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Anna Marchi

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Via Cosimo Ridolfi, 10 – 56124 PISA – Tel. Segr. Amm. 050 945231 Segr. Stud. 050 945317 Fax 050 945375
Cod. Fisc. 80003670504 - P. IVA 00286820501 - Web http://statmat.ec.unipi.it/ - E-mail: dipstat@ec.unipi.it

Chapter 1

ON THE MIX EFFICIENT POINTS

Anna Marchi*
Dept. of Statistics and Applied Mathematics
University of Pisa, Italy

Abstract In this paper the concepts of mix semi-efficient point and mix quasi-convex set are introduced. By means of these concepts we will investigate the connectedness of the mix semi-efficient frontier for vector maximization problems, defined by quasi-concave, strictly and strong quasi-concave functions. We prove that the mix semi-efficient frontier of a mix quasi concave problem is a closed and connected set.

Keywords: vector maximization problem, efficient frontier

Mathematics Subject Classification (2000)

1. Introduction

Many authors have studied the connectedness and closure of the efficient and weakly efficient frontier for a vector maximization problem. These topological properties are very important, from an algorithmic point of view, in continuously generating the efficient frontier of a vector problem. In Marchi (1992) the knowledge of the efficient frontier of a vector problem allows us to solve suitable scalar problems. Warburton (1983) proved that if the vector function F of the problem is strongly quasi-concave then the (Paretian) efficient frontier is closed and connected, while if F is quasi-concave, then the weakly efficient frontier is closed and connected. After this paper, several authors tried to relax the concavity assumption. Hu et al. (1993) proved that if F is a strictly quasi-concave function and the efficient frontier is a closed set, then this

*email:marchiae@ec.unipi.it

frontier is connected. Sun (1996) proved that if F is a strictly quasi-concave function and at least one function is strongly quasi-concave, then the efficient frontier is connected. Finally, Benoist (1998) proved that if F is a strictly quasi-concave function then the efficient frontier is a connected set. However, if we relax the assumptions on the function F, to obtain the connectedness of the efficient frontier, we lose the property of closure of the set. In this paper, we consider a mix quasi-concave function in which there are quasi-concave, strictly quasi-concave and strongly quasi-concave functions because, in the applications, we can have different types of functions. In section 2, we relax the assumptions on the Paretian cone, by introducing the definitions of mix and mix semi-efficient point. In section 3, we study the properties of mix quasi-concave sets because, obviously, the image of a problem defined by a mix quasi-concave function is a mix quasi-concave set. Finally, we prove the closure and connectedness of mix semi-efficient frontier.

In this paper, we consider a convex cone K in $\mathbb{R}^m_+$ such that $intK \neq \emptyset$, $(-K) \cap K = \{0\}$. The sets clK, $intK$ denote the closure, interior of K, respectively. For any $y, z \in \mathbb{R}^m$, $I = \{1, 2, .., m\}$ we have that:

$y \geqq z$ means $y - z \in \mathbb{R}^m$, $y_i \geqq z_i$ for any $i \in I$,

$y \geq z$ means $y - z \in \mathbb{R}^m \setminus \{0\}$, $y_i \geq z_i$ for any $i \in I, y \neq z$,

$y > z$ means $y - z \in int\mathbb{R}^m$, $y_i > z_i$ for any $i \in I$,

$y \bar{>} z$ means $y - z \in \mathbb{R}^m \setminus H$, $y_i > z_i$ for any $i \neq j \in I$, $y_j = z_j$ for the only component j, where $H = \cup_{h<s-1} F_h$, $s \geq 3$, F_h is the h-dimensional face of $\mathbb{R}^s_+$.

2. Mix-problem

Consider the vector maximization problem with respect to the ordering cone K:

$$P_k : K - \max F(x), \ x \in X$$

where $F = (f_1, f_2, ...f_m) : X \rightarrow \mathbb{R}^m$ is a vector continuous function, $X \subset \mathbb{R}^n$ is a convex compact set and K is a convex cone such that $intK \neq \emptyset$ and $(-K \cap K) = \{\emptyset\}$.

