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Mixed Type Duality for Multiobjective Optimization Problems with set constraints

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Abstract

We propose a pair of vector dual programs where the primal has a feasible region defined by a set constraint, equality and inequality constraints, while the dual can be classified as a "mixed type" one. The duality results are proved under suitable generalized concavity properties. Different dual problems can be obtained as the parameters in the "mixed type" dual program take different values. In this framework, different generalized concavity properties can be used in order to get the duality results. In particular, we deepen on the role of the generalized ρ-concavity and, in the case the feasible region has no set constraints, we state duality results assuming the generalized (F, ρ)-concavity.
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1 Introduction

Duality for vector optimization problems has been one of the main issue throughout different fields such as operation research, economic theory, location theory, management science, theory of computational algorithms. Among the very large literature, different aspects of duality theory have been investigated; we can find contributions on duality for both smooth and non smooth functions, particular objective functions such as bicriteria functions, vector fractional functions, generalized fractional functions have been analyzed and various different approaches have been proposed.

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At least at the best of our knowledge, despite of a very large number of papers on duality, the most part of the recent literature deals with vector optimization problems with a feasible region defined by equality and inequality constraint or by a compact set (for this latter case the reader can see for example the leading article by Tanino and Sawaragy [29]).

In this paper we aim to deal with a vector optimization problem where the feasible region is defined by equality constraints, inequality and a set constraint; the set constraint is not required, a priori, to be open, closed or convex. Besides that, we assume that the image space of the primal objective function is ordered by an arbitrary convex, closed and pointed cone C, which is not necessarily the Paretian cone. Since our duality results are related to the concepts of C-maximal and weakly C-maximal point we first recall this definition and then some necessary optimality conditions which can be classified as a maximum principle conditions. Starting from these results we introduce a "general" dual problem that can be classified as a mixed type one; specifying the value of the parameters in the dual problem, we analyze different kinds of duality, such as Wolfe-type and Mond-Weir-type. Quite recently, mixed type duality has been introduced by Xu [32] for a primal problem where the feasible region is defined only by inequality constraint, while Aghezzaf and Hachimi [3] and Mishra [24] suggest mixed type dual problems for a primal feasible region defined by equality and inequality constraints ([1]). In our mixed type dual program both constraint functions (equality and inequality) and set constraint are considered.

For a feasible region without set constraint, there are many duality results dealing with several kind of generalized concavity properties such as invexity, generalized invexity (see for all [2, 6, 7, 16, 19, 22, 28]), ρ-concavity ([14, 20, 30]), F-concavity (see for example [18]) and, more recently, $(F, \rho)-$concavity.

We first state weak and strong duality under the generalized ρ-concavity; scalar ρ-concavity and ρ-quasiconcavity have been introduced by Vial [30] for nonsmooth functions while ρ-pseudoconcavity are due to Jeyakumar [20]. In this paper we refer to differentiable scalar generalized ρ-concavity (see for example [15]). Furthermore, as far as we know, the existing vector generalized ρ-concavity properties refer to the Paretian cone (some authors simply deal with the componentwise ρ-concavity) and hence we propose the definitions of vector ρ-quasiconcavity and ρ-pseudoconcavity which are strictly related to the ordering cone C. The suggested vector generalized (C^*, Θ)-concavity can be seen as a generalization of the so called vector generalized C^*-concavity studied by Cambini [8, 11].

In the case the primal feasible region is defined simply by equality and inequality constraints, we state duality results even under the generalized

[1] The concept of mixed type duality has been also extended to the class of multiobjective variational problem in [26].

$(F, \rho)-$concavity properties. Starting from the leading article by Preda [27] many studies has been developed on the relationship between duality and generalized $(F, \rho)-$concavity (see for example [3, 4, 5, 24, 26, 27, 32]); as in the ρ-concavity case, unlike the existing definitions, our vector generalized $(F, \rho)-$concavity properties are related to an arbitrary ordering cone C which is not necessarily the Paretian cone.

In the last section of the paper we provide some simple generalized concavity properties implying the required generalized ρ-concavity (or the $(F, \rho)-$concavity). We focus our attention on the different kind of the dual problems that can be obtained according with the different values of the parameters in the "mixed" dual problem. In particular we prove that as far we move from the Mond-Weir type dual problem towards the Wolfe type one, as weaker are the required generalized concavity assumptions.

2 Definitions and preliminary results

It is worth pointing out the concepts of efficiency which are used in the paper. Let $f : A \to \Re^s$, $A \subseteq \Re^n$, let $C \subset \Re^s$ be a closed convex pointed cones with nonempty interior and let $S \subset A$ be a set. Let us use also the notation $C^0 = C \setminus \{0\}$ and let $C^* \subseteq C$ be a cone such that $C^* = C$ or $\text{Int}(C) \subseteq C^* \subseteq C^0$. Define the following multiobjective problem:

$$M_P : \begin{cases} C^*\_\max / C^*\_\min \;\; f(x) \\ x \in S \end{cases}$$

A point $x_0 \in S$ is said to be *a C^*-maximal [C^*-minimal] point for M_P* if

$$\nexists y \in S,\ y \neq x_0, \text{ such that } f(y) \in f(x_0) + C^* \quad [f(y) \in f(x_0) - C^*]$$

in this case we say that

$$x_0 \in C^*\_\arg\max(P) \quad [x_0 \in C^*\_\arg\min(P)].$$

Note that in the case $C^* = C^0$ the previous definition is nothing but the known concept of efficiency, while when $C^* = \text{Int}(C)$ it is the concept of weak efficiency.

In this paper duality results for the following multiobjective nonlinear problem P will be studied.

Definition 2.1 (Primal Problem)

$$P : \begin{cases} C\_\max \;\; f(x) \\ x \in S_P \end{cases} \equiv \begin{cases} C\_\max \;\; f(x) & \\ g(x) \in V & \textit{inequality constraints} \\ h(x) = 0 & \textit{equality constraints} \\ x \in X & \textit{set constraint} \end{cases}$$

where

$$S_P = \{x \in A : \ g(x) \in V, \ h(x) = 0, \ x \in X\},$$

$A \subseteq \Re^n$ is an open convex set, $f : A \longrightarrow \Re^s$ and $g : A \longrightarrow \Re^m$ are Gâteaux differentiable functions, $h : A \longrightarrow \Re^p$ is a Fréchet differentiable function. Moreover $C \subset \Re^s$ and $V \subset \Re^m$ are closed convex pointed cones with nonempty interior (that is to say convex pointed solid cones), and $X \subseteq A$ is a set which is not required to be open or convex or with nonempty interior.

