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Alberto Cambini *- Laura Martein †- Siegfried Schaible ‡

Abstract

In [7] Charnes and Cooper reduce a linear fractional program to a linear program with help of a suitable transformation of variables. We show that this transformation preserves pseudoconvexity of a differentiable function.

KeyWords Fractional programming, pseudoconvexity, Charnes-Cooper type transformation.
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1 Introduction

In this note we will show that the Charnes-Cooper transformation [7] as well as a generalized version of it preserve the pseudoconvexity of a differentiable function f.

Such a result generalizes the previous one given by the authors in the case where f is a twice differentiable function [16]. The result will be derived by showing that a pseudomonotone gradient of a differentiable function turns into a pseudomonotone gradient of the transformed function where the Charnes-Cooper type transformation is applied to the function (not to the gradient). This implies the result of the paper since pseudomonotonicity of the gradient corresponds to pseudoconvexity of a differentiable function [17].

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2 The Charnes-Cooper transformation

Consider the following transformation from $\Re^n$ to $\Re^n$

$$y = \frac{Ax}{b^T x + b_0} \tag{2.1}$$

defined on the set $\Gamma = \{x \in \Re^n : b^T x + b_0 > 0\}$ where A is a nonsingular matrix of order n, $b \in \Re^n$ and $b_0 \neq 0$.

Lemma 2.1 *The inverse of the transformation (2.1) is given by*

$$x = \frac{b_0 A^{-1} y}{1 - b^T A^{-1} y} \tag{2.2}$$

$$y \in \Gamma^* = \{y \in \Re^n : \frac{b_0}{1 - b^T A^{-1} y} > 0\} \tag{2.3}$$

Proof *From (2.1) we have* $A^{-1}y = \frac{x}{b^T x + b_0}$, $b^T A^{-1} y = \frac{b^T x}{b^T x + b_0} = 1 - \frac{b_0}{b^T x + b_0}$, *so that*

$$\frac{1}{b^T x + b_0} = \frac{1 - b^T A^{-1} y}{b_0} \tag{2.4}$$

The thesis is achieved taking into account that $b^T x + b_0 > 0$ *implies* $\frac{b_0}{1 - b^T A^{-1} y} > 0$ *and that* $x = (b^T x + b_0) A^{-1} y$. □

Denote with J_x, J_y the Jacobian matrices of the transformation (2.1), (2.2), respectively; from differential calculus rules, we have

$$J_x = \frac{A}{b^T x + b_0}[I - \frac{x b^T}{b^T x + b_0}] \tag{2.5}$$

$$J_y = \frac{b_0 A^{-1}}{1 - b^T A^{-1} y}[I + \frac{y b^T A^{-1}}{1 - b^T A^{-1} y}] \tag{2.6}$$

Let $J_{x(y)}$ ($J_{y(x)}$) the Jacobian matrix J_x (J_y) evaluated at $x = \frac{b_0 A^{-1} y}{1 - b^T A^{-1} y}$ ($y = \frac{Ax}{b^T x + b_0}$).
The following lemma holds.

Lemma 2.2
i)

$$J_{x(y)} = \frac{1 - b^T A^{-1} y}{b_0}[A - y b^T] \tag{2.7}$$

$$J_{y(x)} = (b^T x + b_0)[A^{-1} + \frac{x b^T A^{-1}}{b_0}] \tag{2.8}$$

ii) $J_y J_{x(y)} = I, \quad J_x J_{y(x)} = I.$

Proof

i) (2.7) follows directly from (2.5) substituting (2.4) and (2.2). In a similar way (2.8) follows.

ii) $J_y J_{x(y)} = \frac{b_0 A^{-1}}{1-b^T A^{-1} y}[I + \frac{y b^T A^{-1}}{1-b^T A^{-1} y}] \cdot \frac{1-b^T A^{-1} y}{b_0}[A - y b^T] =$

$=[A^{-1} + \frac{A^{-1} y b^T A^{-1}}{1-b^T A^{-1} y}] \cdot [A - y b^T] = I - A^{-1} y b^T + \frac{A^{-1} y b^T}{1-b^T A^{-1} y} - \frac{A^{-1} y b^T A^{-1} y b^T}{1-b^T A^{-1} y} =$

$= I - A^{-1} y b^T + \frac{A^{-1} y b^T}{1-b^T A^{-1} y} - \frac{A^{-1} y b^T (b^T A^{-1} y)}{1-b^T A^{-1} y} = I - A^{-1} y b^T + \frac{(1-b^T A^{-1} y) A^{-1} y b^T}{1-b^T A^{-1} y} = I.$

In a analogous way it can be proved that $J_x J_{y(x)} = I$. □

In the following we will utilize the following lemma.