Definition 2.1 *$x^0 \in X$ is an* **efficient point** *for problem P_M with respect to the ordering generated by $K \Longleftrightarrow$ there does not exist a $x \in X$ such that $F(x) \in F(x^0) + K \setminus \{0\}$.*

We recall that when $K = \mathbb{R}^m_+$ or $K = int\mathbb{R}^m_+$ we have the usual definition of the Paretian efficiency and weak efficiency, respectively. Let us consider the sets of all efficient and weakly efficient points of $F(X)$ with respect to the ordering generated by K denoted by $E(F(X), K)$ and $WE(F(X), K)$ respectively, i.e.

$$E(F(X),K) = \{y \in F(X) \mid z - y \notin K\backslash 0, \forall\, z \in F(X)\}$$
$$WE(F(X),K) = \{y \in F(X) \mid z - y \notin intK, \forall\, z \in F(X)\}$$

where $F(X) \subset \mathbb{R}^m$ is the outcome of F. Set $F(x) = (F_Q(x), F_S(x), F_T(x))$ where $F_Q(x)=(f_1(x),..,f_q(x))$, $F_S(x)=(f_{q+1}(x)..,f_{q+s}(x))$, $F_T(x)=(f_{q+s+1}(x),..,f_{q+s+t}(x))$ for any $x \in X$ and $m = q + s + t$. In this paper we consider a vector mix quasi-concave function:

Definition 2.2 *If $F_Q(x)$ is a quasi-concave, $F_S(x)$ is a strictly quasi-concave and $F_T(x)$ is a strongly quasi-concave function then $F(x)$ is said to be a* **mix quasi-concave** *function on X.*

where F_Q, F_S, F_T have the well-known definitions:

- If $F_Q[\lambda x_1 + (1-\lambda)x_2] \geq min[F_Q(x_1), F_Q(x_2)]$ for any $x_1, x_2 \in X$, $\lambda \in (0,1)$ then F_Q is said to be **quasi-concave** on X,
- If for any f_i, $i \in I$ satisfies $f_i(\lambda x_1 + (1-\lambda)x_2) > min[f_i(x_1), f_i(x_2)]$ for any $x_1, x_2 \in X$, $f_i(x_1) \neq f_i(x_2)$, $\lambda \in (0,1)$ then F_S is said to be **strictly quasi-concave** on X,
- If $F_T(\lambda x_1 + (1-\lambda)x_2) > min[F_T(x_1), F_T(x_2)]$ for any $x_1, x_2 \in X$, $\lambda \in (0,1)$ then F_T is said to be **strongly quasi-concave** on X.

Set $I = \{1, 2, \cdots, m\}$, $Q = \{1,..,q\}$, $S = \{q+1,...,q+s\}$, $T = \{q+s+1,..,m\}$, $f_i(\overline{x}) = \max\{f_i(x) \mid x \in X\}$, $S^i = \{F(x) \mid f_i(x) = f_i(\overline{x})\}$ for any $i \in I$, we have:

Lemma 2.1 *If F is a mix quasi-concave function then, for any $i \in I$, S^i is a connected set, in particular, for any $i \in T$, S^i has a unique element.*

Proof. Directly, taking into account the continuity and definition of a mix quasi-concave function. □

Set $K = int\mathbb{R}^q_+ \times \mathbb{R}^s_+ \backslash H \times \mathbb{R}^t_+$, where $H = \cup_{h<s-1} F_h$, $s \geq 3$, F_h is the h-dimensional face of $\mathbb{R}^s_+$, we introduce the following definitions:

Definition 2.3 *$x^0 \in X$ is a* **mix efficient point** *for problem $P_M \Longleftrightarrow$ there does not exist a $x \in X$ such that $F(x) \in F(x^0) + \{int\mathbb{R}^q_+ \times \mathbb{R}^s_+ \backslash \{0\} \times \mathbb{R}^t_+\}$.*

Definition 2.4 *$x^0 \in X$ is a* **mix semi-efficient point** *for problem $P_M \Longleftrightarrow$ there does not exist a $x \in X$ such that $F(x) \in F(x^0) + \{int\mathbb{R}^q_+ \times \mathbb{R}^s_+ \backslash H \times \mathbb{R}^t_+\}$.*

We will study the properties of closure and connectedness of the mix semi-efficient frontier of $F(X)$.