Throughout the paper we will denote with C^+ and V^+ the positive polar cones of C and V, respectively.

The following necessary optimality condition of the maximum principle type holds for problem P (see [12] which generalizes the results in [21]).

Theorem 2.1 *Consider problem P and assume X to be convex. If the feasible point $x_0 \in X$ is a local efficient point then:*

(C_X) $\exists\,(\alpha_f, \alpha_g, \alpha_h) \in (C^+ \times V^+ \times \Re^p)$, $(\alpha_f, \alpha_g, \alpha_h) \neq 0$, *such that:*

$$\alpha_g^T g(x_0) = 0 \quad and \quad [\alpha_f^T J_f(x_0) + \alpha_g^T J_g(x_0) + \alpha_h^T J_h(x_0)] \in -T(X, x_0)^+$$

Moreover, the following further results hold:

i) if $p = 0$ or $J_h(x_0)[T(X, x_0)] = \Re^p$ then condition (C_X) is verified with $(\alpha_f, \alpha_g) \neq 0$,

ii) if the constraint qualification

$$J_{g,h}(x_0)[T(X, x_0)] = \Re^{m+p} \quad where \quad J_{g,h}(x_0) = \begin{bmatrix} J_g(x_0) \\ J_h(x_0) \end{bmatrix}$$

holds, then condition (C_X) is verified with $\alpha_f \neq 0$,

iii) let $p = 0$ or $J_h(x_0)[T(X, x_0)] = \Re^p$, if the constraint qualification

$$\{d \in \Re^n : J_g(x_0)d \in \mathrm{Int}(V)\} \cap \mathrm{Ker}(J_h(x_0)) \cap T(X, x_0) \neq \emptyset$$

holds, then condition (C_X) is verified with $\alpha_f \neq 0$.

Remark 2.1 In particular, note that condition (C_X) implies that:

- $\exists\,(\alpha_f, \alpha_g, \alpha_h) \in (C^+ \times V^+ \times \Re^p)$, $(\alpha_f, \alpha_g, \alpha_h) \neq 0$, such that:

  $$\alpha_g^T g(x_0) = 0 \ \text{ and } \ [\alpha_f^T J_f(x_0) + \alpha_g^T J_g(x_0) + \alpha_h^T J_h(x_0)](y - x_0) \leq 0 \ \ \forall y \in X$$

3 Duality

Starting from the necessary optimality condition of Theorem 2.1 we define a "general" dual problem for P and we state the related duality results.

3.1 Dual problem

Definition 3.1 Consider problem P, a $0-1$ parameter δ (i.e. $\delta \in \{0,1\}$), and $c \in \text{Int}(C)$. Given the set of indices $\mathcal{P} = \{1,\ldots,p\}$, let $\mathcal{J} = \{J_1, J_2, J_3, J_4\}$ be a partition of $\mathcal{P}$, and accordingly with the partition $\mathcal{J}$, define $h(x) = [h_1(x), h_2(x), h_3(x), h_4(x)]$ and $\alpha_h = (\alpha_{h_1}, \alpha_{h_2}, \alpha_{h_3}, \alpha_{h_4})$. The following Dual problem of P can be introduced

$$D : \left\{ \begin{array}{c} C\_\min \;\; \mathcal{L}(x,\alpha_f,\alpha_g,\alpha_h) = f(x) + \frac{c}{\alpha_f^T c}[\delta\alpha_g^T g(x) + \alpha_{h_1}^T h_1(x)] \\ (x,\alpha_f,\alpha_g,\alpha_h) \in S_D \end{array} \right. ,$$

where

$$S_D = \left\{ \begin{array}{c} (x,\alpha_f,\alpha_g,\alpha_h) \in (A \times C^+ \times V^+ \times \Re^p),\ \alpha_f \neq 0, \\ \left[\alpha_f^T J_f(x) + \alpha_g^T J_g(x) + \alpha_h^T J_h(x)\right](y-x) \leq 0\ \forall y \in X, \\ (1-\delta)\alpha_g^T g(x) + \alpha_{h_2}^T h_2(x) \leq 0 \\ \alpha_{h_3}^T h_3(x) = 0,\ \ \alpha_{h_4}^T h_4(x) \leq 0 \end{array} \right\}$$

For the sake of convenience, the following scalar function is also defined:

$$\Delta(x,\alpha_g,\alpha_h) = (1-\delta)\alpha_g^T g(x) + \alpha_h^T h(x) - \alpha_{h_1}^T h_1(x)$$

Remark 3.1 Note that several different dual problems can be obtained depending on the value of the parameter δ and on the choice of the partition $\mathcal{J} = \{J_1, J_2, J_3, J_4\}$. In particular:

- when $\delta = 0$ and $J_3 = \{1,\ldots,p\}$ (hence $J_1 = J_2 = J_4 = \emptyset$) we generalize the Mond-Weir dual to minimum principle problems,
- for $\delta = 1$ and $J_1 = \{1,\ldots,p\}$ (hence $J_2 = J_3 = J_4 = \emptyset$) we provide a generalization of the Wolfe dual to minimum principle problems and to vector valued objective functions

In the other cases we have a sort of mixed type dual problem ([2]). It can be easily seen that when $\delta = 1$ and $J_1 = \{1,\ldots,p\}$ the dual problem has the most "complex" objective function while with $\delta = 0$ and $J_1 = \emptyset$, it has the simplest one. Moreover, in the case $\delta = 1$ and $J_1 = \{1,\ldots,p\}$ the feasible region of the dual problem is the biggest one, while in the case $\delta = 0$ and $J_1 = \emptyset$, $J_3 = \{1,\ldots,p\}$, the feasible region is the smallest one.

As the reader will see, whenever you get duality results by defining a simpler objective function (see the case $\delta = 0$ and $J_1 = \emptyset$, $J_3 = \{1,\ldots,p\}$), the feasible region of the dual problem is smaller and viceversa a bigger feasible region (see $\delta = 1$ and $J_1 = \{1,\ldots,p\}$) is "paid" by a more complex objective function.

[2] In the case C and V are the Paretian cone and a primal feasible region is defined by inequality constraint, mixed type dual problems have been firstly introduced by Xu in [32]. In the case of a primal feasible region with both equality and inequality constraint, similar kind of mixed type of dual problems have been proposed in [2, 3, 24].

