac{y_1}{t_1}t_2+y_1\right)+g\left(\frac{y_1}{t_1}\right)(t_2-t_1) \\ &= t_2J_g\left(\frac{y_1}{t_1}\right)\left(\frac{y_2}{t_2}-\frac{y_1}{t_1}\right)+g\left(\frac{y_1}{t_1}\right)(t_2-t_1) \end{aligned} \quad (1.4)$$

Since $t_2>0$, $\frac{y_2}{t_2},\frac{y_1}{t_1}\in X$ and g is V-convex in A, it holds

$$t_2g\left(\frac{y_2}{t_2}\right)-t_2g\left(\frac{y_1}{t_1}\right)-t_2J_g\left(\frac{y_1}{t_1}\right)\left(\frac{y_2}{t_2}-\frac{y_1}{t_1}\right)\in V \quad (1.5)$$

Therefore, from (1.4) we obtain

$$t_2J_g\left(\frac{y_1}{t_1}\right)\left(\frac{y_2}{t_2}-\frac{y_1}{t_1}\right)=J_{\overline{g}}(y_1,t_1)\begin{bmatrix} y_2-y_1 \\ t_2-t_1\end{bmatrix}-t_2g\left(\frac{y_1}{t_1}\right)+t_1g\left(\frac{y_1}{t_1}\right) \quad (1.6)$$

and so substituting (1.6) in (1.5) we get

$$t_2g\left(\frac{y_2}{t_2}\right)-t_2g\left(\frac{y_1}{t_1}\right)-J_{\overline{g}}(y_1,t_1)\begin{bmatrix} y_2-y_1 \\ t_2-t_1\end{bmatrix}+t_2g\left(\frac{y_1}{t_1}\right)-t_1g\left(\frac{y_1}{t_1}\right)\in V$$

that is

$$\overline{g}\,(y_2,t_2)-\overline{g}\,(y_1,t_1)-J_{\overline{g}}(y_1,t_1)\begin{bmatrix} y_2-y_1 \\ t_2-t_1\end{bmatrix}\in V$$

and then we are done.

3. It is a particular case of 2.
4. It trivially holds that

$$\overline{d}(y,t) = t \cdot \left(d^T \left(\frac{y}{t}\right) + d_0\right) - 1 = d^T y + d_0 t - 1.$$

The affinity of $\overline{h}$ is obtained with the same argument. □

From Remark 2.3 and Lemma 3.1, the duality results proved in the previous section can now be applied to $\overline{P}$. With this aim let us introduce the following notations:

$$\begin{aligned} L_F(x,\alpha_g,\alpha_h,\lambda) &= n(x) + \alpha_g^T g(x) + \alpha_h^T h(x) + \lambda d(x) \\ \nabla_x L_F(x,\alpha_g,\alpha_h,\lambda)^T &= \nabla n(x)^T + \alpha_g^T J_g(x) + \alpha_h^T B + \lambda d^T \end{aligned}$$

To simplify the notations we will use also the following shortcuts:

$$L_F(x) = L_F(x,\alpha_g,\alpha_h,\lambda) \quad \text{and} \quad \nabla_x L_F(x) = \nabla_x L_F(x,\alpha_g,\alpha_h,\lambda)$$

The dual problem of $\overline{P}$ results:

$$\left\{ \begin{array}{c} \max \ tZ(y,t) - t\delta - \lambda \\ t\nabla_y Z(y,t)^T(y_1 - y) + [Z(y,t) - \delta + t\nabla_t Z(y,t)]\,(t_1 - t) \geq 0 \ \forall (y_1,t_1) \in \Pi_X \\ (y,t) \in \Pi_A, \ \alpha_g \in V^+, \ \alpha_h \in \mathbb{R}^p, \ \lambda \in \mathbb{R}, \ \delta \geq 0 \end{array} \right. \tag{1.7}$$

where $Z(y,t) = L_F(\frac{y}{t},\alpha_g,\alpha_h,\lambda)$. Since problems $\overline{P}$ and P_F are equivalent, we can consider a problem equivalent to (1.7) as the dual of P_F. We now present the following technical lemma.

Lemma 3.2 *The following conditions are equivalent:*

i) $\nabla_x L_F(x)\,(x_1 - x) + [L_F(x) - \delta]\left(1 - \frac{t}{t_1}\right) \geq 0 \ \forall x_1 \in X, \ \forall t_1 > 0,$

ii) $0 \leq [\delta - L_F(x)] \leq \nabla_x L_F(x)\,(x_1 - x) \ \forall x_1 \in X.$

Proof. *i)⇒ii)* Since *i)* holds $\forall t_1 > 0$ it results:

$$\begin{aligned} 0 &\leq \lim_{t_1 \to +\infty} \nabla_x L_F(x)\,(x_1 - x) + [L_F(x) - \delta]\left(1 - \frac{t}{t_1}\right) \\ &= \nabla_x L_F(x)\,(x_1 - x) + [L_F(x) - \delta]\,. \end{aligned}$$

Suppose now by contradiction that $[L_F(x) - \delta] > 0$, then $t > 0$ implies

$$\lim_{t_1 \to 0^+} \nabla_x L_F(x)\,(x_1 - x) + [L_F(x) - \delta]\left(1 - \frac{t}{t_1}\right) = -\infty$$

which contradicts *i)*.

ii)$\Rightarrow$*i)* Since $t > 0$ and $t_1 > 0$ it is $\left(1 - \frac{t}{t_1}\right) < 1$, so that *ii)* implies

$$[\delta - L_F(x)]\left(1 - \frac{t}{t_1}\right) \leq [\delta - L_F(x)] \leq \nabla_x L_F(x)\,(x_1 - x)\ \ \forall x_1 \in X$$

and the result is proved. □

We are now able to suggest the following dual problem of P_F.