3. Mix quasi-concave sets

Let $Y \subset \mathbb{R}^m$ be a non-empty closed set. The following definition is introduced in Benoist (1998):

Definition 3.1 *Y is said to be* **sequentially strictly quasi-concave** *(**SQV**) if, for any $y^1, y^2 \in Y$ there exists a sequence$\left\{y^L\right\} \to y^1$, $y^L \in Y$ such that, for any $L \in \aleph$, $y_i^L > min\{y_i^1, y_i^2\}$ if $y_i^1 \neq y_i^2$, $y_i^L \geqq y_i^1 = y_i^2$ if $y_i^1 = y_i^2$.*

Set $\bar{y}_i = \max\{y_i \mid y \in Y\}$, $M^i = \{y \in Y \mid y_i = \bar{y}_i\}$ $i \in I$.

Obviously (see Benoist (1998)) we have that if F is a quasi-concave function, then its outcome $F(X)$ is a sequentially quasi-concave set.

We will extend the definition given by Benoist for the class of strictly quasi-concave sets to the class of mix quasi-concave sets.

Let $z = (z_Q, z_S, z_T)$ be an element of $\mathbb{R}^m$, such that $z_Q \in \mathbb{R}^q$, $z_S \in \mathbb{R}^s$, $z_T \in \mathbb{R}^t$, we define:

Definition 3.2 *Y is said to be* **sequentially mix quasi-concave** *(MQV) if, for any $\hat{y}, \check{y} \in Y$, there exists a sequence$\left\{y^L\right\} \to \hat{y}$, $y^L \in Y$ such that for any $L \in \aleph$,*

$$\begin{aligned} y_Q^L &\geq min\{\hat{y}_Q, \check{y}_Q\} \qquad (1.1)\\ y_i^L &> min\{\hat{y}_i, \check{y}_i\} \text{ for any } i \in \{i | \hat{y}_i \neq \check{y}_i\}\ ,\ y_i^L \geqq \hat{y}_i = \check{y}_i,\ i \in S \\ y_T^L &> min\{\hat{y}_T, \check{y}_T\}. \end{aligned}$$

Obviously, we have that if F is a mix quasi-concave function, then its outcome $F(X)$ is a mix quasi-concave set.

Lemma 3.1 *M^i is a connected set for any $i \in I$.*

Proof. For lemma 2.1, $F(X)$ has the property that M^i is a connected set for any $i \in I$. □

3.1 Mix semi-efficient elements

Let $Y \subset \mathbb{R}^m$ be a non-empty closed set. Let us consider the sets of all efficient and weakly efficient elements of Y denoted by $E(Y, \mathbb{R}_+^m)$ and $WE(Y, \mathbb{R}_+^m)$ respectively, i.e.

$$E(Y, \mathbb{R}_+^m) = \{y \in Y \mid\ \nexists\ z \geq y\ ,\ \forall\ z \in Y\}$$

$$WE(Y,\mathbb{R}^m_+) = \{y \in Y \mid \nexists\, z > y,\ \forall\, z \in Y\}.$$

If we define:

Definition 3.3 *For any $y^1, y^2 \in \mathbb{R}^m$, $y^1 \succeq y^2$ means $y^1 - y^2 \in \{int\mathbb{R}^q_+ \times \mathbb{R}^s_+ \backslash H \times \mathbb{R}^t_+\}$, that is $y^1_Q > y^2_Q$, $y^1_S \overline{>} y^2_S$, $y^1_T \geqq y^2_T$.*

Definition 3.4 *For any $y^1, y^2 \in \mathbb{R}^m$, $y^1 \underline{\succeq} y^2$ means $y^1 - y^2 \in \{int\mathbb{R}^q_+ \times \mathbb{R}^s_+ \backslash \{0\} \times \mathbb{R}^t_+\}$, that is $y^1_Q > y^2_Q$, $y^1_S \geq y^2_S$, $y^1_T \geqq y^2_T$.*

we have that:

Definition 3.5 *$y^0 \in Y$ is a* **mix semi-efficient element** *for $Y \Longleftrightarrow$ there does not exist a $y \in Y$ such that $y \succeq y^0$.*