Since duality theorems are stated under suitable generalized concavity properties of functions $\mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h)$ and $\Delta(x, \alpha_g, \alpha_h)$, we will use the concepts of ρ-quasiconcavity and ρ-pseudoconcavity for differentiable functions. With this aim let us recall that these generalized concavity properties have been already studied in [15, 20, 30]. For the readers' convenience we provide the following definitions which will be used throughout the paper.

Definition 3.2 Let $f : A \to \Re$, with $A \subseteq \Re^n$ open and convex, be a differentiable scalar function. Given a value $\rho \in \Re$ function f is said to be *ρ-quasiconcave* in A if the following implication holds $\forall x_1, x_2 \in A$, $x_1 \neq x_2$:

$$f(x_1) \geq f(x_2) \quad \Rightarrow \quad \nabla f(x_2)^T(x_1 - x_2) \geq \rho \|x_2 - x_1\|^2$$

while it is said to be *strictly ρ-pseudoconcave* in A if the following implication holds $\forall x_1, x_2 \in A$, $x_1 \neq x_2$:

$$f(x_1) \geq f(x_2) \quad \Rightarrow \quad \nabla f(x_2)^T(x_1 - x_2) > \rho \|x_2 - x_1\|^2$$

At the best of our knowledge, the existing ρ-generalized concavity properties for multiobjective functions are related to the Paretian cone. On the other hand, in our primal and dual problems we refer to an arbitrary closed, convex and pointed cone C; due to this we suggest the following new definitions of vector valued generalized (C^*, Θ)-concavity which directly takes into account the order relation induced by the cone C.

Definition 3.3 Let $f : A \to \Re^s$, with $A \subseteq \Re^n$ open and convex, be a differentiable vector valued function, $C \subset \Re^s$ be a closed convex pointed cones with nonempty interior. Given a vector $\Theta \in \Re^n$ function f is said to be *(C^*, Θ)-quasiconcave* in A if the following implication holds $\forall x_1, x_2 \in A$, $x_1 \neq x_2$:

$$f(x_1) \in f(x_2) + C^* \quad \Rightarrow \quad J_f(x_2)(x_1 - x_2) \in \Theta \|x_2 - x_1\|^2 + C$$

while it is said to be *(C^*, Θ)-pseudoconcave* in A if the following implication holds $\forall x_1, x_2 \in A$, $x_1 \neq x_2$:

$$f(x_1) \in f(x_2) + C^* \quad \Rightarrow \quad J_f(x_2)(x_1 - x_2) \in \Theta \|x_2 - x_1\|^2 + \text{Int}(C)$$

Remark 3.2 Scalar ρ-quasiconcave and strictly ρ-pseudoconcave functions are nothing but a generalization of the very well known quasiconcave and strictly pseudoconcave functions, respectively, which can be obtained just as a particular case assuming $\rho = 0$ (see for all [1]). On the other hand, the vector valued (C^*, Θ)-quasiconcave and (C^*, Θ)-pseudoconcave functions are a generalization of weakly (C^*, C)-quasiconcave (also known as differentiable C^*-quasiconcave) and $(C^*, \text{Int}(C))$-pseudoconcave functions ([3]), respectively, which can be obtained as a particular case assuming $\Theta = 0$ (see [8, 11]). Note finally that in Definition 3.3 the bigger is the cone C^* the stronger is the generalized concavity property of the function.

The following generalized concavity properties in the set A with respect to the variable x are going to be used in the next sections regarding to functions $\mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h)$ and $\Delta(x, \alpha_g, \alpha_h)$:

(C_1) $\forall(\alpha_f, \alpha_g, \alpha_h) \in (C^+ \times V^+ \times \Re^p)$, $\alpha_f \neq 0$, function $\mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h)$ is (C^*, Θ)-quasiconcave and function $\Delta(x, \alpha_g, \alpha_h)$ is ρ-quasiconcave with $\rho + \alpha_f^T \Theta > 0$;

(C_2) $\forall(\alpha_f, \alpha_g, \alpha_h) \in (C^+ \times V^+ \times \Re^p)$, $\alpha_f \neq 0$, function $\mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h)$ is (C^*, Θ)-pseudoconcave and function $\Delta(x, \alpha_g, \alpha_h)$ is ρ-quasiconcave with $\rho + \alpha_f^T \Theta \geq 0$;

(C_3) $\forall(\alpha_f, \alpha_g, \alpha_h) \in (C^+ \times V^+ \times \Re^p)$, $\alpha_f \neq 0$, function $\mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h)$ is (C^*, Θ)-quasiconcave and function $\Delta(x, \alpha_g, \alpha_h)$ is strictly ρ-pseudoconcave with $\rho + \alpha_f^T \Theta \geq 0$.

3.2 Weak Duality

Let us now prove the weak duality for the pair of dual problems introduced so far. With this aim, the following preliminary results are proved.

Lemma 3.1 *Let us consider the primal problem P and the dual problem D. The following statements hold:*

i) $\mathcal{L}(x_1, \alpha_f, \alpha_g, \alpha_h) \in f(x_1) + C$ $\forall(\alpha_f, \alpha_g, \alpha_h) \in (C^+ \times V^+ \times \Re^p)$, $\alpha_f \neq 0$, $\forall x_1 \in S_P$;

ii) $\Delta(x_2, \alpha_g, \alpha_h) \leq 0 \leq \Delta(x_1, \alpha_g, \alpha_h)$ $\forall(x_2, \alpha_f, \alpha_g, \alpha_h) \in S_D$ *and* $\forall x_1 \in S_P$.