Lemma 2.3 *Let* $x, \bar{x} \in \Gamma$, $y, \bar{y} \in \Gamma^*$ *such that* $x = \frac{b_0 A^{-1} y}{1-b^T A^{-1} y}$, $\bar{x} = \frac{b_0 A^{-1} \bar{y}}{1-b^T A^{-1} \bar{y}}$. *We have*

$$(\bar{x} - x)^T J^T_{x(y)} = \frac{1 - b^T A^{-1} y}{1 - b^T A^{-1} \bar{y}} (\bar{y} - y)^T \tag{2.9}$$

$$(\bar{x} - x)^T = \frac{1 - b^T A^{-1} y}{1 - b^T A^{-1} \bar{y}} (\bar{y} - y)^T J^T_y \tag{2.10}$$

$$(\bar{x} - x)^T J^T_{x(\bar{y})} = \frac{1 - b^T A^{-1} \bar{y}}{1 - b^T A^{-1} y} (\bar{y} - y)^T \tag{2.11}$$

$$(\bar{x} - x)^T = \frac{1 - b^T A^{-1} \bar{y}}{1 - b^T A^{-1} y} (\bar{y} - y)^T J^T_{\bar{y}}. \tag{2.12}$$

Proof *We have* $(\bar{x} - x)^T J^T_{x(y)} = \bar{x}^T J^T_{x(y)} - x^T J^T_{x(y)}$. *On the other hand:*

- $x^T J^T_{x(y)} = \frac{b_0 y^T (A^{-1})^T}{1-b^T A^{-1} y} (\frac{1-b^T A^{-1} y}{b_0})(A^T - b y^T) = (1 - b^T A^{-1} y) y^T.$
- $\bar{x}^T J^T_{x(y)} = \frac{b_0 \bar{y}^T (A^{-1})^T}{1-b^T A^{-1} \bar{y}} (\frac{1-b^T A^{-1} y}{b_0})(A^T - b y^T) = \frac{1-b^T A^{-1} y}{1-b^T A^{-1} \bar{y}} [\bar{y}^T - \bar{y}^T (A^{-1})^T b y^T].$

Consequently $(\bar{x} - x)^T J^T_{x(y)} = \frac{1-b^T A^{-1} y}{1-b^T A^{-1} \bar{y}} [\bar{y}^T - \bar{y}^T (A^{-1})^T b y^T - (1 - b^T A^{-1} \bar{y}) y^T] =$
$\frac{1-b^T A^{-1} y}{1-b^T A^{-1} \bar{y}} [\bar{y}^T - y^T - \bar{y}^T (A^{-1})^T b y^T + b^T A^{-1} \bar{y} y^T].$

Since $b^T A^{-1} \bar{y} = \bar{y}^T (A^{-1})^T b$, *(2.9) is achieved.*

Taking into account that the inverse of $J_{x(y)}$ *is* J_y, *(2.10) follows.*

From (2.9), substituting $\bar{x}$ *with* x *and* x *with* $\bar{x}$, *we obtain* $(x - \bar{x})^T J^T_{x(\bar{y})} = \frac{1-b^T A^{-1} \bar{y}}{1-b^T A^{-1} y} (y - \bar{y})^T$, *which is equivalent to (2.11). Finally, (2.12) follows taking into account that* $J_{x(\bar{y})}$ *is the inverse of* $J_{\bar{y}}$. □

3 Pseudoconvexity under the Charnes-Cooper transformation

Let $f(x)$ be a differentiable real-valued function defined on an open and convex subset S of $\Re^n$ and consider the function $\psi(y)$ obtained applying the previous Charnes- Cooper transformation to $f(x)$. Obviously we have $f(x(y)) = \psi(y)$ and $f(x) = \psi(y(x))$.
By the differential calculus rules we have
$\nabla f(x) = J_x^T \nabla\psi(y(x))$, $\nabla\psi(y) = J_y^T \nabla f(x(y))$.
We recall that a function F is pseudomonotone on an open and convex set $C \subseteq \Re^n$ if and only if for every pair of distinct points $v, w \in C$ we have

$$(w-v)^T F(v) > 0 \Rightarrow (w-v)^T F(w) > 0. \tag{3.1}$$

The following theorem points out that the Charnes-Cooper transformation preserves the pseudomonotonicity of the gradient of the function.