Definition 3.6 *$y^0 \in Y$ is a* **mix efficient element** *for $Y \Longleftrightarrow$ there does not exist a $y \in Y$ such that $y \underline{\succeq} y^0$.*

The sets of all mix efficient and mix semi-efficient elements of Y denoted by:

$$MSE(Y,\mathbb{R}^m_+) = \{y \in Y \mid \nexists\, z \succeq y\ ,\ \forall\, z \in Y\},$$
$$ME(Y,\mathbb{R}^m_+) = \{y \in Y \mid \nexists\, z \underline{\succeq} y,\ \forall\, z \in Y\},$$

respectively. It is easy to prove that among these sets the following relationships hold:

$$E(Y,\mathbb{R}^m_+) \subseteq ME(Y,\mathbb{R}^m_+) \subseteq MSE(Y,\mathbb{R}^m_+) \subseteq WE(Y,\mathbb{R}^m_+).$$

We will prove that if $F(X)$ is a mix quasi-concave set then $MSE(F(X),\mathbb{R}^m_+)$ is a closed and connected set, while the connectedness of the mix efficient frontier is yet an open problem.

Lemma 3.2 *$MSE(M^i,\mathbb{R}^m_+)$ is connected for any $i \in Q \cup T$.*

Proof. For any $i \in T$, M^i has a unique element which is a mix semi-efficient element. For any $i \in Q$, $MSE(M^i,\mathbb{R}^m_+)$ is a connected set because $MSE(M^i,\mathbb{R}^m_+) = M^i$ and M^i is a connected set for Lemma 3.1. □

Lemma 3.3 *If Y is a MQV and $\tilde{y}, \bar{y} \in MSE(Y,\mathbb{R}^m_+)$, $\tilde{y}_j = \bar{y}_j$ $j \in Q \cup S$ implies the existence of a natural number N and a sequence $\left\{y^k\right\} \to \tilde{y}$ such that $y^k \in MSE(Y,\mathbb{R}^m_+)$ $y^k_j = \tilde{y}_j = \bar{y}_j$, $k \in \aleph$.*

Proof. Directly, from definitions of MQV set and mix semi-efficient element. □

Theorem 3.1 *If Y is a* **mix quasi-concave** *set then $MSE(Y, \mathbb{R}^m_+)$ is closed.*

Proof. Let us suppose that $\bar{y} \in Y, \bar{y} \notin MSE(Y, \mathbb{R}^m_+)$ and there exists a sequence $\{y^k\} \to \bar{y}$ such that $y^k \in MSE(Y, \mathbb{R}^m_+)$ for any $k = N, N+1...$ If $\bar{y}$ is not mix semi-efficient element for Y there exists a $\hat{y} \in Y$ such that $\hat{y} \succeq \bar{y}$. Since Y is a MQV set then there exists a $z \in Y$ such that conditions (1.1) hold. Hence, there exists a $I_{\bar{y}} \subset z - int\mathbb{R}^m_+$ such that $I_{\bar{y}} \cap MSE(Y, \mathbb{R}^m_+) = \emptyset$. This is absurd since there exist a sequence of mix semi-efficient elements converging to $\bar{y}$. □

Theorem 3.2 *If Y is a MQV set then $\tilde{y} \in MSE(Y, \mathbb{R}^m_+)$, $\tilde{y} \notin M^j$ $j \in I$ implies: i) the existence of a natural number N and a sequence$\left\{y^k\right\} \to \tilde{y}$ such that $y^k \in MSE(Y, \mathbb{R}^m_+)$ $y^k_j > \tilde{y}_j$ $j \in T$, $y^k_j \geq \tilde{y}_j$ $\forall j \in Q \cup S$, $k \in \aleph$, ii) if $j \in Q \cup S$ then there exists $y^0 \in MSE(Y, \mathbb{R}^m_+)$, $y^0_j = \tilde{y}_j$ and a sequence$\left\{y^k\right\} \to y^0$ such that $y^k \in MSE(Y, \mathbb{R}^m_+)$ $y^k_j > y^0_j$.*