[3][8, 11] Let $f : A \to \Re^s$, with $A \subseteq \Re^n$ open and convex, be a differentiable function, $C \subset \Re^s$ be a closed convex pointed cones with nonempty interior. Function f is said to be $(C^*, \text{Int}(C))$-*pseudoconcave* in A if the following implication holds $\forall x_1, x_2 \in A$, $x_1 \neq x_2$:

$$f(x_1) \in f(x_2) + C^* \quad \Rightarrow \quad J_f(x_2)(x_1 - x_2) \in \text{Int}(C)$$

while it is said to be *weakly* (C^*, C)-*quasiconcave* in A if the following implication holds $\forall x_1, x_2 \in A$, $x_1 \neq x_2$:

$$f(x_1) \in f(x_2) + C^* \quad \Rightarrow \quad J_f(x_2)(x_1 - x_2) \in C.$$

Proof *i)* Since $x_1 \in S_P$ it is $g(x_1) \in V$ and $h(x_1) = 0$, so that $\alpha_g \in V^+$ implies $\alpha_g^T g(x_1) \geq 0$ and hence $\delta \alpha_g^T g(x_1) + \alpha_{h_1}^T h_1(x_1) \geq 0$. By definition it is

$$\mathcal{L}(x_1, \alpha_f, \alpha_g, \alpha_h) = f(x_1) + \frac{c}{\alpha_f^T c}[\delta \alpha_g^T g(x_1) + \alpha_{h_1}^T h_1(x_1)]$$

with $c \in \text{Int}(C)$, so that the result follows since $\alpha_f \in C^+ \setminus \{0\}$ implies $\frac{c}{\alpha_f^T c} \in \text{Int}(C)$.

ii) Follows directly from the definitions. □

The following theorem provides the weak duality result for the pair of dual problems previously defined; it is worth noticing that we do not need to assume the convexity of the set X.

Theorem 3.1 (Weak duality) *Let us consider the primal problem P and the dual problem D, assuming that at least one of conditions (C_1), (C_2) and (C_3) is verified. Then $\forall x_1 \in S_P$ and $\forall (x_2, \alpha_f, \alpha_g, \alpha_h) \in S_D$, it is*

$$f(x_1) \notin \mathcal{L}(x_2, \alpha_f, \alpha_g, \alpha_h) + C^*,$$

where in the case $C^* = C$ it is also assumed that $x_1 \neq x_2$.

Proof Assume condition (C_1) holds and suppose by contradiction that $\exists x_1 \in S_P$ and $\exists (x_2, \alpha_f, \alpha_g, \alpha_h) \in S_D$ such that

$$f(x_1) \in \mathcal{L}(x_2, \alpha_f, \alpha_g, \alpha_h) + C^*$$

For condition *i)* of Lemma 3.1 we have

$$\mathcal{L}(x_1, \alpha_f, \alpha_g, \alpha_h) \in f(x_1) + C$$

so that, since C is a closed convex pointed cone with nonempty interior, it results

$$\mathcal{L}(x_1, \alpha_f, \alpha_g, \alpha_h) \in \mathcal{L}(x_2, \alpha_f, \alpha_g, \alpha_h) + C^*$$

It can be easily seen that $x_1 \neq x_2$, in fact if $C^* = C$ this is guaranteed by the hypothesis, while if $C^* \neq C$, that is $C^* \subseteq C^0$, this is implied by the previous condition. Since $\mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h)$ is (C^*, Θ)-quasiconcave with respect to the variable x it yields

$$J_{\mathcal{L}}(x_2, \alpha_f, \alpha_g, \alpha_h)(x_1 - x_2) \in \Theta \|x_2 - x_1\|^2 + C$$

hence, since $\alpha_f \in C^+ \setminus \{0\}$, it results

$$\alpha_f^T J_{\mathcal{L}}(x_2, \alpha_f, \alpha_g, \alpha_h)(x_1 - x_2) \geq \alpha_f^T \Theta \|x_2 - x_1\|^2$$

As a consequence we get

$$[\alpha_f^T J_f(x_2) + \delta\alpha_g^T J_g(x_2) + \alpha_{h_1}^T J_{h_1}(x_2)](x_1 - x_2) \geq \alpha_f^T \Theta \|x_2 - x_1\|^2 \quad (3.1)$$

From property *ii)* of Lemma 3.1 and the ρ-quasiconcavity of $\Delta(x, \alpha_g, \alpha_h)$ we have

$$[(1-\delta)\alpha_g^T J_g(x_2) + \alpha_h^T J_h(x_2) - \alpha_{h_1}^T J_{h_1}(x_2)](x_1 - x_2) \geq \rho\|x_2 - x_1\|^2 \quad (3.2)$$

so that, by adding (3.1) and (3.2), since $\rho + \alpha_f^T\Theta > 0$ we get:

$$[\alpha_f^T J_f(x_2) + \alpha_g^T J_g(x_2) + \alpha_h^T J_h(x_2)]\,(x_1 - x_2) \geq (\rho + \alpha_f^T\Theta)\|x_2 - x_1\|^2 > 0$$

This implies that $(x_2, \alpha_f, \alpha_g, \alpha_h) \notin S_D$, which is a contradiction.

The proofs for the assumptions (C_2) and (C_3) are analogous and hence they are omitted. □

Note that, in the previous theorem, the bigger is the cone C^* (and hence the stronger is the generalized concavity property of $\mathcal{L}(x_1, \alpha_f, \alpha_g, \alpha_h)$), the stronger is the proved necessary condition.

Corollary 3.1 *Let us consider the primal problem P and the dual problem D, assuming that at least one of conditions (C_1), (C_2) and (C_3) is verified. If $(x, \alpha_f, \alpha_g, \alpha_h) \in S_D$ with $\delta\alpha_g^T g(x) = 0$ and $x \in S_P$ then*

$$x \in C^*\_\arg\max(P) \text{ and } (x, \alpha_f, \alpha_g, \alpha_h) \in C^*\_\arg\min(D).$$

Proof As a preliminary result, note that $x \in S_P$ and $\delta\alpha_g g(x) = 0$ imply $\mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h) = f(x)$.
Suppose by contradiction that $x \notin C^*\_\arg\max(P)$, that is there exists $y \in S_p$ such that $f(y) \in f(x) + C^*$; hence $f(y) \in \mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h) + C^*$ and this contradicts the weak duality result.
Now suppose by contradiction that $(x, \alpha_f, \alpha_g, \alpha_h) \notin C^*\_\arg\min(D)$, that is there exists $(\widehat{x}, \widehat{\alpha}_f, \widehat{\alpha}_g, \widehat{\alpha}_h) \in S_D$ such that $\mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h) \in \mathcal{L}(\widehat{x}, \widehat{\alpha}_f, \widehat{\alpha}_g, \widehat{\alpha}_h) + C^*$; hence $f(x) \in \mathcal{L}(\widehat{x}, \widehat{\alpha}_f, \widehat{\alpha}_g, \widehat{\alpha}_h) + C^*$ and this contradicts the weak duality result. □

3.3 Strong Duality

We are now ready to prove the following results related to strong duality. With this aim, in this subsection we will assume the set X to be convex.