Theorem 3.1 *$\nabla f(x)$ is pseudomonotone if and only if $\nabla\psi(y)$ is pseudomonotone.*
Proof *Assume the pseudomonotonicity of the gradient of $f(x)$. We must show that $(\bar{y}-y)^T\nabla\psi(y) > 0$ implies $(\bar{y}-y)^T\nabla\psi(\bar{y}) > 0$. Taking into account Lemma (2.3) and ii) of Lemma (2.2), we have*
$(\bar{y}-y)^T\nabla\psi(y) = (\bar{y}-y)^T J_y^T \nabla f(x(y)) = \frac{1-b^TA^{-1}\bar{y}}{1-b^TA^{-1}y}(\bar{x}-x)^T J_{x(y)}^T J_y^T \nabla f(x) =$
$= \frac{1-b^TA^{-1}\bar{y}}{1-b^TA^{-1}y}(\bar{x}-x)^T\nabla f(x)$. *Since* $\frac{1-b^TA^{-1}\bar{y}}{1-b^TA^{-1}y} > 0$, *the inequality* $(\bar{y}-y)^T\nabla\psi(y) > 0$ *implies* $(\bar{x}-x)^T\nabla f(x) > 0$ *so that, for the pseudomonotonicity of* $\nabla f(x)$, *we have* $(\bar{x}-x)^T\nabla f(\bar{x}) > 0$. *From (2.12) we have*
$(\bar{x}-x)^T\nabla f(\bar{x}) = \frac{1-b^TA^{-1}\bar{y}}{1-b^TA^{-1}y}(\bar{y}-y)^T J_{\bar{y}}^T \nabla f(y(\bar{x})) = \frac{1-b^TA^{-1}\bar{y}}{1-b^TA^{-1}y}(\bar{y}-y)^T\nabla\psi(\bar{y})$, *so that* $(\bar{x}-x)^T\nabla f(\bar{x}) > 0$ *implies* $(\bar{y}-y)^T\nabla\psi(\bar{y}) > 0$ *and the thesis follows.*
Assume now the pseudomonotonicity of the gradient of $\psi(y)$. *We must prove that* $(\bar{x}-x)^T\nabla f(x) > 0$ *implies* $(\bar{x}-x)^T\nabla f(\bar{x}) > 0$. *Taking into account (2.10), we have*
$(\bar{x}-x)^T\nabla f(x) = \frac{1-b^TA^{-1}y}{1-b^TA^{-1}\bar{y}}(\bar{y}-y)^T J_y^T \nabla f(y(x)) = \frac{1-b^TA^{-1}y}{1-b^TA^{-1}\bar{y}}(\bar{y}-y)^T\nabla\psi(y)$ *and thus* $(\bar{x}-x)^T\nabla f(x) > 0$ *implies* $(\bar{y}-y)^T\nabla\psi(y) > 0$; *for the pseudomonotonicity of* $\nabla\psi(y)$, *we have* $(\bar{y}-y)^T\nabla\psi(\bar{y}) > 0$. *Taking into account (2.11) we have*
$(\bar{y}-y)^T\nabla\psi(\bar{y}) = \frac{1-b^TA^{-1}y}{1-b^TA^{-1}\bar{y}}(\bar{x}-x)^T J_{x(\bar{y})}^T \nabla\psi(\bar{y}) = \frac{1-b^TA^{-1}y}{1-b^TA^{-1}\bar{y}}(\bar{x}-x)^T J_{x(\bar{y})}^T J_{\bar{y}}^T \nabla f(\bar{x}) =$
$\frac{1-b^TA^{-1}y}{1-b^TA^{-1}\bar{y}}(\bar{x}-x)^T\nabla f(\bar{x})$.
Consequently, $(\bar{y}-y)^T\nabla\psi(\bar{y}) > 0$ *implies*
$(\bar{x}-x)^T\nabla f(\bar{x}) > 0$ *and the thesis is complete.*

□

Corollary 3.1 *The Charnes-Cooper transformation (2.1) preserves the pseudoconvexity of a function f, that is $f(x)$ is pseudoconvex if and only if $\psi(y)$ is pseudoconvex.*
Proof *It is sufficient to note that a function is pseudoconvex if and only if its gradient map is pseudomonotone.*

Remark 3.1 *While the pseudoconvexity of a function is preserved under a Charnes-Cooper transformation, the pseudomonotonicity of a map is not preserved under a Charnes- Cooper transformation. Consider for instance, the pseudomonotone function $f(x) = x^3$. For $y = -\frac{x}{x+1}$ $(x > -1)$, hence $x = -\frac{y}{y+1}$ $(y > -1)$, we obtain $f(x(y)) = (-\frac{y}{y+1})^3$ which is not pseudomonotone. To see this, let $y = -\frac{1}{2}$, $z = 1$. Then $(z-y)f(x(y)) = \frac{3}{2} > 0$, but $(z-y)f(x(z)) = -\frac{3}{16} < 0$.*
To avoid a misinterpretation of Theorem 3.1, we emphasize that the Charnes-Cooper transformation is applied to the function $f(x)$, not to its gradient. In other words, the pseudomonotonicity of the gradient map is just a means to obtain the main result given in Corollary 3.1.

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