Theorem 3.1 *The dual of problem P_F is:*

$$D_F : \begin{cases} \max\ -\lambda\ \equiv\ -\min\ \lambda \\ L_F(x) \geq 0 \\ \nabla_x L_F(x)\,(x_1 - x) \geq 0\ \forall x_1 \in X \\ x \in A,\ \ \alpha_g \in V^+,\ \ \alpha_h \in \mathbb{R}^p,\ \ \lambda \in \mathbb{R} \end{cases} \tag{1.8}$$

Proof. Since

$$\nabla_y Z(y,t)^T = \frac{1}{t}\nabla_x L_F\left(\frac{y}{t}, \alpha_g, \alpha_h, \lambda\right)$$
$$\nabla_t Z(y,t) = -\frac{1}{t^2}\nabla_x L_F\left(\frac{y}{t}, \alpha_g, \alpha_h, \lambda\right) y$$

and using the notations $x = \frac{y}{t}$ and $x_1 = \frac{y_1}{t_1}$, the dual problem (1.7) can be rewritten as:

$$\begin{cases} \max\ \ tL_F(x) - t\delta - \lambda \\ \nabla_x L_F(x)\,(x_1 - x) + [L_F(x) - \delta]\left(1 - \frac{t}{t_1}\right) \geq 0\ \forall x_1 \in X,\ \ \forall t_1 > 0 \\ x \in A,\ \ t > 0,\ \ \alpha_g \in V^+,\ \ \alpha_h \in \mathbb{R}^p,\ \ \lambda \in \mathbb{R},\ \ \delta \geq 0 \end{cases} \tag{1.9}$$

From Lemma 3.2 problem (1.9) is equivalent to the following one:

$$\begin{cases} \max\ \ -t\,[\delta - L_F(x)] - \lambda \\ \nabla_x L_F(x)\,(x_1 - x) \geq [\delta - L_F(x)]\ \ \forall x_1 \in X \\ [\delta - L_F(x)] \geq 0 \\ x \in A,\ \ \alpha_g \in V^+,\ \ \alpha_h \in \mathbb{R}^p,\ \ t > 0,\ \ \lambda \in \mathbb{R},\ \ \delta \geq 0 \end{cases} \tag{1.10}$$

Observe that for any optimal solution of problem (1.10) it is $[\delta - L_F(x)] = 0$. Suppose otherwise; then there exists an optimal solution $(\tilde{x}, \tilde{\alpha}_g, \tilde{\alpha}_h, \tilde{\lambda}, \tilde{t}, \tilde{\delta})$ such that $\left[\tilde{\delta} - L_F(\tilde{x})\right] > 0$. Since $\tilde{t} > 0$, for any $0 < \epsilon < \tilde{t}$ the vector $(\tilde{x}, \tilde{\alpha}_g, \tilde{\alpha}_h, \tilde{\lambda}, \tilde{t} - \epsilon, \tilde{\delta})$ is feasible for problem (1.10) and is better than $(\tilde{x}, \tilde{\alpha}_g, \tilde{\alpha}_h, \tilde{\lambda}, \tilde{t}, \tilde{\delta})$, which is a contradiction. Therefore, the result follows from (1.10) being $[\delta - L_F(x)] = 0$ and $\delta \geq 0$. □

In the case $X = \mathbb{R}^n$ (that is P_F has no set constraints), problem D_F coincides with the one already studied in the literature (see Jagannathan (1973); Schaible (1976a); Schaible (1976b)).

Corollary 3.1 *In the case $X = A = \mathbb{R}^n$ it results:*

$$D_F : \begin{cases} \max\ -\lambda \ \equiv\ -\min\ \lambda \\ L_F(x) \geq 0 \\ \nabla_x L_F(x) = 0 \\ x \in \mathbb{R}^n,\ \alpha_g \in V^+,\ \alpha_h \in \mathbb{R}^p,\ \lambda \in \mathbb{R} \end{cases}$$

Proof. Follows from Theorem 3.1 noticing that, being $X = A = \mathbb{R}^n$, condition $\nabla_x L_F(x)\,(x_1 - x) \geq 0$ holds $\forall x_1 \in \mathbb{R}^n$ if and only if $\nabla_x L_F(x) = 0$. □

Notes

1. We denote with $Co(X)$ the convex hull of a set X.

2. Let $A \subset \mathbb{R}^n$ be convex and $V \subset \mathbb{R}^m$ a convex cone; a function $g : A \longrightarrow \mathbb{R}^m$ is said to be V-convex at $x_2 \in A$ if

$$g(x_1) - g(x_2) - Jg(x_2)\,(x_1 - x_2) \in V \quad \forall x_1 \in A.$$

Function g is said to be V-convex in A if it is V-convex at x_2 for every $x_2 \in A$.

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