Theorem 3.2 (Strong duality) *Let us consider the primal problem P and the dual problem D, assuming that at least one of conditions (C_1), (C_2) and*

(C_3) *is verified. Assume also that X is convex and a constraint qualification holds for problem P. Then $\forall x \in C^0\_\arg\max(P)$ $\exists\alpha_f \in C^+ \setminus \{0\}$, $\exists\alpha_g \in V^+$, $\exists\alpha_h \in \Re^p$ such that:*

$$(x, \alpha_f, \alpha_g, \alpha_h) \in C^*\_\arg\min(D) \quad \textit{and} \quad f(x) = \mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h)$$

Proof Let $x \in C^0\_\arg\max(P)$; by means of Theorem 2.1 $\exists\alpha_f \in C^+ \setminus \{0\}$, $\exists\alpha_g \in V^+$, $\exists\alpha_h \in \Re^p$ such that $\alpha_g^T g(x) = 0$ and

$$[\alpha_f^T J_f(x)^T + \alpha_g^T J_g(x) + \alpha_h^T J_h(x)](y - x) \leq 0 \ \forall y \in X.$$

Since $h(x) = 0$ and $\alpha_g^T g(x) = 0$ it results $f(x) = \mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h)$ and $(x, \alpha_f, \alpha_g, \alpha_h) \in S_D$. For the weak duality theorem $\nexists(\widehat{x}, \widehat{\alpha}_f, \widehat{\alpha}_g, \widehat{\alpha}_h) \in S_D$ such that

$$\mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h) = f(x) \in \mathcal{L}(\widehat{x}, \widehat{\alpha}_f, \widehat{\alpha}_g, \widehat{\alpha}_h) + C^*$$

In other words, $\nexists(\widehat{x}, \widehat{\alpha}_f, \widehat{\alpha}_g, \widehat{\alpha}_h) \in S_D$ such that

$$\mathcal{L}(\widehat{x}, \widehat{\alpha}_f, \widehat{\alpha}_g, \widehat{\alpha}_h) \in \mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h) - C^*$$

and hence $(x, \alpha_f, \alpha_g, \alpha_h) \in C^*\_\arg\min(D)$. □

The following result follows directly from Theorem 3.2.

Corollary 3.2 *Let us consider the primal problem P and the dual problem D, assuming that at least one of conditions (C_1), (C_2) and (C_3) is verified. Assume also that X is convex and a constraint qualification holds for problem P. If $C^*\_\arg\min(D) = \emptyset$, then $C^0\_\arg\max(P) = \emptyset$.*

The following further duality result follows from the weak and the strong duality theorems.

Corollary 3.3 *Let us consider the primal problem P and the dual problem D, assuming that at least one of conditions (C_1), (C_2) and (C_3) is verified. Suppose that X is convex and a constraint qualification holds for problem P; then*

$$f(x_1) - \mathcal{L}(x_2, \alpha_f, \alpha_g, \alpha_h) \notin (C^* \cup -C^*)$$

$\forall x_1 \in C^0\_\arg\max(P)$ *and* $\forall(x_2, \alpha_f, \alpha_g, \alpha_h) \in C^*\_\arg\min(D)$.

Proof Let $x_1 \in C^0\_\arg\max(P)$ and $(x_2, \alpha_f, \alpha_g, \alpha_h) \in C^*\_\arg\min(D)$; for the weak duality theorem it is

$$f(x_1) - \mathcal{L}(x_2, \alpha_f, \alpha_g, \alpha_h) \notin C^*$$

For the strong duality theorem $\exists \alpha_f \in C^+ \setminus \{0\}$, $\exists \alpha_g \in V^+$, $\exists \alpha_h \in \Re^p$ such that $(x_1, \alpha_f, \alpha_g, \alpha_h) \in C^*\_\arg\min(D)$ and $f(x_1) = \mathcal{L}(x_1, \alpha_f, \alpha_g, \alpha_h)$. As a consequence, condition $(x_2, \alpha_f, \alpha_g, \alpha_h) \in C^*\_\arg\min(D)$ implies

$$\mathcal{L}(x_1, \alpha_f, \alpha_g, \alpha_h) \notin \mathcal{L}(x_2, \alpha_f, \alpha_g, \alpha_h) - C^*$$

and hence for the equality $f(x_1) = \mathcal{L}(x_1, \alpha_f, \alpha_g, \alpha_h)$ we have

$$f(x_1) - \mathcal{L}(x_2, \alpha_f, \alpha_g, \alpha_h) \notin -C^*$$

which proves the result. □

4 The case $X = A$ and (F, ρ)-quasiconcavity

In the recent literature, when C and V are the Paretian cone and $X = A$, some duality results have been stated with the use of generalized (F, ρ)-concave functions (see for example [3, 4, 5, 27, 24, 32]), which are nothing but a generalization of the generalized ρ-concave functions previously handled. It is worth noticing that an interesting study about relationship between (F, ρ)-convexity and $\rho-$invexity appeared in [13]. Due to the widespread interest for this kind of generalized concavity properties, we aim to deal with this issue too; we consider the case $X = A$, but we still assume that C and V are arbitrarily closed convex pointed cones.

Definition 4.1 Consider problem P where $X = A$; the dual problem reduces to the following form.

$$D^* : \begin{cases} C\_\min \;\; \mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h) = f(x) + \frac{c}{\alpha_f^T c}[\delta \alpha_g^T g(x) + \alpha_{h_1}^T h_1(x)] \\ \qquad (x, \alpha_f, \alpha_g, \alpha_h) \in S_{D^*} \end{cases},$$

where

$$S_{D^*} = \begin{Bmatrix} (x, \alpha_f, \alpha_g, \alpha_h) \in (A \times C^+ \times V^+ \times \Re^p), \;\; \alpha_f \neq 0, \\ \alpha_f^T J_f(x) + \alpha_g^T J_g(x) + \alpha_h^T J_h(x) = 0, \\ (1 - \delta)\alpha_g^T g(x) + \alpha_{h_2}^T h_2(x) \leq 0 \\ \alpha_{h_3}^T h_3(x) = 0, \;\; \alpha_{h_4}^T h_4(x) \leq 0 \end{Bmatrix}$$

For the sake of completeness we recall the following definitions which are going to be used in the duality theorems (see for example [3, 4, 27, 32]).