Proof. i) Since $\tilde{y} \notin M^j$ there exists a $\hat{y} \in M^j$, $\hat{y} \in MSE(Y, \mathbb{R}^m_+)$, such that $\hat{y}_j > \tilde{y}_j$. If Y is a MQV set then there exists a sequence $\left\{y^L\right\} \to \tilde{y}$, $y^L \in Y$ such that conditions (1.1) hold. For any $L \in \aleph$, set $Y^L = Y \cap \left\{y^L + \mathbb{R}^m_+\right\}$. Since Y^L is a compact set then $E(Y^L, \mathbb{R}^m_+) \neq \emptyset$ and there exists a $y^k \in MSE(Y, \mathbb{R}^m_+)$ such that $y^k \geqq y^L$. Since $\left\{y^L\right\} \to \tilde{y}$, $\left\{y^k\right\} \to \tilde{y}$ with $y^k_j \geq \tilde{y}_j$ $\forall j \in Q \cup S$, $y^k_j > \tilde{y}_j$ $\forall j \in T$, ii) From i) of this theorem, Lemma 3.3 and Theorem 3.1. □

Theorem 3.3 *Let Y be a MQV set. If $Q \cup T \neq \emptyset$ then $MSE(Y, \mathbb{R}^m_+)$ is a connected set.*

Proof. Let us suppose that $MSE(Y, \mathbb{R}^m_+)$ is disconnected. Since, for Theorem 3.1, $MSE(Y, \mathbb{R}^m_+)$ is a closed set, then there exist two closed sets E^1, E^2 of $MSE(Y, \mathbb{R}^m_+)$ such that $MSE(Y, \mathbb{R}^m_+) = E^1 \cup E^2$, $E^1 \cap E^2 = \emptyset$. Let $\hat{y}^1_i = \max\{y_i | \, y \in E^1\}$ and $\hat{y}^2_i = \max\{y_i | \, y \in E^2\}$ with $\hat{y}^1 \neq \hat{y}^2 \in Y$, $i \in Q \cup T$. Since $\forall i \in Q \cup T$ $MSE(M^i, \mathbb{R}^m_+)$ is a connected set, then $\hat{y}^1_i \neq \hat{y}^2_i$. Suppose $\hat{y}^1_i > \hat{y}^2_i$, then, for Theorem 3.2, there exists a sequence $\{y^L\} \to \hat{y}^2$ such that $y^L \in MSE(Y, \mathbb{R}^m_+)$ with $y^L_i > \hat{y}^2_i$ if $i \in T$, so $y^L \in E^1$. This is absurd because E^1 is a closed set. If $i \in Q$, then there exists $y^0 \in E^2$ with $y_i^0 = \hat{y}_i^2$ and a sequence $\left\{y^k\right\} \to y^0$ such that $y^k \in MSE(Y, \mathbb{R}_+^m)$ $y_i^k > y_i^0$, so $y^k \in E^1$. This is absurd because E^1 is a closed set. □

The following example, by ?, shows $E(F(X), \mathbb{R}_+^m)$ may be disconnected when one component of F is merely quasi-concave, while $MSE(F(X), \mathbb{R}_+^m)$ is a connected set.

Example 3.1 *Let* $X = [0, 2]$. *Define* $F(x) = (f_1(x), f_2(x), ...f_m(x))$ *with* $\mathbf{f}_i(\mathbf{x}) = \mathbf{x}$ *for any* $i \in \{1, 2, .., m-1\}$, $\mathbf{f}_m(\mathbf{x}) = \mathbf{2} - \mathbf{x}$ *if* $0 \leq x \leq 1$, $\mathbf{1}$ *if* $1 < x \leq 2$. *We can easily verify that* $E(F(X), \mathbb{R}_+^m) = \{F(x) \mid x \in [0, 1) \cup \{2\}\}$ *is disconnected while* $MSE(F(X), \mathbb{R}_+^m) = \{F(x) \mid x \in [0, 2]\}$ *is a connected set. In this case,* $MSE(F(X), \mathbb{R}_+^m) = ME(F(X), \mathbb{R}_+^m)$ *because there are no strictly quasi-concave functions.*

4. Conclusions

In this paper, we have given the definitions of mix efficient and mix semi-efficient points for a vector maximization problem. By means of these concepts, we have proved the properties of closure and connnectedness of the mix semi-efficient frontier for a mix quasi-concave problem. The connnectedness of the mix efficient frontier remains an open problem.

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