Definition 4.2 A functional $F : X \times X \times \Re^n \to \Re$ is said to superlinear if for every x_1, $x_2 \in X$ it is

$$F(x_1, x_2, a_1 + a_2) \geq F(x_1, x_2, a_1) + F(x_1, x_2, a_2) \;\; \forall a_1, a_2 \in \Re^n$$
$$F(x_1, x_2, \alpha a) = \alpha F(x_1, x_2, a) \;\; \forall a \in \Re^n, \forall \alpha \in \Re, \; \alpha \geq 0.$$

Definition 4.3 Let $f : A \rightarrow \Re$, with $A \subseteq \Re^n$ open and convex, be a differentiable scalar function. Given a value $\rho \in \Re$ and a superlinear functional F, function f is said to be *(F,ρ)-quasiconcave* in A if the following implication holds $\forall x_1, x_2 \in A$, $x_1 \neq x_2$:

$$f(x_1) \geq f(x_2) \quad \Rightarrow \quad F\left(x_1, x_2, \nabla f(x_2)\right) \geq \rho \|x_2 - x_1\|^2$$

while it is said to be *strictly (F,ρ)-pseudoconcave* in A if the following implication holds $\forall x_1, x_2 \in A$, $x_1 \neq x_2$:

$$f(x_1) \geq f(x_2) \quad \Rightarrow \quad F\left(x_1, x_2, \nabla f(x_2)\right) > \rho \|x_2 - x_1\|^2$$

Similarly to the case of generalized ρ-concavity, the existing (F,ρ)-generalized concavity properties for multiobjective functions are related to the Paretian cone and often defined componentwisely. Therefore we suggest new definitions of both *(C^*, F, Θ)-quasiconcavity* and *(C^*, F, Θ)-pseudoconcavity* which directly take into account the order relation induced by the cone C.

Definition 4.4 Let $f : A \rightarrow \Re^s$, with $A \subseteq \Re^n$ open and convex, be a differentiable vector valued function, $C \subset \Re^s$ be a closed convex pointed cones with nonempty interior. Let F be a superlinear functional and denote

$$F\left(x_1, x_2, J_f(x_2)\right) = \begin{bmatrix} F\left(x_1, x_2, \nabla f_1(x_2)\right) \\ F\left(x_1, x_2, \nabla f_2(x_2)\right) \\ \cdots \\ F\left(x_1, x_2, \nabla f_n(x_2)\right) \end{bmatrix}$$

Assume that F satisfies the following condition :

$$F\left(x_1, x_2, \alpha_f^T J_f(x_2)\right) \geq \alpha_f^T F\left(x_1, x_2, J_f(x_2)\right) \ \forall \alpha_f \in C^+ \qquad (4.1)$$

Given a vector $\Theta \in \Re^n$ function f is said to be *(C^*, F, Θ)-quasiconcave* in A if the following implication holds $\forall x_1, x_2 \in A$, $x_1 \neq x_2$:

$$f(x_1) \in f(x_2) + C^* \quad \Rightarrow F\left(x_1, x_2, J_f(x_2)\right) \in \Theta \|x_2 - x_1\|^2 + C$$

while it is said to be *(C^*, F, Θ)-pseudoconcave* in A if the following implication holds $\forall x_1, x_2 \in A$, $x_1 \neq x_2$:

$$f(x_1) \in f(x_2) + C^* \quad \Rightarrow F\left(x_1, x_2, J_f(x_2)\right) \in \Theta \|x_2 - x_1\|^2 + \mathrm{Int}(C)$$

In the case C is the Paretian cone, Condition (4.1) is directly implied by the superlineary of F. If in addiction it is $C^* = C^0$ $[C^* = C]$, the *(C^*, F, Θ)-quasiconcavity* coincide with the weak *(F, Θ)-quasiconcavity* *[(F, Θ)-quasiconcavity]* used in [3, 27]. On the other hand if C is the Paretian cone and $C^* = C^0$ $[C^* = \mathrm{Int}(C)]$, the *(C^*, F, Θ)-pseudoconcavity* is the weak strictly *(F, Θ)-pseudoconcavity* *[(F, Θ)-pseudoconcavity]* analyzed in [3, 27].

Theorem 4.1 *Let us consider the primal problem P with $X = A$ and the dual problem D^*. Assume also that at least one of the following generalized convexity properties in the set A with respect to the variable x holds for functions $\mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h)$ and $\Delta(x, \alpha_g, \alpha_h)$:*

(F_1) *$\forall(\alpha_f, \alpha_g, \alpha_h) \in (C^+ \times V^+ \times \Re^p)$, $\alpha_f \neq 0$, function $\mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h)$ is (C^*, F, Θ)-quasiconcave and function $\Delta(x, \alpha_g, \alpha_h)$ is (F, ρ)-quasiconcave with $\rho + \alpha_f^T \Theta > 0$;*

(F_2) *$\forall(\alpha_f, \alpha_g, \alpha_h) \in (C^+ \times V^+ \times \Re^p)$, $\alpha_f \neq 0$, function $\mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h)$ is (C^*, F, Θ)-pseudoconcave and function $\Delta(x, \alpha_g, \alpha_h)$ is (F, ρ)-quasiconcave with $\rho + \alpha_f^T \Theta \geq 0$;*

(F_3) *$\forall(\alpha_f, \alpha_g, \alpha_h) \in (C^+ \times V^+ \times \Re^p)$, $\alpha_f \neq 0$, function $\mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h)$ is (C^*, F, Θ)-quasiconcave and function $\Delta(x, \alpha_g, \alpha_h)$ is strictly (F, ρ)-pseudoconcave with $\rho + \alpha_f^T \Theta \geq 0$.*

Then the following condition holds $\forall x_1 \in S_P$ and $\forall(x_2, \alpha_f, \alpha_g, \alpha_h) \in S_{D^*}$:

$$f(x_1) \notin \mathcal{L}(x_2, \alpha_f, \alpha_g, \alpha_h) + C^*$$

where in the case $C^* = C$ it is also assumed that $x_1 \neq x_2$.

Proof Assume (F_1) holds and suppose by contradiction that $\exists x_1 \in S_P$ and $\exists(x_2, \alpha_f, \alpha_g, \alpha_h) \in S_{D^*}$ such that

$$f(x_1) \in \mathcal{L}(x_2, \alpha_f, \alpha_g, \alpha_h) + C^*.$$

For condition *i)* of Lemma 3.1 we have

$$\mathcal{L}(x_1, \alpha_f, \alpha_g, \alpha_h) \in f(x_1) + C$$

so that, since C is a closed convex pointed cone with nonempty interior, it results

$$\mathcal{L}(x_1, \alpha_f, \alpha_g, \alpha_h) \in \mathcal{L}(x_2, \alpha_f, \alpha_g, \alpha_h) + C^*.$$

It can be easily seen that $x_1 \neq x_2$, in fact if $C^* = C$ this is guaranteed by the hypothesis, while if $C^* \neq C$, that is $C^* \subseteq C^0$, this is implied by the previous condition. Since $\mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h)$ is (C^*, F, Θ)-quasiconcave with respect to the variable x it yields

$$F(x_1, x_2, J_{\mathcal{L}}(x_2, \alpha_f, \alpha_g, \alpha_h)) \in \Theta\|x_2 - x_1\|^2 + C. \tag{4.2}$$

Since $\alpha_f \in C^+ \setminus \{0\}$, from Conditions (4.1) and (4.2) it results

$$\begin{aligned} F\left(x_1, x_2, \alpha_f^T J_{\mathcal{L}}(x_2, \alpha_f, \alpha_g, \alpha_h)\right) &\geq \alpha_f^T F(x_1, x_2, J_{\mathcal{L}}(x_2, \alpha_f, \alpha_g, \alpha_h)) \\ &\geq \alpha_f^T \Theta\|x_2 - x_1\|^2. \end{aligned} \tag{4.3}$$

From property *ii)* of Lemma 3.1 and the (F, ρ)-quasiconcavity of $\Delta(x, \alpha_g, \alpha_h)$ we have

$$F(x_1, x_2, \nabla\Delta(x_2, \alpha_g, \alpha_h)) \geq \rho\|x_2 - x_1\|^2. \tag{4.4}$$

Observe that since $(x_2, \alpha_f, \alpha_g, \alpha_h) \in S_{D^*}$ it is

$$\nabla\Delta(x_2, \alpha_g, \alpha_h) + \alpha_f^T J_{\mathcal{L}}(x_2, \alpha_f, \alpha_g, \alpha_h) = 0.$$

By adding (4.3) and (4.4), since $\rho + \alpha_f^T\Theta > 0$ it follows:

$$\begin{aligned} 0 = F(x_1, x_2, 0) &= F\left(x_1, x_2, \nabla\Delta(x_2, \alpha_g, \alpha_h) + \alpha_f^T J_{\mathcal{L}}(x_2, \alpha_f, \alpha_g, \alpha_h)\right) \geq \\ &\geq F(x_1, x_2, \nabla\Delta(x_2, \alpha_g, \alpha_h)) + F\left(x_1, x_2, \alpha_f^T J_{\mathcal{L}}(x_2, \alpha_f, \alpha_g, \alpha_h)\right) \\ &\geq \left(\rho + \alpha_f^T\Theta\right)\|x_2 - x_1\|^2 > 0 \end{aligned}$$

which can not be true.

The proofs for the assumptions (F_2) and (F_3) are analogous and hence they are omitted. □

Obviously, the statement of Corollary 3.1 still holds if conditions (C_1), (C_2) and (C_3) are replaced by (F_1), (F_2) and (F_3) and again, the strong duality results similarly follow as the ones already proved in Section 3.3.

5 Generalized concavity of f, g, h

In the previous sections duality results are proved under very general conditions, that is Conditions (C_1), (C_2),(C_3) and Conditions (F_1), (F_2),(F_3). As a conclusion of the paper we aim to investigate the role of generalized concavity properties which are less general but, at the same time, easier to be checked. With this regards, the following proposition holds.

Proposition 5.1 *Let us consider the primal problem P and the dual problem D. The following statements hold:*

- *i) if function $\mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h)$ is $(C^*, \mathrm{Int}(C))$-pseudoconcave in A with respect to the variable x and function $\Delta(x, \alpha_g, \alpha_h)$ is quasiconcave in A with respect to the variable x then conditions (C_2) and (F_2) hold;*
- *ii) if function $\mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h)$ is weakly (C^*, C)-quasiconcave in A with respect to the variable x and function $\Delta(x, \alpha_g, \alpha_h)$ is strictly pseudoconcave in A with respect to the variable x then conditions (C_3) and (F_3) hold.*

Proof They follow directly from the definitions assuming $\rho = 0$, $\Theta = 0$ and, if the case, $F(x_1, x_2, \nabla f(x_2)) = \nabla f(x_2)^T(x_1 - x_2)$ or $F(x_1, x_2, J_f(x_2)) = J_f(x_2)(x_1 - x_2)$. □

Since the quasiconcavity of function $\Delta(x, \alpha_g, \alpha_h)$ plays a fundamental role in the previous result, we are interested in finding conditions which imply this crucial property; with this aim we give the following proposition.

Proposition 5.2 *Consider Problem P and the dual Problem D.*
If g is polarly V-quasiconcave ([4]) *in A and h is affine then* $\Delta(x, \alpha_g, \alpha_h)$ *is quasiconcave in A with respect to the variable x.*

Proof Since h is affine, it is quasiconcave; recalling that the sum of two quasiconcave functions is quasiconcave, the result follows directly from the definition of polarly V-quasiconcavity. □

Observe that Proposition 5.1 and 5.2 hold regardless the specific form of the dual problem D. Nevertheless according with the value of δ and the choice of $\mathcal{J}$, we have different kind of generalized concavity properties of functions f, g and h, that guarantee one of the conditions (C_2) ,(C_3) (or equivalently (F_2),(F_3)). In this section we give sufficient conditions for the pseudoconcavity of function $\mathcal{L}$ and hence for Condition (C_2) ([5]); we prove that as you move from the case $\delta = 1$ and $J_1 = \{1, \ldots, p\}$ to the one with $\delta = 0$ and $J_1 = \emptyset$, you can require weaker generalized concavity assumptions in order to guarantee the pseudoconcavity of $\mathcal{L}$. We are going to deal with the case $\delta = 1$ and $\delta = 0$ separately.

Proposition 5.3 *Consider Problem P and the dual Problem D where function h is affine.*

i) Assume that $\delta = 1$. $\mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h)$ *is* $(C^*, \mathrm{Int}(C))$*-pseudoconcave in A with respect to the variable x if one of the following conditions is verified:*

i.a) f is C-concave ([6]) *in A, g is V-concave in A and* $C^* = \mathrm{Int}(C)$.

i.b) f is $\mathrm{Int}(C)$*-concave in A and g is V-concave in A.*

[4]A function f is said to be *polarly C-quasiconcave* if and only if $\phi(x) = \alpha^T f(x)$ is quasiconcave $\forall \alpha \in C^+$, $\alpha \neq 0$, that is to say if and only if $\forall \alpha \in C^+$, $\alpha \neq 0$, $\forall x, y \in A$, $x \neq y$, it holds:

$$\alpha^T f(y) \geq \alpha^T f(x) \quad \Rightarrow \quad \alpha^T J_f(x)(y - x) \geq 0.$$

[5]Similar results can be easily stated and proved for Conditions (C_3) or (F_3).

[6]Let $f : A \to \Re^s$, with $A \subseteq \Re^n$ open and convex, be a differentiable function, $C \subset \Re^s$ be a closed convex pointed cones with nonempty interior. Function f is said to be

i.c) f is C-concave in A and g is $\mathrm{Int}(V)$-concave in A.

ii) Assume that $\delta = 0$ and $h_1 \neq \emptyset$. $\mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h)$ is $(C^*, \mathrm{Int}(C))$-pseudoconcave in A with respect to the variable x if one of the following conditions is verified:

ii.a) f is C-concave in A and $C^* = \mathrm{Int}(C)$.

ii.b) f is $\mathrm{Int}(C)$-concave in A.

iii) Assume that $\delta = 0$ and $h_1 = \emptyset$.
If f is $(C^*, \mathrm{Int}(C))$-pseudoconcave in A then $\mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h)$ is $(C^*, \mathrm{Int}(C))$-pseudoconcave in A with respect to the variable x.

Proof i) When $\delta = 1$, the dual problem is specified as follows

$$D: \left\{ \begin{array}{ll} C\_\min & \mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h) = f(x) + \frac{c}{\alpha_f^T c}[\alpha_g^T g(x) + \alpha_{h_1}^T h_1(x)] \\ & (x, \alpha_f, \alpha_g, \alpha_h) \in S_D \end{array} \right. ,$$

where

$$S_D = \left\{ \begin{array}{c} (x, \alpha_f, \alpha_g, \alpha_h) \in (A \times C^+ \times V^+ \times \Re^p), \ \alpha_f \neq 0, \\ \left[\alpha_f^T J_f(x) + \alpha_g^T J_g(x) + \alpha_h^T J_h(x)\right](y - x) \leq 0 \ \forall y \in X, \\ \alpha_{h_2}^T h_2(x) \leq 0 \\ \alpha_{h_3}^T h_3(x) = 0, \ \alpha_{h_4}^T h_4(x) \leq 0 \end{array} \right\}$$

We are have to show that $\mathcal{L}$ is $(C^*, \mathrm{Int}(C))$-pseudoconcave in A. Consider $x_1, x_2 \in A$ such that $\mathcal{L}(x_1, \alpha_f, \alpha_g, \alpha_h) \in \mathcal{L}(x_2, \alpha_f, \alpha_g, \alpha_h) + \mathrm{Int}(C)$, that is $\mathcal{L}(x_1, \alpha_f, \alpha_g, \alpha_h) - \mathcal{L}(x_2, \alpha_f, \alpha_g, \alpha_h) \in \mathrm{Int}(C)$. From the concavity assumptions of f and g and from the affinity of h we get

$$\begin{array}{l} f(x_1) = f(x_2) + J_f(x_2)(x_1 - x_2) - c_1 \\ g(x_1) = g(x_2) + J_g(x_2)(x_1 - x_2) - v_1 \\ h(x_1) = h(x_2) + J_h(x_2)(x_1 - x_2) \end{array}$$

where $c_1 \in C$ and $v_1 \in V$. Therefore

$$\begin{array}{rl} & \mathcal{L}(x_1, \alpha_f, \alpha_g, \alpha_h) - \mathcal{L}(x_2, \alpha_f, \alpha_g, \alpha_h) = \\ = & f(x_1) - f(x_2) + \frac{c}{\alpha_f^T c}[\alpha_g^T g(x_1) + \alpha_{h_1}^T h_1(x_1) - \alpha_g^T g(x_2) - \alpha_{h_1}^T h_1(x_2)] \\ = & J_f(x_2)(x_1 - x_2) - c_1 + \frac{c}{\alpha_f^T c}[J_g(x_2)(x_1 - x_2) - \alpha_g^T v_1 + J_h(x_2)(x_1 - x_2)] \\ = & \left[J_f(x_2) + \frac{c}{\alpha_f^T c}[J_g(x_2) + J_h(x_2)\right](x_1 - x_2) - c_1 - \frac{c}{\alpha_f^T c}\left(\alpha_g^T v\right). \end{array}$$

C^*-concave if and only if $\forall x, y \in A$, $x \neq y$, it holds:

$$f(y) - f(x) - J_f(x)(y - x) \in -C^*.$$

Since $\alpha_g^T v \geq 0$, it results $\frac{c}{\alpha_f^T c}\left(\alpha_g^T v\right) \in C$;
from $\mathcal{L}(x_1, \alpha_f, \alpha_g, \alpha_h) - \mathcal{L}(x_2, \alpha_f, \alpha_g, \alpha_h) \in \mathrm{Int}(C)$, there exists $\overline{c} \in \mathrm{Int}(C)$ such that $\left[J_f(x_2) + \frac{c}{\alpha_f^T c}[J_g(x_2) + J_h(x_2)\right](x_1 - x_2) = \overline{c} + c_1 + \frac{c}{\alpha_f^T c}\left(\alpha_g^T v\right)$
and therefore

$$J_{\mathcal{L}}(...)\,(x_1 - x_2) = \left[J_f(x_2) + \frac{c}{\alpha_f^T c}[J_g(x_2) + J_h(x_2)\right](x_1 - x_2) \in \mathrm{Int}(C)$$

i.b)-i.c) The proof is analogous of case i.a).
ii) The result follows along the lines of i).
ii.b) In this case $\mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h) = f(x)$ and hence the result is trivial.
iv) Obvious since $\mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h) = f(x)$. □

Remark 5.1 It is worth noticing that since a C^*-concave function is also polarly quasiconcave (see [9, 10]), the conditions stated in Proposition 5.3-i) guarantee the quasiconcavity of $\Delta(x, \alpha_g, \alpha_h)$ with respect to the variable x.

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